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SOME REMARKS ON APPROXIMATION OF PLURISUBHARMONIC FUNCTIONS

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1. Introduction

Let Ω be a domain in $\mathbf{C}^n$. An upper semicontinuous function $u : \Omega \to [-\infty, \infty)$ is said to be *plurisubharmonic* if the restriction of u to each complex line is subharmonic (we allow the function identically $-\infty$ to be plurisubharmonic). We say that u is strictly plurisubharmonic if for every $z_0 \in \Omega$ there is a neigbourhood U of z_0 and $\lambda > 0$ such that $u(z) - \lambda|z|^2$ is plurisubharmonic on U. We write $\mathcal{PSH}(\Omega)$ for the set of plurisubharmonic functions on $\Omega, \mathcal{PSH}^-(\Omega)$ for the subset of bounded from above and $\mathcal{PSH}^c(\Omega)$ for the set of continuous functions on $\overline{\Omega}$ which are plurisubharmonic on Ω. It is a natural question whether it is possible to approximate elements of one of these classes by elements of a smaller one. It may be useful to recall some known facts regarding this problem. Richberg [5] showed that for every strictly plurisubharmonic continuous function u on Ω and every positive continuous function ε on Ω, there exists a $\mathcal{C}^\infty$ smooth strictly plurisubharmonic $\tilde{u}$ on Ω such that $0 < u - \tilde{u} < \varepsilon$ on Ω. Later, Fornaess and Narasimhan proved that every plurisubharmonic function u on a pseudoconvex domain Ω can be approximated from above by a sequence of $\mathcal{C}^\infty$ smooth strictly plurisubharmonic functions. A remarkable example of Fornaess [1] shows that the above conclusion fails without the assumption on pseudoconvexity of Ω. In fact, the domain Ω in Fornaess'example is a smoothly bounded Hartogs domain in $\mathbf{C}^2$. Recall that Ω is said to be Hartogs if $(z, w) \in \Omega \Rightarrow (z, w') \in \Omega$ provided that $|w| = |w'|$. Regarding approximation on smoothly bounded domain, Fornaess and Wiegerinck [2] proved that every continuous function on $\overline{\Omega}$ which is plurisubharmonic on Ω, can be approximated uniformly on $\overline{\Omega}$ by $\mathcal{C}^\infty$ smooth plurisubharmonic functions on neigbourhoods of $\overline{\Omega}$. Besides, Fornaess and Wiegerinck show that every plurisubharmonic function u on a bounded Reinhardt domain Ω can be approximated from above by a sequence of smooth strictly plurisubharmonic functions on Ω. Recall that Ω is said to be Reinhardt if $(z_1, \cdots, z_n) \in \Omega \Rightarrow (z'_1, \cdots, z'_n) \in \Omega$ provided that $|z'_i| = |z_i|$ for every $1 \leqslant i \leqslant n$. This result is very interesting in comparison with the mentioned above example of Fornaess.

The aim of the present paper is to study the problem of approximation on $\overline{\Omega}$ of the upper regularization u^* of a given function $u \in \mathcal{PSH}^-(\Omega)$ by elements in $\mathcal{PSH}^c(\Omega)$, where $u^*(z) = \limsup\limits_{\xi \to z} u(\xi)$. This problem has been considered by in [7]

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and then in [3] and [4]. One of the results in [7] which is relevant to our note is the following: *If Ω is a bounded $B-$regular domains in $\mathbf{C}^n$ and $u \in \mathcal{PSH}^-(\Omega)$ then there is a sequence $\{u_j\}$ of functions in $\mathcal{PSH}^c(\Omega)$ decreasing to u^* on $\overline{\Omega}$.* (See the next section for the definition of $B-$regular domains.) The first result of the present work is a variant of Wikstrom's theorem where the domain Ω is assumed to be strictly pseudoconvex. Under this stronger assumption, we show that the approximating functions u_j can be chosen to be smooth and plurisubharmonic on a fixed neigbourhood of $\overline{\Omega}$. The next result deals with the case where Ω is an *arbitrary* Reinhardt domain. Then using some ideas from [2], we show that the same conclusion as in Wikstrom's theorem holds if u^* is continuous at every point of $\partial\Omega$.

This work may be considered as an addendum to [3] and [4]. Those papers deal with the same problem of approximation by using more technical tools. The most important one is a duality theorem of Edwards that allows one to express the upper envelope of a family of plurisubharmonic functions as the lower envelope of a family of integrals.

2. Preliminaries

We first recall the general process of smoothing plurisubharmonic functions. Let ρ be a $\mathcal{C}^\infty$ smooth function with compact support in the unit ball $B(0,1)$, and $\int\limits_{\mathbf{C}^n} \rho d\lambda = 1$. We put $\rho_\varepsilon(z) = \rho(z/\varepsilon)$. It is well known that

$$(u * \rho_\varepsilon)(z) = \int\limits_{\Omega} u(z-w)\rho(w)d\lambda(w),$$

where λ denotes the Lebesugue measure, is $\mathcal{C}^\infty$ and plurisubharmonic on the domain $\Omega_\varepsilon = \{z : \text{dist } (z, \partial\Omega) > \varepsilon\}$. Moreover, $u * \rho_\varepsilon$ decreases to u as ε tends to 0. Observe that $u * \rho_\varepsilon$ is defined on a strictly smaller domain than Ω. Now if Ω is a bounded domain in $\mathbf{C}^n$, then following Sibony [6], we say that Ω is $B-$regular if for every continuous function φ on $\partial\Omega$, there is a continuous function $\hat{\varphi}$ on $\overline{\Omega}$ which is plurisubharmonic on Ω such that $\hat{\varphi} = \varphi$ on $\partial\Omega$. It is well known that if Ω is $B-$regular then we can choose such a function $\hat{\varphi}$ with the additional property that

$$\hat{\varphi} = \sup\{u : u \in \mathcal{PSH}(\Omega), u^* \leqslant \varphi \text{ on } \partial\Omega\}. \tag{1}$$

A special but important class of $B-$regular domains is that of strictly pseudoconvex domains. Recall that a bounded domain Ω in $\mathbf{C}^n$ is said to be strictly pseudoconvex if there is a strictly plurisubharmonic function φ on a neigbourhood of $\overline{\Omega}$ such that

$$\Omega = \{z : \varphi(z) < 0\}, \partial\Omega = \{z : \varphi(z) = 0\}.$$

Observe that we allow strictly pseudoconvex domains with possibly non smooth boundaries.

3. Results

We start with the following result, which is similar to Theorem 4.1 in [7]

Theorem 3.1. *Let Ω be a bounded strictly pseudoconvex domain in $\mathbf{C}^n$ such that $\Omega = \{z \in U : \varphi(z) < 0\}, \partial\Omega = \{z \in U : \varphi(z) = 0\}$, where φ is strictly plurisubharmonic on a neigbourhood U of $\overline{\Omega}$. Then for every $u \in \mathcal{PSH}^-(\Omega)$ and every neigbourhood V of $\overline{\Omega}$ relatively compact in U we can find a neigbourhood U' of $\overline{V}$ and a sequence*

$$\{u_j\}_{j\geqslant 1} \subset \mathcal{PSH}(U') \cap \mathcal{C}^\infty(U')$$

satisfying
(i) $\lim_{j\to\infty} u_j = u^*$ *on* $\overline{\Omega}$.
(ii) *For each compact subset E of Ω and every $\varepsilon > 0$ there exists $j(E)$ so that $u_j \geqslant u_{j+1}$ for all $j \geqslant j(E)$.*

Notice that under such a strong assumption on Ω, we get a fixed neigbourhood U' for all function u and that the approximation occurs everywhere on $\partial\Omega$. It is desirable to know whether the sequence can be chosen to be decreasing on $\overline{\Omega}$.

Proof of Theorem 3.1. Since u is bounded from above, the function u^* is upper semicontinuous and bounded from above on $\partial\Omega$. So we can choose a sequence $\varphi_j \in \mathcal{C}(\partial\Omega)$ decreasing to u^* on $\partial\Omega$. Set

$$\max_{\partial\Omega}(u^* - \varphi_j) := -\varepsilon_j < 0.$$

For each j, choose $\tilde{\varphi}_j \in \mathcal{C}^2(U)$ such that $|\varphi_j - \tilde{\varphi}_j| < \varepsilon_j/2$ on $\partial\Omega$. As Ω is strictly pseudoconvex, according to (1) we may extend φ_j to a continuous function $\hat{\varphi}_j$ on $\overline{\Omega}$ which is plurisubharmonic on Ω. Since φ is strictly plurisubharmonic on U, we can choose λ_j large enough so that the function $\lambda_j\varphi + \tilde{\varphi}_j$ is plurisubharmonic on some neigbourhood $\tilde{V}$ of $\overline{\Omega}$ such that $\overline{V} \subset \tilde{V} \subset\subset U$. It follows from (1) that

$$\psi_j = \begin{cases} \hat{\varphi}_j & \text{on } \Omega \\ \lambda_j\varphi + \tilde{\varphi}_j & \text{on } U'\backslash\overline{\Omega} \end{cases}$$

is plurisubharmonic on V. From (1) and the inequality $u^* \leqslant \varphi_j - \varepsilon_j \leqslant \tilde{\varphi}_j - \varepsilon_j/2$ on $\partial\Omega$ we get

$$u^* \leqslant \hat{\varphi}_j - \varepsilon_j \text{ on } \overline{\Omega}.$$

Now we claim that there exists a decreasing sequence $\{r_j\}$ such that

$$0 < r_j < d_j := \text{dist}\,\Big(\Big\{\varphi(z) < -\frac{1}{2j^2}\Big\}, \partial\Omega\Big),$$

and that $u * \rho_{r_j} \leqslant \hat{\varphi}_j - \varepsilon_j/2$ on $\Omega_{r_j}, |\varphi(z)| < \frac{1}{2j^2}$ if dist $(z, \partial\Omega) < r_j$, where $\Omega_\delta = \{z \in \Omega : \text{dist}\,(z, \partial\Omega) > \delta\}$. The first condition on r_j can be fulfilled because the sequence d_j converges to 0. For the second one, observe that φ is upper semicontinuous and

$$u * \rho_\delta \leqslant (\hat{\varphi}_j - \varepsilon_j) * \rho_\delta = (\hat{\varphi}_j * \rho_\delta) - \varepsilon_j < \hat{\varphi}_j - \varepsilon_j/2$$

if δ is chosen sufficiently small, as $\hat{\varphi}_j * \rho_\delta$ converges uniformly to $\hat{\varphi}_j$ as δ goes to 0.

Define

$$\tilde{u}_j = \begin{cases} \max\{u * \rho_{r_j}, j(\varphi * \rho_{r_j}) + \hat{\varphi}_j + 1/j\} & \text{on } \Omega_{r_j} \\ j(\varphi * \rho_{r_j}) + \hat{\varphi}_j + 1/j & \text{on } \tilde{V}\backslash\overline{\Omega_{r_j}}. \end{cases}$$

Notice that on $\partial\Omega_{r_j}$ we have $\varphi \equiv -\frac{1}{2j^2}$, so

$$u * \rho_{r_j} \leqslant \hat{\varphi}_j - \varepsilon_j/2 \leqslant j\varphi + \hat{\varphi}_j + 1/j \leqslant j(\varphi * \rho_{r_j}) + \hat{\varphi}_j + 1/j.$$

This implies $\tilde{u}_j \in \mathcal{PSH}(\tilde{V}) \cap \mathcal{C}(\tilde{V})$. Since $\{u * \rho_{r_j}\}$ decreases to u on Ω and $|\varphi * \rho_{r_j}| \leqslant \frac{1}{2j^2}$ on $\partial\Omega$ we deduce $\{\tilde{u}_j\} \to u^*$ pointwise on $\overline{\Omega}$. Since each function $\tilde{u}_j$ is continuous on $\tilde{V}$, by convolving with a suitable standard regularizing kernel, we can find a neigbourhood U' of $\overline{\Omega}$ and $u_j \in \mathcal{PSH}(U') \cap \mathcal{C}^\infty(U')$ such that $|\tilde{u}_j - u'_j| < 1/j$ on $\overline{\Omega}$. The proof is thereby completed. $\square$

Now we turn to Reinhardt domains.

Theorem 3.2. *Let Ω be a bounded Reinhardt domain in $\mathbf{C}^n$ and $u \in \mathcal{PSH}^-(\Omega)$. Assume that $\lim_{\xi\to z} u(\xi) = u^*(z) > -\infty$ for every $z \in \partial\Omega$. Then there exists a sequence $\{u_j\}_{j\geqslant 1} \in \mathcal{PSH}^c(\Omega)$ such that $u_j^* \downarrow u^*$ on $\overline{\Omega}$.*

The proof relies heavily on ideas given in Section 4 in [2]. First recall that if u is a plurisubharmonic function on a Reinhardt domain Ω then, as in [2] (p. 264), for $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n), \varepsilon_i > 0$ we put

$$u^\varepsilon(z) = \frac{1}{2^n \varepsilon_1 \cdots \varepsilon_n} \int_{-\varepsilon_1}^{\varepsilon_1} \cdots \int_{-\varepsilon_n}^{\varepsilon_n} u(e^{i\theta_1} z_1, \cdots, e^{i\theta_n} z_n) d\theta_1 \cdots d\theta_n. \tag{2}$$

We first check that u^ε is plurisubharmonic on Ω. It is true if u is $\mathcal{C}^2$, since in this case we can differentiate under the integral sign of (2) to find out that u^ε is a $\mathcal{C}^2$ smooth function which is subharmonic on every complex line cutting Ω. The general case follows by considering a smoothing sequence $\{u_\delta\}$ of u and observing that $(u_\delta)^\varepsilon \downarrow u^\varepsilon$ when δ tends to 0. The continuity of u^ε is a bit more delicate and this fact was proved in Lemma 4 of [2].

The following lemma is crucial in the proof.

Lemma 3.1. *Let Ω be a bounded Reinhardt domain in $\mathbf{C}^n$ and $u \in \mathcal{PSH}^-(\Omega)$, $u \geqslant 0$. Then we have*
(a) *If $\lim_{z\to a} u(z) = u^*(a) > -\infty, a \in \partial\Omega$ then*

$$\lim_{(\varepsilon,z)\to(0,a)} u^\varepsilon(z) = u^*(a), \quad \lim_{z\to a} u^\varepsilon(z) = (u^\varepsilon)^*(a) > -\infty.$$

(b) *For each compact subset K of Ω, there exists a sequence $\{\varepsilon_j\}_{j\geqslant 1} = \{(\varepsilon_j^1, \cdots, \varepsilon_j^n)\}$, $\varepsilon_j^i > 0$ converging to 0 such that the sequence $u_{\varepsilon_j} + 1/j$ decreases to u on K.*

Proof. (a) follows from (2) and the Lebesgue dominated convergence theorem, (b) is precisely Lemma 5 in [2]. $\square$

Proof of Theorem 3.2. Let $\{K_m\}_{m\geqslant 1}$ be a sequence of compact subsets of Ω such that $K_m \subset \operatorname{int} K_{m+1}, \cup K_m = \Omega$, where $\operatorname{int} K$ denotes the interior of K. For each m, by Lemma 3.1(b), there exists a sequence $\{u_{k,m}\}_{k\geqslant 1} \subset \mathcal{PSH}(\Omega) \cap \mathcal{C}(\Omega)$ which decreases to $\max(u, -m)$ on K_m, where

$$u_{k,m} = (\max(u, -m))^{\varepsilon_{k,m}} + 1/k \tag{3}$$

and $\{\varepsilon_{k,m}\}_{k\geqslant 1} = \{(\varepsilon^1_{k,m}, \cdots, \varepsilon^n_{k,m})\}_{k\geqslant 1}$ is a sequence in $\mathbf{R}^n$ satisfying $0 < \varepsilon^i_{k,m} < \min(1/k, 1/m)$. It follows that the sequence $\max(u_{p,m}, u_{m,l})$ decreases to $u_{m,l}$ on K_l for all $1 \leqslant l \leqslant m$ as $p \to \infty$. By Dini's lemma, for each m, we can choose $k(m) > m$ large enough so that

$$u_{k(m),m} \leqslant u_{m,l} + 1/m \quad \text{on} \quad K_l, \;\; 1 \leqslant l \leqslant m. \tag{4}$$

Let

$$v_j = \sup_{m\geqslant j} u_{k(m),m}.$$

Being the upper envelope of a uniformly bounded from above family of continuous functions, v_j is real valued, lower semicontinuous on Ω. We claim that v_j is upper semicontinuous on Ω. Fix $z_0 \in \Omega, \varepsilon > 0$. Choose $p > j$ so that $z_0 \in \operatorname{int} K_p$ and $1/p < \varepsilon$. For $m \geqslant k(p)$ and $w \in K_p$, using (4) we have

$$u_{k(m),m}(w) \leqslant u_{m,p}(w) + 1/m \leqslant u_{k(p),p}(w) + \varepsilon.$$

It follows that

$$\limsup_{w\to z} v_j(w) \leqslant v_j(z_0) + \varepsilon.$$

The claim is valid. Thus $v_j \in \mathcal{PSH}^-(\Omega) \cap \mathcal{C}(\Omega)$. We infer also from (4) that v_j decreases to u on Ω. Next we show that $\lim_{z\to a} v_j(z) = v^*_j(a)$ for all $a \in \partial\Omega$. If not, then we could find $\varepsilon > 0$ and two sequences $\{z_q\}_{q\geqslant 1}$ and $\{\tilde{z}_q\}_{q\geqslant 1}$ tending to $a \in \partial\Omega$ such that $v_j(z_q) \leqslant v_j(\tilde{z}_q) + \varepsilon$ for all q. Thus there exists a sequence $\{m_q\}_{q\geqslant 1}$ so that

$$u_{k(m_q),m_q}(z_q) \leqslant u_{k(m_q),m_q}(\tilde{z}_q) + 2\varepsilon, \quad \forall q.$$

As $\lim_{z\to a} u(z) = u^*(a) > -\infty$, u is locally bounded from below near a. Combining (3) and Lemma 3.1(a), we arrive at a contradiction. Next we show that $v^*_j \downarrow u^*$ on $\partial\Omega$. For this, we argue by contradiction. Assume otherwise, then there exist $\varepsilon > 0, a \in \partial\Omega$ and a sequence $\{z_j\}_{j\geqslant 1}$ tending to a such that $v_j(a_j) > u^*(a) + \varepsilon$ for all j. Thus there exists a sequence $\{m_j\}$ tending to ∞ such that

$$u_{k(m_j),m_j}(a_j) > u^*(a) + \varepsilon/2, \quad \forall j.$$

Applying again Lemma 3.1(a) and (3) we also reach a contradiction. The proof is complete. □

Remark. The following simple example of Wikstrom (see Section 4 in [7]) shows that the assumption on continuity of u^* at boundary points in Theorem 3.2 can not be removed. Let Ω be the Hartogs triangle, $\Omega = \{(z, w) : |z| < |w| < 1\}$ and $u(z, w) = |z/w|$. Then Ω is a bounded pseudoconvex Reinhardt domain in $\mathbf{C}^2$ and $u \in \mathcal{PSH}^-(\Omega), u^*$ is continuous everywhere on $\overline{\Omega}$ except at the origin. Assume that there is a sequence $u_j \in \mathcal{PSH}^c(\Omega)$ such that $u_j \downarrow u^*$ on $\overline{\Omega}$. Observe that the "disk" $K = \{(z, w) : z = 0, |w| = 1/2\}$ is included in Ω and $u \equiv 0$ on K. By Dini's lemma, u_j converges uniformly to 0 on K. Applying the maximum principle, we get $u^* = 0$. This is absurd.

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