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ON CERTAIN SUBCLASS OF MULTIVALENT FUNCTIONS OF COMPLEX ORDER

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Abstract. The aim of the present paper is to show several properties of functions belonging to a subclass $M^p_{n,\Omega}(A,B,\lambda,b)$ (where b is complex number with $\mathbf{Re}(b)>0$ and A ve B are two arbitrary constants with $-1\le B<A\le 1$). Coefficient estimates and some distortion theorems for this class of functions are obtained. For this class we also derive the radii of close-to-convexity, starlikeness, and convexity. Further, an application involving fractional calculus for functions in $M^p_{n,\Omega}(A,B,\lambda,b)$ is given.

1. Introduction

Let $A_p(n)$ denote the family of functions of the form:

$$f(z)=z^p-\sum_{k=n+p}^{\infty}a_kz^k \qquad (a_k\ge 0;\ n,p\in N=\{1,2,3,...\}), \tag{1.1}$$

which are analytic and $p-$valent in the open unit disk

$$U=\{z: z\in C \text{ and } |z|<1\}.$$

Then the Hadamard product (or convolution) of a function $f\in A_p(=A_p(1))$ defined by (1.1) and a function $g\in A_p$ given by

$$g(z)=z^p-\sum_{k=n+p}^{\infty}b_kz^k \qquad (b_k\ge 0;\ n,p\in N), \tag{1.2}$$

is defined by

$$(f*g)(z)=z^p+\sum_{k=1+p}^{\infty}a_kb_kz^k=(g*f)(z). \tag{1.3}$$

The extended linear derivative operator of Ruscheweyh [1] type $D^{\lambda,p}:A_p\to A_p$ is defined by setting

$$D^{\lambda,p}f(z)=\frac{z^p}{(1-z)^{\lambda+p}}*f(z) \quad (\lambda>-p;\ f\in A_p). \tag{1.4}$$

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One has

$$D^{\lambda,p}f(z)=z^p-\sum_{k=1+p}^{\infty}\left(\begin{array}{c}\lambda+k-1\\ k-p\end{array}\right)a_k z^k \quad (\lambda>-p;\ f\in A_p). \tag{1.5}$$

In particular, if we choose $\lambda=n \ \ (n\in N)$, then $D^{\lambda,p}f(z)=D^{n,p}f(z)$ and

$$D^{n,p}f(z)=\frac{z^p(z^{n-p}f(z))^{(n)}}{n!} \qquad (n,p\in\mathbb{N}). \tag{1.6}$$

We have

$$D^{1,p}f(z)=(1-p)f(z)+zf'(z), \tag{1.7}$$

$$D^{2,p}f(z)=\frac{(1-p)(2-p)f(z)}{2!}+(2-p)zf'(z)+\frac{z^2f''(z)}{2!}, \tag{1.8}$$

and so forth.

We denote by $M^p_{n,\Omega}(A,B,\lambda,b)$ the class of functions $f\in A_p(n)$ that satisfy the condition

$$1+\frac{1}{b}[\frac{z(D^{\lambda,p}f(z))^{\Omega+1}}{(D^{\lambda,p}f(z))^{\Omega}}-(p-\Omega)]\prec\frac{1+Az}{1+Bz}, \tag{1.9}$$

where $\prec$ denotes subordination, $b\neq 0$ is any complex number with $\mathbf{Re}b>0,\ A$ and B are arbitrary fixed numbers, $-1\leq B<A\leq 1$. Some special cases of our results can be found in [4]. Therefore, this paper presents the generalization of the results in [4].

2. Coefficient estimates

We begin by proving a coefficient inequality.

Theorem 2.1. *A necessary and sufficient condition for a function $f\in A_p(n)$ to be in the class $M^p_{n,\Omega}(A,B,\lambda,b)$ is*

$$\frac{\sum\limits_{k=n+p}^{\infty}[(k-p)+|b(A-B)-B(k-p)|]\left(\begin{array}{c}\lambda+k-1\\ k-p\end{array}\right)\frac{k!(p-\Omega)!}{p!(k-\Omega)!}|a_k|}{|b|\,(A-B)}\leq 1. \tag{2.1}$$

Proof. ($\Rightarrow$) By definition of subordination we can write (1.9) as

$$1+\frac{1}{b}[\frac{z(D^{\lambda,p}f(z))^{\Omega+1}}{(D^{\lambda,p}f(z))^{\Omega}}-(p-\Omega)]=\frac{1+Aw(z)}{1+Bw(z)} \qquad (w(z)\in U),$$

$$\frac{z(D^{\lambda,p}f(z))^{\Omega+1}}{(D^{\lambda,p}f(z))^{\Omega}}-(p-\Omega)=[b(A-B)-B\Big(\frac{z(D^{\lambda,p}f(z))^{\Omega+1}}{(D^{\lambda,p}f(z))^{\Omega}}-(p-\Omega)\Big)]w(z), \tag{2.2}$$

$$\frac{(p-\Omega)\begin{pmatrix} p \\ \Omega \end{pmatrix} z^{p-\Omega} - \sum_{k=n+p}^{\infty} \begin{pmatrix} \lambda+k-1 \\ k-p \end{pmatrix} \begin{pmatrix} k \\ \Omega \end{pmatrix} (k-\Omega) a_k z^{k-\Omega}}{\begin{pmatrix} p \\ \Omega \end{pmatrix} z^{p-\Omega} - \sum_{k=n+p}^{\infty} \begin{pmatrix} \lambda+k-1 \\ k-p \end{pmatrix} \begin{pmatrix} k \\ \Omega \end{pmatrix} a_k z^{k-\Omega}} - (p-\Omega)$$

$$= \Bigg(b(A-B)$$

$$- B\Bigg(\frac{(p-\Omega)\begin{pmatrix} p \\ \Omega \end{pmatrix} z^{p-\Omega} - \sum_{k=n+p}^{\infty} \begin{pmatrix} \lambda+k-1 \\ k-p \end{pmatrix} \begin{pmatrix} k \\ \Omega \end{pmatrix} (k-\Omega) a_k z^{k-\Omega}}{\begin{pmatrix} p \\ \Omega \end{pmatrix} z^{p-\Omega} - \sum_{k=n+p}^{\infty} \begin{pmatrix} \lambda+k-1 \\ k-p \end{pmatrix} \begin{pmatrix} k \\ \Omega \end{pmatrix} a_k z^{k-\Omega}}$$

$$- (p-\Omega)\Bigg)\Bigg) w(z).$$

That is,

$$\frac{\sum_{k=n+p}^{\infty} \begin{pmatrix} \lambda+k-1 \\ k-p \end{pmatrix} \frac{k!(p-\Omega)!}{p!(k-\Omega)!} (p-k) a_k z^{k-p}}{1 - \sum_{k=n+p}^{\infty} \begin{pmatrix} \lambda+k-1 \\ k-p \end{pmatrix} \frac{k!(p-\Omega)!}{p!(k-\Omega)!} a_k z^{k-p}}$$

$$= (b(A-B) - B \left(\frac{\sum_{k=n+p}^{\infty} \begin{pmatrix} \lambda+k-1 \\ k-p \end{pmatrix} \frac{k!(p-\Omega)!}{p!(k-\Omega)!} (p-k) a_k z^{k-p}}{1 - \sum_{k=n+p}^{\infty} \begin{pmatrix} \lambda+k-1 \\ k-p \end{pmatrix} \frac{k!(p-\Omega)!}{p!(k-\Omega)!} a_k z^{k-p}} \right)) w(z).$$

Since $|w(z)| < 1$,

$$\left| \sum_{k=n+p}^{\infty} \begin{pmatrix} \lambda+k-1 \\ k-p \end{pmatrix} \frac{k!(p-\Omega)!}{p!(k-\Omega)!} (p-k) a_k z^{k-p} \right|$$

$$\leq \left| b(A-B) - \sum_{k=n+p}^{\infty} [b(A-B) - B(k-p)] \begin{pmatrix} \lambda+k-1 \\ k-p \end{pmatrix} \frac{k!(p-\Omega)!}{p!(k-\Omega)!} a_k z^{k-p} \right|.$$

Letting $z \to 1^-$ through real values we have

$$\sum_{k=n+p}^{\infty} [(k-p) + |b(A-B) - B(k-p)|] \begin{pmatrix} \lambda+k-1 \\ k-p \end{pmatrix} \frac{k!(p-\Omega)!}{p!(k-\Omega)!} |a_k| \leq |b| (A-B),$$

That is,

$$\frac{\sum_{k=n+p}^{\infty} [(k-p) + |b(A-B) - B(k-p)|] \begin{pmatrix} \lambda+k-1 \\ k-p \end{pmatrix} \frac{k!(p-\Omega)!}{p!(k-\Omega)!} |a_k|}{|b| (A-B)} \leq 1.$$

($\Leftarrow$) Let (2.1) be true. Since $|w(z)| < 1$, from (2.2) we see that

(2.3)
$$\left|\frac{z(D^{\lambda,p}f(z))^{\Omega+1}-(p-\Omega)(D^{\lambda,p}f(z))^{\Omega}}{b(A-B)(D^{\lambda,p}f(z))^{\Omega}-B[z(D^{\lambda,p}f(z))^{\Omega+1}-(p-\Omega)(D^{\lambda,p}f(z))^{\Omega}]}\right|$$
$$=\left|\frac{\sum_{k=n+p}^{\infty}\begin{pmatrix}\lambda+k-1\\ k-p\end{pmatrix}\frac{(p-k)k!(p-\Omega)!}{p!(k-\Omega)!}a_kz^{k-p}}{b(A-B)-\sum_{k=n+p}^{\infty}[b(A-B)-B(k-p)]\begin{pmatrix}\lambda+k-1\\ k-p\end{pmatrix}\frac{k!(p-\Omega)!}{p!(k-\Omega)!}a_kz^{k-p}}\right|<1.$$

We must show that (2.3) is true. By applying the hypothesis (2.1) and letting $|z| = 1$ we find that

$$\left|\frac{\sum_{k=n+p}^{\infty}\begin{pmatrix}\lambda+k-1\\ k-p\end{pmatrix}\frac{k!(p-\Omega)!}{p!(k-\Omega)!}(p-k)a_kz^{k-p}}{b(A-B)-\sum_{k=n+p}^{\infty}[b(A-B)-B(k-p)]\begin{pmatrix}\lambda+k-1\\ k-p\end{pmatrix}\frac{k!(p-\Omega)!}{p!(k-\Omega)!}a_kz^{k-p}}\right|$$
$$\leq\frac{\sum_{k=n+p}^{\infty}\begin{pmatrix}\lambda+k-1\\ k-p\end{pmatrix}\frac{k!(p-\Omega)!}{p!(k-\Omega)!}(k-p)\left|a_k\right|}{|b|(A-B)-\sum_{k=n+p}^{\infty}|b(A-B)-B(k-p)|\begin{pmatrix}\lambda+k-1\\ k-p\end{pmatrix}\frac{k!(p-\Omega)!}{p!(k-\Omega)!}\left|a_k\right|}$$
$$\leq\frac{|b|(A-B)-\sum_{k=n+p}^{\infty}|b(A-B)-B(k-p)|\begin{pmatrix}\lambda+k-1\\ k-p\end{pmatrix}\frac{k!(p-\Omega)!}{p!(k-\Omega)!}\left|a_k\right|}{|b|(A-B)-\sum_{k=n+p}^{\infty}|b(A-B)-B(k-p)|\begin{pmatrix}\lambda+k-1\\ k-p\end{pmatrix}\frac{k!(p-\Omega)!}{p!(k-\Omega)!}\left|a_k\right|}\leq 1.$$

Hence we find that (2.3) is true. Therefore $f \in M^p_{n,\Omega}(A, B, \lambda, b)$. □

3. Distortion theorems

In this section we shall prove some distortion thorems for functions belonging to the class $M^p_{n,\Omega}(A, B, \lambda, b)$.

Theorem 3.1. *If $f \in M^p_{n,\Omega}(A, B, \lambda, b)$ then*

$$r^p-r^{p+n}\frac{|b|(A-B)}{(n+|b(A-B)-Bn|)\begin{pmatrix}\lambda+n+p-1\\ n\end{pmatrix}\frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}}\leq|f(z)|$$
$$\leq r^p+r^{p+n}\frac{|b|(A-B)}{(n+|b(A-B)-Bn|)\begin{pmatrix}\lambda+n+p-1\\ n\end{pmatrix}\frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}}$$

$(|z| = r)$, *with equality for*

$$f(z) = z^p - z^{p+n} \frac{|b|(A-B)}{(n + |b(A-B) - Bn|) \begin{pmatrix} \lambda + n + p - 1 \\ n \end{pmatrix} \frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}}$$

Proof. By (2.1) we have

$$\sum_{k=n+p}^{\infty} ((k-p) + |b(A-B) - B(k-p)|) \begin{pmatrix} \lambda + k - 1 \\ k-p \end{pmatrix} \frac{k!(p-\Omega)!}{p!(k-\Omega)!} |a_k| \leq |b|(A-B).$$

$$\sum_{k=n+p}^{\infty} |a_k| \leq \frac{|b|(A-B)}{(n + |b(A-B) - Bn|) \begin{pmatrix} \lambda + n + p - 1 \\ n \end{pmatrix} \frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}}. \tag{3.1}$$

From (1.2) and (3.1) it follows that

$$|f(z)| \geq |z|^p - \sum_{k=n+p}^{\infty} |a_k| |z|^k \geq r^p - r^{n+p} \sum_{k=n+p}^{\infty} |a_k|$$

$$\geq r^p - r^{n+p} \frac{|b|(A-B)}{(n + |b(A-B) - Bn|) \begin{pmatrix} \lambda + n + p - 1 \\ n \end{pmatrix} \frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}}.$$

Similarly,

$$|f(z)| \leq |z|^p + \sum_{k=n+p}^{\infty} |a_k| |z|^k \leq r^p + r^{n+p} \sum_{k=n+p}^{\infty} |a_k|$$

$$\leq r^p + r^{n+p} \frac{|b|(A-B)}{[n + |b(A-B) - Bn|] \begin{pmatrix} \lambda + n + p - 1 \\ n \end{pmatrix} \frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}}.$$

This completes the proof. □

Theorem 3.2. *If* $f \in M^p_{n,\Omega}(A, B, \lambda, b)$ *then*

$$pr^{p-1} - r^{n+p-1} \frac{(p+n)|b|(A-B)}{(n + |b(A-B) - Bn|) \begin{pmatrix} \lambda + n + p - 1 \\ n \end{pmatrix} \frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}} \leq |f'(z)|$$

$$\leq pr^{p-1} + r^{n+p-1} \frac{(p+n)|b|(A-B)}{(n + |b(A-B) - Bn|) \begin{pmatrix} \lambda + n + p - 1 \\ n \end{pmatrix} \frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}}$$

$(|z| = r)$, *with equality for*

$$f(z) = z^p - z^{n+p} \frac{|b|(A-B)}{(n + |b(A-B) - Bn|) \begin{pmatrix} \lambda + n + p - 1 \\ n \end{pmatrix} \frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}}.$$

Proof. By (3.1) we have

(3.2)
$$\sum_{k=n+p}^{\infty} k\,|a_k| \leq \frac{(p+n)\,|b|\,(A-B)}{(n+|b(A-B)-Bn|)\begin{pmatrix}\lambda+n+p-1\\ n\end{pmatrix}\frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}}.$$

From (1.1) and (3.2) it follows that

$$\begin{aligned}\left|f'(z)\right| &\geq p\,|z|^{p-1} - \sum_{k=n+p}^{\infty} k\,|a_k|\,|z|^{k-1} \geq pr^{p-1} - r^{n+p-1}\sum_{k=n+p}^{\infty} k\,|a_k| \\ &\geq pr^{p-1} - r^{n+p-1}\frac{(p+n)\,|b|\,(A-B)}{(n+|b(A-B)-Bn|)\begin{pmatrix}\lambda+n+p-1\\ n\end{pmatrix}\frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}}.\end{aligned}$$

Similarly,

$$\begin{aligned}\left|f'(z)\right| &\leq p\,|z|^{p-1} + \sum_{k=n+p}^{\infty} k\,|a_k|\,|z|^{k-1} \leq pr^{p-1} + r^{n+p-1}\sum_{k=n+p}^{\infty} k\,|a_k| \\ &\leq pr^{p-1} + r^{n+p-1}\frac{(p+n)\,|b|\,(A-B)}{(n+|b(A-B)-Bn|)\begin{pmatrix}\lambda+n+p-1\\ n\end{pmatrix}\frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}}.\end{aligned}$$

The proof is complete. □

4. Close-to-convexity, starlikeness and convexity

In this section, radii of close-to cenvexity, convexity and starlikeness are derived for the class $M_{n,\Omega}^{p}(A,B,\lambda,b)$.

A function $f \in A_p(n)$ is said to be close-to-convex of order δ $(0 \leq \delta < 1)$ if

$$\mathbf{Re}\{f'(z)\} > \delta \tag{4.1}$$

for all $z \in U$. A function $f \in A_p(n)$ is said to be starlike of order δ if

$$\mathbf{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \delta. \tag{4.2}$$

A function $f \in A_p(n)$ is said to be convex of order δ if and only if $zf'(z)$ is starlike of order δ, that is,

$$\mathbf{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \delta. \tag{4.3}$$

Theorem 4.1. *If $f \in M^p_{n,\Omega}(A,B,\lambda,b)$, then f is close-to-convex of order δ in $|z| < r_1(p,n,\Omega,A,B,b,\lambda,\delta)$ where*

$$r_1(p,n,\Omega,A,B,b,\lambda,\delta) =$$

$$\inf_k \left(\frac{(p-\delta)[(p-k)+|p(A-B)-B(k-p)|] \begin{pmatrix} \lambda+k-1 \\ k-p \end{pmatrix} \frac{k!(p-\Omega)!}{p!(k-\Omega)!}}{k\,|b|\,(A-B)} \right)^{\frac{1}{k-p}}.$$

Proof. It is sufficient to show that

$$\left| \frac{zf'(z)}{z^{p-1}} - p \right| \leq \sum_{k=n+p}^{\infty} k\,|a_k|\,|z|^{k-p} \leq p-\delta. \tag{4.4}$$

By (2.1) we have

$$\sum_{k=n+p}^{\infty} [(k-p)+|b(A-B)-B(k-p)|] \begin{pmatrix} \lambda+k-1 \\ k-p \end{pmatrix} \frac{k!(p-\Omega)!}{p!(k-\Omega)!} |a_k| \leq |b|\,(A-B). \tag{4.5}$$

Observe that (4.4) is true if

$$\frac{k\,|z|^{k-p}}{p-\delta} \leq \frac{[(k-p)+|b(A-B)-B(k-p)|] \begin{pmatrix} \lambda+k-1 \\ k-p \end{pmatrix} \frac{k!(p-\Omega)!}{p!(k-\Omega)!}}{|b|\,(A-B)}. \tag{4.6}$$

Solving (4.6) for $|z|$ we obtain

$$|z| \leq \left(\frac{(p-\delta)[(k-p)+|b(A-B)-B(k-p)|] \begin{pmatrix} \lambda+k-1 \\ k-p \end{pmatrix} \frac{k!(p-\Omega)!}{p!(k-\Omega)!}}{k\,|b|\,(A-B)} \right)^{\frac{1}{k-p}}$$

$(p \neq k; p,k \in N)$, which completes the proof. □

Theorem 4.2. *If $f \in M^p_{n,\Omega}(A,B,\lambda,b)$, then f is starlike of order δ in $|z| < r_2(p,n,\Omega,A,B,b,\lambda,\delta)$ where*

$$r_2(p,n,\Omega,A,B,b,\lambda,\delta)$$

$$= \inf_k \left(\frac{(p-\delta)\,((k-p)+|p(A-B)-B(k-p)|) \begin{pmatrix} \lambda+k-1 \\ k-p \end{pmatrix} \frac{k!(p-\Omega)!}{p!(k-\Omega)!}}{(k-\delta)\,(|b|\,(A-B))} \right)^{\frac{1}{k-p}}$$

Proof. We must show that

$$\left|\frac{zf'(z)}{f(z)}-p\right| \leq \frac{\sum_{k=p+n}^{\infty}(k-p)\,|a_k|\,|z|^{k-p}}{1-\sum_{k=p+n}^{\infty}|a_k|\,|z|^{k-p}} \leq p-\delta. \tag{4.7}$$

We see from (4.5) that (4.7) is true if

$$\frac{(k-\delta)\,|z|^{k-p}}{p-\delta} \leq \frac{[(k-p)+|b(A-B)-B(k-p)|]\binom{\lambda+k-1}{k-p}\frac{k!(p-\Omega)!}{p!(k-\Omega)!}}{|b|\,(A-B)}. \tag{4.8}$$

Solving (4.8) for $|z|$ we obtain

$$|z| \leq \left(\frac{(p-\delta)[(p-k)+|p(A-B)-B(k-p)|]\binom{\lambda+k-1}{k-p}\frac{k!(p-\Omega)!}{p!(k-\Omega)!}}{(k-\delta)[|b|\,(A-B)]}\right)^{\frac{1}{k-p}}$$

$(p \neq k; p, k \in N)$. The proof is complete. □

Theorem 4.3. *If $f \in M^p_{n,\Omega}(A,B,\lambda,b)$, then f is convex of order δ in $|z| < r_3(p,n,\Omega,A,B,b,\lambda,\delta)$ where*

$$r_3(p,n,\Omega,A,B,b,\lambda,\delta)$$
$$=\inf_k\left(\frac{p(p-\delta)\left((k-p)+|p(A-B)-B(k-p)|\right)\binom{\lambda+k-1}{k-p}\frac{k!(p-\Omega)!}{p!(k-\Omega)!}}{k(k-\delta)[|p|\,(A-B)]}\right)^{\frac{1}{k-p}}.$$

Proof. We must prove that

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| \leq \frac{\sum_{k=p+n}^{\infty}k(k-p)\,|a_k|\,|z|^{k-p}}{p-\sum_{k=p+n}^{\infty}k\,|a_k|\,|z|^{k-p}} \leq p-\delta. \tag{4.9}$$

From (4.5) we see that (4.9) is true if

(4.10)

$$\frac{k(k-\delta)\,|z|^{k-p}}{p(p-\delta)} \leq \frac{[(k-p)+|b(A-B)-B(k-p)|]\binom{\lambda+k-1}{k-p}\frac{k!(p-\Omega)!}{p!(k-\Omega)!}}{|b|\,(A-B)}.$$

Solving (4.10) for $|z|$ we obtain

$$|z| \le \left(\frac{p(p-\delta)\left((k-p)+|b(A-B)-B(k-p)|\right)\binom{\lambda+k-1}{k-p}\frac{k!(p-\Omega)!}{p!(k-\Omega)!}}{k(k-\delta)\left(|b|\,(A-B)\right)} \right)^{\frac{1}{k-p}}$$

$(p \neq k; p, k \in N)$, which is completes the proof. □

5. An application in the fractional calculus

Let us begin by recalling the following definitions of the fractional calculus which were introduced by Owa in [3].

Definition 5.1. The fractional integral of order δ is defined, for a function $f(z)$, by $D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int\limits_0^z \frac{f(t)}{(z-t)^{1-\delta}} dt$, where $\delta > 0$, $f(z)$ is an analytic function in a simply connected region of the $z-$plane containing the origin, and the multiplicity of $(z-t)^{\delta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-t) > 0$.

Definition 5.2. The fractional derivative of order δ is defined, for a function $f(z)$, by $D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int\limits_0^z \frac{f(t)}{(z-t)^{\delta}} dt$, where $0 \le \delta < 1$, $f(z)$ is an analytic function in a simply connected region of the $z-$plane containing the origin, and the multiplicity of $(z-t)^{-\delta}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-t) > 0$.

Definition 5.3. Under the condition of Definition 5.2, the fractional derivative of order $n+\delta$ is defined by $D^{n+\delta} f(z) = \frac{d^n}{dz^n} D_z^{\delta} f(z)$, where $0 \le \delta < 1$ and $n = 0, 1, 2, \ldots$

Using the above definitions we can the following results.

Theorem 5.1. *If $f \in A_p(n)$ is in the class $M_{n,\Omega}^{p}(A, B, \lambda, b)$ then*

$$\left| D_z^{-\delta} f(z) \right| \le \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} |z|^{p+\delta} \left(1 + \frac{(p+n)\left(|b|\,(A-B)\right)}{(p+n+\delta)\left(n+|b(A-B)-Bn|\right)\binom{\lambda+n+p-1}{n}\frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}} |z| \right) \tag{5.1}$$

and

$$\left| D_z^{-\delta} f(z) \right| \ge \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} |z|^{p+\delta} \left[1 - \frac{(p+n)[|b|\,(A-B)]}{(p+n+\delta)\left(n+|b(A-B)-Bn|\right)\binom{\lambda+n+p-1}{n}\frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}} |z| \right]. \tag{5.2}$$

Proof. From Definition 5.1 we see that

$$D_z^{-\delta} f(z) = \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} z^{p+\delta} - \sum_{k=p+n}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+\delta+1)} a_k z^{k+\delta}, \tag{5.3}$$

$$(\delta > 0;\ k \geq p+n;\ p, n \in N)$$

For convenience, let

$$\phi(k) = \frac{\Gamma(k+1)}{\Gamma(k+\delta+1)}.$$

It is clear that $\phi(k)$ is a decreasing function of k and

$$0 < \phi(k) \leq \phi(p+n) = \frac{\Gamma(p+n+1)}{\Gamma(p+n+\delta+1)}.$$

By (2.1) we have

$$\sum_{k=n+p}^{\infty} |a_k| \leq \frac{|b|\,(A-B)}{(n+|b(A-B)-Bn|)\begin{pmatrix} \lambda+n+p-1 \\ n \end{pmatrix} \frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}}. \tag{5.4}$$

From (5.3) and (5.4) it follows that

$$\left|D_z^{-\delta} f(z)\right| \leq |z|^{p+\delta} \left(\frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} + \phi(p+n)\,|z| \sum_{k=p+n}^{\infty} |a_k| \right)$$

$$\leq \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} |z|^{p+\delta} \left(1+ \right.$$

$$\left. + \frac{(p+n)[|b|\,(A-B)]}{(p+n+\delta)\,(n+|b(A-B)-Bn|)\begin{pmatrix} \lambda+n+p-1 \\ n \end{pmatrix} \frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}} |z| \right)$$

which is equivalent to (5.1) and

$$\left|D_z^{-\delta} f(z)\right| \geq |z|^{p+\delta} \left(\frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} - \phi(p+n)\,|z| \sum_{k=p+n}^{\infty} |a_k| \right)$$

$$\geq \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} |z|^{p+\delta} \left(1- \right.$$

$$\left. - \frac{(p+n)[|b|\,(A-B)]}{(p+n+\delta)\,(n+|b(A-B)-Bn|)\begin{pmatrix} \lambda+n+p-1 \\ n \end{pmatrix} \frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}} |z| \right)$$

which is equivalent to (5.2). $\square$

Theorem 5.2. *If* $f \in A_p(n)$ *is in the class* $M^p_{n,\Omega}(A, B, \lambda, b)$, *then*

$$\left|D^\delta_z f(z)\right| \le \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} |z|^{p-\delta} \Bigg(1+ \tag{5.5}$$

$$+\frac{(p+n)[|b|(A-B)]}{(p+n-\delta)(n+|b(A-B)-Bn|)\begin{pmatrix}\lambda+n+p-1\\ n\end{pmatrix}\frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}}|z|\Bigg).$$

and

$$\left|D^\delta_z f(z)\right| \ge \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} |z|^{p-\delta} \Bigg(1- \tag{5.6}$$

$$-\frac{(p+n)[|b|(A-B)]}{(p+n-\delta)(n+|b(A-B)-Bn|)\begin{pmatrix}\lambda+n+p-1\\ n\end{pmatrix}\frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}}|z|\Bigg).$$

Proof. By using Definition 5.2 we have

$$D^\delta_z f(z) = \frac{\Gamma(p+1)}{\Gamma(p-\delta+1)} z^{p-\delta} - \sum_{k=p+n}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\delta+1)} a_k z^{k-\delta}, \tag{5.7}$$

$$(0 \le \delta < 1;\ k \ge n+p;\ n,\, p \in N).$$

Let

$$\psi(k) = \frac{\Gamma(k+1)}{\Gamma(k-\delta+1)}.$$

Since $\psi(k)$ is a decreasing function of k we have

$$0 < \psi(k) \le \psi(p+n) = \frac{\Gamma(p+n+1)}{\Gamma(p+n-\delta+1)}.$$

By (5.4) we have

$$\sum_{k=n+p}^{\infty} k|a_k| \le \frac{(p+n)|b|(A-B)}{(n+|b(A-B)-Bn|)\begin{pmatrix}\lambda+n+p-1\\ n\end{pmatrix}\frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}}. \tag{5.8}$$

From (5.7) and (5.8) it follows that

$$\left|D^\delta_z f(z)\right| \le |z|^{p-\delta}\left(\frac{\Gamma(p+1)}{\Gamma(p-\delta+1)} + \psi(p+n)|z|\sum_{k=p+n}^{\infty} k|a_k|\right)$$

$$\le \frac{\Gamma(p+1)}{\Gamma(p-\delta+1)}|z|^{p-\delta}\Bigg(1+$$

$$+\frac{(p+n)(|b|(A-B))}{(p+n-\delta)(n+|b(A-B)-Bn|)\begin{pmatrix}\lambda+n+p-1\\ n\end{pmatrix}\frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}}|z|\Bigg)$$

which is equivalent to (5.5) and

$$\left|D_z^{\delta} f(z)\right| \geq |z|^{p-\delta} \left(\frac{\Gamma(p+1)}{\Gamma(p-\delta+1)} - \psi(p+n)\,|z| \sum_{k=p+n}^{\infty} k\,|a_k| \right)$$

$$\geq \frac{\Gamma(p+1)}{\Gamma(p-\delta+1)} |z|^{p-\delta} \left(1 - \frac{(p+n)[|b|\,(A-B)]}{(p+n-\delta)\,(n+|b(A-B)-Bn|) \binom{\lambda+n+p-1}{n} \frac{(p+n)!(p-\Omega)!}{p!(p+n-\Omega)!}} |z| \right)$$

which is equivalent to (5.6). □

Remark. Putting $n = 1$, $\Omega = 0$ and $\lambda = n + 1$ in Theorems 5.1 and 5.2 we obtain the results of [4] .

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