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CONVERGENCE OF TWO-PARAMETER MULTIVALUED MARTINGALES IN THE LIMIT

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Abstract. In this paper, some convergence results of two-parameter multivalued martingales in the limit are presented.

1. Introduction

Multivalued martingales were introduced at the end of the sixties by Van Custem and were studied by several authors. There are several extensions of the notion of martingale such as quasi-martingales, asymptotic martingales, etc. The notion of an one-parameter multivalued martingale in the limit (mil) has been introduced by Castaing and Ezzaki [1]. Recently, Krupa [8] presents new convergence results on one-parameter multivalued mils in the case when the Banach space without the Radon-Nikodym property.

The purpose of the present paper is to go on with the study of two-parameter multivalued mils in a Banach space.

2. Notations and definitions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We shall denote by $\mathfrak{X}$ a separable (real) Banach space and $wkc(\mathfrak{X})$ the collection of all nonempty, weakly compact and convex subsets of $\mathfrak{X}$. Further, let us denote by $\mathbb{N}$ the set of all nonnegative integers and $J = \mathbb{N}\times\mathbb{N}$. Being endowed with the order "$\leqslant$" given by $s = (s_1, s_2) \leqslant t = (t_1, t_2)$ iff $s_1 \leqslant t_1$ and $s_2 \leqslant t_2$, J is a directed set. Let $(\mathcal{F}_t)_{t\in J}$ be a complete stochastic basis of $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., a nondecreasing family of complete sub-fields of $\mathcal{F}$ with $\mathcal{F} = \bigvee_{t\in J} \mathcal{F}_t$.

Given subsets A, B and C of $\mathfrak{X}$, the distance function $d(., C)$, the support function $s(., C)$, the Hausdorff distance and measurable multifunction are defined as in Hiai-Umegaki [7], Hess [6] and Krupa [8]. We say that $(C_t)_{t\in J}$ is *weakly convergent to* C and write $C_t \overset{w}{\to} C$, $t \in J$, if for each $x^* \in \mathfrak{X}^*$ we have

$$s(x^*, C_t) \to s(x^*, C),$$

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for every $t \in J$. Now, let us set

$$s\text{-}\liminf_{t\in J} C_t := \{x \in \mathfrak{X} : \lim d(x, C_t) = 0\},$$
$$w\text{-}\limsup_{t\in J} C_t := \{x \in \mathfrak{X} : \exists\ (t_k)_{k\in\mathbb{N}},\ x_k \to x,\ x_k \in C_{t_k}\},$$

where $(t_k)_{k\in\mathbb{N}}$ is a cofinal subsequence of J.

Finally, (C_t) is said to be *convergent to C in the Mosco sense*, write $M\text{-}\lim_{t\in J} C_t = C$, if $w\text{-}\limsup_{t\in J} C_t = C = s\text{-}\liminf_{t\in J} C_t$. For each $t = (t_1, t_2) \in J$, we put $\mathcal{F}^1_t = \bigvee_{u\in\mathbb{N}} \mathcal{F}_{(t_1,u)}$. A map $\tau : \Omega \to J$ is called an 1-*stopping time* if $[\tau = t] \in \mathcal{F}^1_t,\ t \in J$. The set of all simple 1-stopping times is denoted by T^1. It is well known that equipped with the a.s. order "$\leqslant$" given by $\sigma \leqslant \tau$ iff $\sigma(\omega) \leqslant \tau(\omega)$ a.s., T^1 is a directed set. Moreover, $\overline{\mathbb{N}} := \{(n,n),\ n \in \mathbb{N}\}$ and J can be regarded as two special cofinal subsets of T^1.

3. Main results

From now on, let $\mathcal{L}^1_{wkc(\mathfrak{X})}$ denote the complete metric space of all integrable bounded multifunctions $F : \Omega \to wkc(\mathfrak{X})$.

Unless otherwise stated, we shall consider only processes $(F_t)_{t\in J}$ in $\mathcal{L}^1_{wkc(\mathfrak{X})}$ such that each F_t is $\mathcal{F}_t$-measurable. Related to the constructive results of Talagrand [11] for vector-valued mils, the following definition has been proposed in [13].

Definition 3.1. Let $(F_t)_{t\in J}$ be an adapted sequence of integrably bounded $wkc(\mathfrak{X})$-valued multifunctions. We say that $(F_t)_{t\in J}$ is an optional-*mil*, if $(F_t, \mathcal{F}^1_t)_{t\in J}$ is a mil, i.e., for every $\varepsilon > 0$, there exists $p \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, $\tau \in T^1$, $\overline{p} \leqslant \tau \leqslant \overline{n}$ we have

$$\mathbb{P}(h(F_\tau, \mathbb{E}(F_{\overline{n}}|\mathcal{F}^1_t)) > \varepsilon) < \varepsilon.$$

For multivalued optional-mils, we get the following convergence result, which is better than Theorem 3.2 in [13].

Theorem 3.1. *Let $(F_t)_{t\in J}$ be an uniformly integrable $wkc(\mathfrak{X})$-valued optional-mil. Suppose that*

$$\overline{co} \bigcup_{t\in J} F_t(\omega) \in wkc(\mathfrak{X}), \quad \omega \in \Omega.$$

Then
a) there exists $F \in \mathcal{L}^1_{wkc(\mathfrak{X})}$ such that

$$w\text{-}\lim F_t = F, \quad a.s.,$$

b)

$$\lim_{t\in J} h(F_t, \mathbb{E}(F|\mathcal{F}^1_t)) = 0, \quad a.s.,$$

and

$$M\text{-}\lim_{t\in J} F_t = F, \quad a.s.$$

Proof. We denote by D (D^*) a countable dense subset of the unit ball B (B^*) of $\mathfrak{X}$ ($\mathfrak{X}^*$) in the norm topology of $\mathfrak{X}$ (the Mackey topology of $\mathfrak{X}^*$). Let D_1^* denote the set of all rational linear combinations of members of D^*. Let $(F_t)_{t\in J}$ be given as in the theorem.

a) For any $x^* \in D_1^*$, $(s(x^*, F_t), \mathcal{F}_t^1)_{t\in J}$ is a two-parameter $\mathbb{R}$-valued uniformly integrable mil and the net $(s(x^*, F_\tau))_{\tau\in T^1}$ converges in probability (cf. [13]). Hence, there are a negligible set $N \subset \Omega$ and a $wkc(\mathfrak{X})$-valued multifunction F such that for each $x^* \in D_1$ and any $\omega \in N^c$ such that $\lim_{t\in J} s(x^*, F_t(\omega)) = s(x^*, F(\omega))$. It follows from this and the equicontinuity of the sequence of functions $s(., F_t(\omega))$ that $\lim_{t\in J} s(x^*, F_t(\omega)) = s(x^*, F(\omega))$ for any $\omega \in N^c$ and $x^* \in \mathfrak{X}^*$.

b) First, we note that the weak convergence of (F_t) to F yields that $\|F(\omega)\| \leqslant \liminf_{t\in J} \|F_t(\omega)\|$, a.s.. Fatou's lemma implies $\mathbb{E}\|F\| \leqslant \liminf_{t\in J} \mathbb{E}\|F_t\| < \infty$. Next, we show that

$$\lim_{t\in J} h(F_t, \mathbb{E}(F|\mathcal{F}_t^1)) = 0, \text{ a.s.} \tag{1}$$

To do this, we put

$$X_t(x^*) = s(x^*, F_t) - s(x^*, \mathbb{E}(F|\mathcal{F}_t^1)), \quad x^* \in \mathfrak{X}^*.$$

By (a), there exists a negligible set $N_1 \subset \Omega$ such that for any $\omega \notin N_1$ and any $x^* \in \mathfrak{X}^*$ it holds

$$\lim_{t\in J} s(x^*, F_t(\omega)) = s(x^*, F(\omega)). \tag{2}$$

Since $(s(x^*, \mathbb{E}(F|\mathcal{F}_t^1))_{t\in J}$ is a real-valued martingale and the stochastic basis $(\mathcal{F}_t^1)$ satisfies the condition Vitali (V) (see [10]), for any $x^* \in \mathfrak{X}^*$ by Theorem 4.4.6 in [5] we have

$$\lim_{t\in J} s(x^*, \mathbb{E}(F|\mathcal{F}_t^1)) = s(x^*, F), \text{ a.s.}$$

In view of the properties of the support functions it easy to see that there exists a negligible set $N_2 \subset \Omega$ such that

$$\lim_{t\in J} s(x^*, \mathbb{E}(F|\mathcal{F}_t^1)(\omega)) = s(x^*, F(\omega))$$

for any $x^* \in \mathfrak{X}^*$. Therefore,

$$\lim_{t\in J} X_t(x^*)(\omega) = 0$$

for any $\omega \notin N = N_1 \cup N_2$, $x^* \in \mathfrak{X}^*$. Thus, for any $x^* \in \mathfrak{X}^*$, the real-valued optional-mil $(X_t(x^*))_{t\in J}$ goes a.s. to zero.

Furthermore, it is easy to show that the process $(X_t(.))_{t\in J}$ with values in the Banach space $\mathcal{B}_{\mathbb{R}}(B^*)$ of all bounded real functions defined in B^* is also a optional-mil.

Put

$$\|X_t\|_\infty = \sup_{x^*\in B^*} \|X_t(x^*)\|.$$

Then $\|X_t\|_\infty$ is measurable and integrable. It is clear that $\|X_t\|_\infty = h(F_t, \mathbb{E}(F|\mathcal{F}_t^1))$. We claim that $\lim_{t\in J} \mathbb{E}\|X_t\|_\infty = \infty$ if $(X_t(x^*))_{t\in J}$ does not converge a.s. to zero in the norm $\|.\|_\infty$. Indeed, there exist $a > 0$ and a set $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$ such that $\limsup_{t\in J} \|X_t\|_\infty(\omega) > a$ for any $\omega \in A$. Applying techniques similar to those used in the proof of Talagrand [11, Theorem 6], we have: for every $n_1 \in \mathbb{N}$ and $0 < \varepsilon < \mathbb{P}(A)/4$ there exists $n_2 \in \mathbb{N}$ such that, for each $D \in \mathcal{F}^1_{\overline{n}_1}$ with $\mathbb{P}(D) < \mathbb{P}(A)/4$ and each $n \geqslant n_2$, there exists a set $E \in \mathcal{F}^1_{\overline{n}_2}$ with $E \cap D = \emptyset$ and $\mathbb{P}(E) < \varepsilon$ such that $\int_E \|X_{\overline{n}}\|_\infty d\mathbb{P} \geqslant a\mathbb{P}(A)/16$.

Now we construct by induction an increasing sequence $(n_p) \subset \mathbb{N}$ and a sequence (D_i) of joint sets such that $D_i \in \mathcal{F}^1_{\overline{n}}$ with $D_1 = \emptyset$, $\mathbb{P}(D_i) < 2^{-i}\mathbb{P}(A)$ and $\int_{D_i} \|X_{\overline{n}}\|_\infty d\mathbb{P} \geqslant a\mathbb{P}(A)/16$ for every $i \leqslant p$ and $n \geqslant n_p$. Then

$$\int_D \|X_{\overline{n}}\|_\infty d\mathbb{P} \geqslant (p-1)a\mathbb{P}(A)/16,$$

where $D = \bigcup_{i\leqslant p} D_i$. Thus $\lim_n \mathbb{E}\|X_{\overline{n}}\|_\infty = \infty$.

Next, since F is an integrably bounded multifunction, there exists a sequence $(f^i, i \geqslant 1) \subset S^1_F(\mathcal{F})$ (see [2]) such that $F(\omega) = cl\{f^i(\omega),\ i \geqslant 1\}$ for all $\omega \in \Omega$ and $\widehat{F}_t = cl\{\mathbb{E}(f^i|\mathcal{F}^1_t),\ i \geqslant 1\}$, a.s. for all $t \in J$, where $\widehat{F}_t = \mathbb{E}(F|\mathcal{F}^1_t)$. By the convergence of vector-valued martingale [10, Theorem 12.4],

$$\lim_{t\in J} \|\mathbb{E}(f^i|\mathcal{F}^1_t) - f^i\| = 0, \text{ a.s.}$$

Thus there exists $N_3 \in \mathcal{F}$, $\mathbb{P}(N_3) = 0$, such that for all $\omega \in N_3^c$ it holds

$$\lim_{t\in J} \mathbb{E}(f^i|\mathcal{F}^1_t)(\omega) = f^i(\omega) \quad (i \geqslant 1)$$

and $\widehat{F}_t(\omega) = cl\{\mathbb{E}(f^i|\mathcal{F}^1_t),\ i \geqslant 1\}$, $t \in J$. Hence, we get

$$F(\omega) \subset s\text{-}\liminf_{t\in J} \widehat{F}_t(\omega),\ \omega \notin N_3. \tag{4}$$

Finally, by (1) there exists $N_4 \in \mathcal{F}$, $\mathbb{P}(N_4) = 0$ such that for any $\omega \notin N_4$, $\lim_{t\in J} h(F_t(\omega), \widehat{F}_t(\omega)) = 0$. It implies that for any $\omega \notin N_4$ and any $x \in \mathfrak{X}$

$$\lim_{t\in J}(d(x, F_t(\omega)) - d(x, \widehat{F}_t(\omega)) = 0. \tag{5}$$

By (4) and (5) we have

$$F(\omega) \subset \liminf_{t\in J} F_t(\omega) \quad \text{for } \omega \notin N_3 \cup N_4. \tag{6}$$

It follows from (1) and (6) that $M\text{-}\lim_{t\in J} F_t = F$, a.s. □

Remark. In Theorem 3.1, the condition (i) plays a crucial role. However, we can replace it by Uhl's condition which is weaker.

A sequence (F_t) is said to satisfy *Uhl's condition in the weak topology* if for any $\varepsilon > 0$ there exists a weak compact set K_ε such that for any $\delta > 0$ there exists a set $A_\delta \in \mathcal{F}$, $A_\delta \subset \Omega$ such that $\mathbb{P}(A_\delta) \geqslant 1 - \varepsilon$ and, for any $A \in A_\delta \cap \mathcal{F}$,

$$\bigcup_{t\in J} \int_A F_t \subset \mathbb{P}(A)K_\varepsilon + \delta B.$$

A similar condition for the case of one-parameter can be found in [9,10].

Theorem 3.2. *Let (F_t) be a uniformly integrable optional-mil with values in $wkc(\mathfrak{X})$. Suppose that $(F_t)_{t\in J}$ satisfies Uhl's condition in the weak topology. Then there exists $F \in \mathcal{L}^1_{wkc(\mathfrak{X})}$ such that*

$$M\text{-}\lim_{t\in J} F_t = F \ \ a.s.$$

Proof. Let $(F_t)_{t\in J}$ be given as in the theorem. Since for any $x^* \in \mathfrak{X}^*$, the process $(s(x^*, F_t), \mathcal{F}^1_t)_{t\in J}$ is a uniformly integrable real-valued mil, Theorem 3.2 in [13] implies that there are a negligible set $N \subset \Omega$ and an integrable function φ_{x^*} such that

$$\lim_{t\in J} s(x^*, F_t(\omega)) = \varphi_{x^*}(\omega) \tag{7}$$

for any $x^* \in D_1^*$ and any $\omega \notin N$.

Using an argument similar to the one used in the proof of Theorem 2.2. in [9] (also, in the proof of Proposition 3.1 in [8]), one can show that by Uhl's condition, for each $\varepsilon > 0$, there exists a weakly compact convex set H_ε and a measurable set B_ε with $\mathbb{P}(B_\varepsilon) \geqslant 1 - \varepsilon$ such that

$$\bigcup_{t\in J} \int_A F_t \subset \mathbb{P}(A) H_\varepsilon, \quad \forall A \in \mathcal{F} \cap B_\varepsilon \tag{8}$$

We put $\Omega_n = B_{1/n}$ and $K_n = H_{1/n}$. Without loss of generality, we can assume that $\Omega_n \subset \Omega_{n+1}$ for any $n \geqslant 1$ (if necessary we replace Ω_n by $\Omega_1 \cup \ldots \cup \Omega_n$ and K_n by the absolutely closed convex hull of $K_1, \ldots , K_n$).

By (8), $\int_A F_t \subset \int_A K_n$ for any $A \in \mathcal{F} \cap \Omega_n$ and any $t \in J$. Therefore, there exists a negligible set $N_n \in \mathcal{F} \cap \Omega_n$ such that for any $\omega \in \Omega_n \cap N_n^c$ and all $t \in J$, $F_t(\omega) \subset K_n$.

We put $M = N \cup \bigcup_{n\geqslant 1} N_n$. Since for any $\omega \in \Omega_n \cap M^c$, and for any $x^* \in D_1^*$, $\lim_{t\in J} s(x^*, F_t(\omega)) = \varphi_{x^*}(\omega)$ and $F_t(\omega) \subset K_n$, by Lemma 5.1 in [6], there exists a measurable multifunction G_n defined in $\Omega_n \cap M^c$ with values in $wkc(\mathfrak{X})$ such that $s(x^*, G_n(\omega)) = \lim_{t\in J} s(x^*, F_t(\omega))$ for any $x^* \in \mathfrak{X}^*$ and $\omega \in \Omega_n \cap M^c$.

From (7) we know that for any $m, n \in \mathbb{N}$, $1 \leqslant m < n$, and for any $x^* \in D_1^*$, $\omega \in \Omega_m \cap M^c$, $s(x^*, G_m(\omega)) = s(x^*, G_n(\omega)) = \varphi_{x^*}(\omega)$. Since for any $\omega \in \Omega_m \cap M^c$ the functions $s(., G_m(\omega))$ and $s(., G_n(\omega))$ are equicontinuous in the Mackey topology, $s(x^*, G_m(\omega)) = s(x^*, G_n(\omega))$ for all $x^* \in \mathfrak{X}^*$. Thus, for any $\omega \in \Omega_m \cap M^c$, $G_m(\omega) = G_n(\omega)$. We define a multifunction F such that $F(\omega) = G_n(\omega)$ for some n with $\omega \in \Omega_n \cap M^c$ and $F(\omega) = \{0\}$ for any $\omega \in M \cup (\bigcap_{n=1}^{\infty} \Omega_n^c) =: \widetilde{N}$. It is clear that F is measurable, bounded integrable and $\lim_{t\in J} s(x^*, F_t(\omega)) = s(x^*, F(\omega))$ for any $\omega \in \widetilde{N}^c$. Thus, since $\mathbb{P}(\widetilde{N}) = 0$, we have $\lim_{t\in J} s(x^*, F_t) = s(x^*, F)$, a.s. Finally, applying the argument used in the proof of the part (b) of Theorem 3.1, we have

$$M - \lim_{t \in J} F_t = F \quad \text{a.s..}$$

The proof is complete. □

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