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SOME REMARKS ON STABILITY IN PERIODIC MULTI-WAVELET DECOMPOSITIONS

DINH DUNG AND VU CONG BANG

ABSTRACT. In the present paper, we established some necessary and sufficient conditions for L_p-stability of periodic multi-wavelet decompositions.

1. The problem of the stability of non-periodic multi-wavelet decomposition was studied by Jia and Micchelli [2]. The necessary and sufficient conditions for the problem of L_2-stability of periodic wavelet decompositions was given by Goh and Yeo [3]. Dinh Dung [1] established sufficient conditions of L_p-stability of mult-wavelet decompositions with the scaling functions periodized from non-periodic functions, where $1 \leqslant p \leqslant \infty$. The present paper extends some results of [1]. We will establish some necessary and sufficient conditions for L_p-stability of periodic multi-wavelet decompositions.

2. Given a function ψ on $\mathbb{R}$ and $u > 0$, the functions $\psi(\cdot - us), s \in \mathbb{Z}$, are called u-step integer translates of ψ. If we set

$$\psi_u = \sum_{s\in\mathbb{Z}} |\psi(\cdot - us)|,$$

then ψ_u is u-periodic. For $1 \leqslant p \leqslant \infty$, define the norm

$$|\psi|_{p,u} := \|\psi_u\|_{L_p([0,u))}.$$

Let $\mathcal{L}_{p,u}(\mathbb{R})$ be the normed space of all functions ψ for which $|\psi|_{p,u} < \infty$. Notice that $|\psi|_{p,u} \leqslant |\psi|_{q,u}$ for $p \leqslant q$.

For a family $\{\psi^i\}_{i=1}^n$ of functions on $\mathbb{R}$, denote by $S_{p,u}(\psi^1, \dots, \psi^n)$ the space of all linear combinations

$$f = \sum_{i=1}^{n} \psi^i * a^i, \tag{1}$$

with $a^i, \dots, a^n \in \ell_p(\mathbb{Z})$, where

$$\psi * a := \sum_{s\in\mathbb{Z}} a(s)\psi(\cdot - us)$$

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is semi-discrete convolution of ψ and a. Here, for $1 \leqslant p < \infty$,

$$\ell_p(\mathbb{Z}) := \{a = \{a(s)\}_{s\in\mathbb{Z}} : \|a\|_p := (\sum_{s\in\mathbb{Z}} |a(s)|^p)^{1/p} < \infty\},$$

and for $p = \infty$

$$\ell_\infty(\mathbb{Z}) := \{a = \{a(s)\}_{s\in\mathbb{Z}} : \|a\|_\infty := \sup_{s\in\mathbb{Z}} |a(s)| < \infty\}.$$

We say that the u-step integer translates of $\psi^1, \dots, \psi^n$, are L_p-stable if there exist positive constants C, C' depending on $p, u, \psi^1, \dots, \psi^n$ such that

$$C \sum_{i=1}^n \|a^i\|_p \leqslant \|f\|_p \leqslant C' \sum_{i=1}^n \|a^i\|_p,$$

for all $f \in S_{p,u}(\psi^1, \dots, \psi^n)$ represented as in (1). If $\psi^1, \dots, \psi^n \in \mathcal{L}_{p,u}(\mathbb{R})$, the inequality on the right hand side holds by the inequality

$$\|\psi * a\|_p \leqslant |\psi|_{p,u} \|a\|_p$$

proved in [2]. The following theorem was proved in [2].

Theorem 1. *Let $1 \leqslant p,\ p' \leqslant \infty$, $1/p + 1/p' = 1$, and $\psi^1, \dots, \psi^n \in \mathcal{L}_{\infty,u}(\mathbb{R})$. Then the u-step integer translates of $\psi^1, \dots, \psi^n$, are L_p-stable if and only if one of the following conditions holds:*

(i) *For any $y \in [0, 2\pi/u)$ the sequences $\{\hat{\psi}^i(y + 2\pi s/u)\}_{s\in\mathbb{Z}}$, $i = 1, \dots, n$, are linearly independent, where $\hat{\psi}^i$ denotes the Fourier transform of ψ^i, $i = 1, \dots, n$.*

(ii) *There exist $g^1, \dots, g^n \in S_{1,u}(\psi^1, \dots, \psi^n)$ such that*

$$\langle g^i, \psi^j(\cdot - us)\rangle = \delta_{ij}\delta_{0s} \quad \text{for } i, j = 1, \dots, n \quad \text{and } s \in \mathbb{Z}.$$

Moreover, if the u-step integer translates of $\psi^1, \dots, \psi^n$, are L_p-stable, then

$$\sum_{i=1}^n \|a^i\|_{\ell_p(\mathbb{Z})} \leqslant \|f\|_p \sum_{i=1}^n |g^i|_{p',u},$$

for all linear combinations (1).

A 2π-periodic function is represented as a function defined on $\mathbb{T} := [-\pi, \pi]$. Let φ be a function on $\mathbb{T}$, and for $m \in \mathbb{N}$, $h = 2\pi/m$. We set

$$\varphi_h := \sum_{s=0}^{m-1} |\varphi(\cdot - hs)|.$$

Then φ_h is a h-periodic function. Similarly to non-periodic functions, for $1 \leqslant p \leqslant \infty$, we defined the norm

$$|\varphi|_{p,h} := \|\varphi_h\|_{L_p([0,h))}.$$

Denote by $\mathcal{L}_{p,h}(\mathbb{T})$ the normed space of all functions φ for which $|\varphi|_{p,h} < \infty$.

Let $\{\varphi_k^j\}_{k\in\mathbb{Z}_+}$, $j=1,\dots,n$, be a finite family of sequences of functions defined on $\mathbb{T}$. Fix $\gamma\in\mathbb{N}$, and for $k\in\mathbb{Z}_+$, $s=0,\dots,\gamma 2^k-1$, put

$$\varphi_{k,s}^j(x) := \varphi_k^j(x-2\pi s/\gamma 2^k), \quad x\in\mathbb{T}.$$

The functions $\varphi_{k,s}^j$, $s=0,\dots,\gamma 2^k-1$, are called $2\pi/\gamma 2^k$-step integer translates of φ_k^j. Suppose that every function $f\in L_p(\mathbb{T})$, $1\leqslant p\leqslant\infty$, can be decomposed into a series:

$$f=\sum_{k=0}^{\infty}\sum_{j=1}^{n}\sum_{s=0}^{\gamma 2^k-1} f_{k,s}^j\varphi_{k,s}^j, \tag{2}$$

converging in the norm of $L_p(\mathbb{T})$, where $f_{k,s}^j=f_{k,s}^j(f)$ are certain coefficient functional of f. This decomposition is called a periodic multi-wavelet decomposition of f. For $k\in\mathbb{Z}_+$, the functions φ_k^j, $j=1,\dots,n$, are called k^{th} scaling functions and $\varphi_{k,s}^j$ wavelets. The periodic multi-wavelet decomposition (2) is called L_p-stable if there exist positive constants C, C' depending on p,γ and n only such that for each linear combination of the wavelets at any dyadic level

$$g:=\sum_{j=1}^{n}\sum_{s=0}^{\gamma 2^k-1} a^j(s)\varphi_{k,s}^j,$$

there hold the inequalities

$$C2^{-k/p}\sum_{j=0}^{n}\Big(\sum_{s=0}^{\gamma 2^k-1}|a^j(s)|^p\Big)^{1/p}\leqslant\|g\|_p\leqslant C'2^{-k/p}\sum_{j=0}^{n}\Big(\sum_{s=0}^{\gamma 2^k-1}|a^j(s)|^p\Big)^{1/p}$$

(the sum is changed to max when $p=\infty$), where $\|\cdot\|_p$ is the usual p-integral norm in $L_p(\mathbb{T})$.

For a sequence $a=\{a(s)\}_{s=0}^{m-1}$, we put

$$\|a\|_p:=\Big(\sum_{s=0}^{m-1}|a(s)|^p\Big)^{1/p}.$$

For a function ψ on $\mathbb{R}$, we define the function $\pi_m(\psi)$ on $\mathbb{T}$, periodized from ψ by

$$\pi_m(\psi,x):=\sum_{k\in\mathbb{Z}}\psi(m(x+2k\pi)).$$

The following theorem was proved in [1].

Theorem 2. *Let $1\leqslant p,\ p'\leqslant\infty$, $1/p+1/p'=1$ and $\psi^j\in\mathcal{L}_{\infty,2\pi/\gamma}(\mathbb{R})$, $j=1,\dots,n$. Assume that every function $f\in L_p(\mathbb{T})$ has a periodic multi-wavelet decompostion* (2) *with the sequences of scaling functions $\{\varphi_k^j\}_{k\in\mathbb{Z}_+}$, $j=1,\dots,n$, defined by*

$$\varphi_k^j(x):=\pi_{2^k}(\psi^j,x).$$

If ψ^j, $j = 1, \dots, n$, have L_p-stable $2\pi/\gamma$-step integer translates, then this decomposition is L_p-stable, that is, for all linear combinations of the wavelets at any dyadic level

$$g = \sum_{j=1}^{n} \sum_{s=0}^{\gamma 2^k - 1} a^j(s)\varphi^j_{k,s},$$

there hold the inequalities

$$C2^{-k/p} \sum_{j=1}^{n} \|a^j\|_p \leqslant \|g\|_p \leqslant C' 2^{-k/p} \sum_{j=1}^{n} \|a^j\|_p,$$

where

$$C = \max_{1\leqslant j\leqslant n} |\psi^j|_{p,2\pi/\gamma}, \quad C' = (n \max_{1\leqslant j\leqslant n} |g^j|_{p',2\pi/\gamma})^{-1},$$

$g^i \in S_{1,2\pi/\gamma}(\psi^1, \dots, \psi^n)$, *$i = 1, \dots, n$, are functions such that*

$$\langle g^i, \psi^j(\cdot - 2\pi s/\gamma)\rangle = \delta_{ij}\delta_{0s} \quad \textit{for } i, j = 1, \dots, n \quad \textit{and } s \in \mathbb{Z}.$$

In the present paper we shall show that the $2\pi/\gamma$-step integer translates of the compactly supported functions $\psi^1, \dots, \psi^n$, are L_p-stable if the multi-wavelet decomposition (2) with the scaling functions periodized from these functions is L_p-stable. Furthermore, we give the necessary and suficient conditions of L_p-stability of multi-wavelet decomposition (2) in the case when

$$\{\varphi^i_k(\cdot - 2\pi s/2^k) : s = 0, \dots, 2^k - 1,\ i = 1, \dots, n\}$$

are orthogonal set for all $k \in \mathbb{Z}_+$. We will treat with univariate functions. However, all our results can be easily extended to multivariate functions.

For $f \in L_p(\mathbb{T})$ and $g \in \mathcal{L}_{p',h}(\mathbb{T})$, $1/p + 1/p' = 1$, $h = 2\pi/m$, we define the sequence $c(f,g)_h = \{c(f,g)_h(s)\}_{s=0}^{m-1}$ by

$$c(f,g)_h(s) := \int_{\mathbb{T}} f(x)\overline{g(x - hs)}dx.$$

We have

$$\|c(f,g)_h\|_p \leqslant \|f\|_p |g|_{p',h} \tag{3}$$

$$\|g * a\|_{p'} \leqslant |g|_{p',h} \|a\|_{p'} \tag{4}$$

The proof of (3) and (4) is similar to the proof of Theorem 3.1 and Theorem 2.1 in [2], respectively. In particular, using (4) with $a(s) = \delta_{0,s}$, $s = 0, \dots, m-1$, we have

$$\|g\|_p \leq |g|_{p,h}; \text{for } 1 \leqslant p \leqslant \infty.$$

3. In the non-periodic case, the stability of integer translates of functions does not imply their linear independence [2]. Let $\varphi^1, \dots, \varphi^n$ be functions defined on $\mathbb{T}$, $m \in \mathbb{N}$. Consider functions of the form

$$f := \sum_{i=1}^{n} \varphi^i * a^i,$$

where $a^i := \{a^i(s)\}_{s=0}^{m-1}$, and

$$\varphi * a := \sum_{s=0}^{m-1} \varphi(\cdot - 2\pi s/m)a(s)$$

is semi-discrete convolution of φ and a. Let $1 \leqslant p \leqslant \infty$, $m \in \mathbb{N}$, $h = 2\pi/m$, and $\varphi^i \in L_p(\mathbb{T})$, $i = 1, \dots, n$. If there exist a positive constant C such that

$$\sum_{i=1}^{n} \|a^i\|_p \leqslant C\|f\|_p, \quad \text{for all } f = \sum_{i=1}^{n} \varphi^i * a^i,$$

then the h-step integer translates of φ^i, $i = 1, \dots, n$, are linearly independent. From this we can see that if a periodic multi-wavelet decomposition (2) is L_p-stable, then the $2\pi/\gamma 2^k$-step integer translates of φ_k^i, $i = 1, \dots, n$, are linearly independent.

Theorem 2 shows that if the scaling functions of a periodic multi-wavelet decomposition (2) are periodized from non-periodic functions $\psi^1, \dots, \psi^n$ having stable $2\pi/\gamma$-step integer translates, then this periodic multi-wavelet decomposition is also L_p-stable. The following theorem shows an inverse assertion for the case when $\psi^1, \dots, \psi^n$ are compactly supported.

Theorem 3. *Let $\gamma \in \mathbb{N}$, and $\psi^1, \dots, \psi^n \in \mathcal{L}_{\infty, 2\pi/\gamma}(\mathbb{R})$ be compactly supported functions. If there exists a positive constant $C = C(p, \psi^1, \dots, \psi^n)$ such that for each $k \in \mathbb{N}$, and for each*

$$f = \sum_{i=1}^{n} \pi_{2^k}(\psi^i) * a^i = \sum_{i=1}^{n} \sum_{s=0}^{\gamma 2^k - 1} a^i(s) \pi_{2^k}(\psi^i, \cdot - hs)$$

with $a^i = \{a^i(s)\}_{s=0}^{\gamma 2^k - 1}$ and $h = 2\pi/\gamma 2^k$, there holds the following inequality

$$2^{-k/p} \sum_{i=1}^{n} \|a^i\|_p \leqslant C\|f\|_p, \tag{5}$$

then the $2\pi/\gamma$-step integer translates of $\psi^1, \dots, \psi^n$, are L_p-stable.

Proof. Without loss the generality, we may assume that there exists $k_0 \in \mathbb{Z}_+$ such that

$$\operatorname{supp}(\psi^i) \subset [-2k_0\pi, 2k_0\pi], \quad i = 1, \dots, n. \tag{6}$$

From the definition of $\pi_{2^k}(\psi, x)$ and (6), we can see that there exists a positive integer $L = (k_0 + 1)$ not depending on k, such that for all $x \in [0, 2\pi)$, $s = 0, \dots, \gamma 2^k - 1$, $i = 1, \dots, n$

$$\pi_{2^k}(\psi^i, x - 2\pi s/\gamma 2^k) = \sum_{|m| \leqslant L} \psi^i(2^k x + 2\pi m 2^k - 2\pi s/\gamma).$$

For $a^i = \{a^i(s)\}_{s\in\mathbb{Z}}$, we consider the function

$$g = \sum_{i=1}^{n} \psi^i * a^i.$$

For any $k \in \mathbb{N}$, $i = 1, \ldots, n$, we set

$$b_k^i(s) := \begin{cases} a^i(s), & \text{if } -\gamma 2^k \leqslant s \leqslant \gamma 2^k - 1; \\ 0, & \text{otherwise}, \end{cases}$$

$$g_k := \sum_{i=1}^{n} \psi^i * b_k^i,$$

and for $s = 0, \ldots, 2\gamma 2^k - 1$, we define

$$c_k^i(s) := \begin{cases} b^i(s), & \text{if } 0 \leqslant s < \gamma 2^k; \\ b^i(-j), & \text{if } s = \gamma 2^{k+1} - j. \end{cases}$$

Then we have $\lim_{k\to\infty} \|g_k\|_p = \|g\|_p$, $\lim_{k\to\infty} \|b_k^i\|_p = \|a^i\|_p$, and

$$\begin{aligned}
(\pi_{2^{k+1}}(\psi^i) * c_k^i)(x) &= \sum_{s=0}^{\gamma 2^{k+1}-1} c_k^i(s)\pi_{2^{k+1}}(\psi^i, x - 2\pi s/\gamma 2^{k+1}) \\
&= \sum_{s=0}^{\gamma 2^k - 1} b_k^i(s)\pi_{2^{k+1}}(\psi^i, x - 2\pi s/\gamma 2^{k+1}) \\
&\quad + \sum_{s=\gamma 2^k}^{\gamma 2^{k+1}-1} c_k^i(s)\pi_{2^{k+1}}(\psi^i, x - 2\pi s/\gamma 2^{k+1}) \\
&= \sum_{s=0}^{\gamma 2^k - 1} b_k^i(s)\pi_{2^{k+1}}(\psi^i, x - 2\pi s/\gamma 2^{k+1}) \\
&\quad + \sum_{j=1}^{\gamma 2^k} b_k^i(-j)\pi_{2^{k+1}}(\psi^i, x + 2\pi j/\gamma 2^{k+1}) \\
&= \sum_{s=0}^{\gamma 2^k - 1} b_k^i(s)\pi_{2^{k+1}}(\psi^i, x - 2\pi s/\gamma 2^{k+1}) \\
&\quad + \sum_{s=-\gamma 2^k}^{-1} b_k^i(s)\pi_{2^{k+1}}(\psi^i, x - 2\pi s/\gamma 2^{k+1}) \\
&= \sum_{s=-\gamma 2^k}^{\gamma 2^k - 1} b_k^i(s)\pi_{2^{k+1}}(\psi^i, x - 2\pi s/\gamma 2^{k+1}).
\end{aligned}$$

Hence, by (5) and $\|b_k^i\|_p = \|c_k^i\|_p$, we have

$$C^{-1}2^{-(k+1)/p}\sum_{i=1}^{n}\|b_k^i\|_p \leqslant \Big(\int_0^{2\pi} |\sum_{i=1}^{n}\pi_{2^{k+1}}(\psi^i,x) * c_k^i|^p dx\Big)^{1/p}$$

$$= \Big(\int_0^{2\pi}\Big|\sum_{i=1}^{n}\sum_{s=-\gamma 2^k}^{\gamma 2^k-1} b_k^i(s)\pi_{2^{k+1}}(\psi^i, x-2\pi s/\gamma 2^{k+1})\Big|^p dx\Big)^{1/p}$$

$$= \Big(\int_0^{2\pi}\Big|\sum_{i=1}^{n}\sum_{s=-\gamma 2^k}^{\gamma 2^k-1} b_k^i(s)\sum_{|m|\leqslant L}\psi^i(2^{k+1}x+2\pi m 2^{k+1}-2\pi s/\gamma)\Big|^p dx\Big)^{1/p}$$

$$= 2^{-(k+1)/p}\Big(\int_0^{2\pi 2^{k+1}}\Big|\sum_{i=1}^{n}\sum_{s=-\gamma 2^k}^{\gamma 2^k-1} b_k^i(s)\sum_{|m|\leqslant L}\psi^i(x+2\pi m 2^{k+1}-2\pi s/\gamma)\Big|^p dx\Big)^{1/p}$$

$$\leqslant 2^{-(k+1)/p}\sum_{|m|\leqslant L}\Big(\int_0^{2\pi 2^{k+1}}\Big|\sum_{i=1}^{n}\sum_{s=-\gamma 2^k}^{\gamma 2^k-1} b_k^i(s)\psi^i(x+2\pi m 2^{k+1}-2\pi s/\gamma)\Big|^p dx\Big)^{1/p}$$

$$= 2^{-(k+1)/p}\sum_{|m|\leqslant L}\Big(\int_{2\pi m 2^{k+1}}^{2\pi 2^{k+1}(m+1)}\Big|\sum_{i=1}^{n}\sum_{s=-\gamma 2^k}^{\gamma 2^k-1} b_k^i(s)\psi^i(x-2\pi s/\gamma)\Big|^p dx\Big)^{1/p}$$

$$\leqslant 2^{-(k+1)/p}(2L+1)\Big(\int_{\mathbb{R}}\Big|\sum_{i=1}^{n}\sum_{s=-\gamma 2^k}^{\gamma 2^k-1} b_k^i(s)\psi^i(x-2\pi s/\gamma)\Big|^p dx\Big)^{1/p}.$$

Therefore,

$$\sum_{i=1}^{n}\|b_k^i\|_p \leqslant C(2L+1)\|g_k\|_p.$$

Letting $k\to\infty$, we have

$$\sum_{i=1}^{n}\|a^i\|_p \leqslant C(2L+1)\|g\|_p.$$

□

Theorem 3 shows that if the multi-wavelet decomposition (2) is L_p-stable and the scaling functions are periodized from non-periodic functions $\psi^1,\dots,\psi^n$ which are compactly supported, then $2\pi/\gamma$-step integer translates of $\psi^1,\dots,\psi^n$, are also L_p-stable. What happens when $\psi^1,\dots,\psi^n$ are not compactly supported? The following counterexample will give an answer to this question.

Counterexample. We consider the case $p=\gamma=2$, $n=1$, we define the function φ by

$$\hat{\varphi}(\xi) = \begin{cases} -\xi/\sqrt{2}+1 & \text{if } 0 \leqslant \xi \leqslant \sqrt{2}; \\ \xi/(2-\sqrt{2})+1 & \text{if } -2+\sqrt{2} \leqslant \xi \leqslant 0; \\ 0 & \text{otherwise.} \end{cases}$$

Then the $2\pi/\gamma$-step integer translates of φ are L_2-unstable. But one can verify that (5) holds. Indeed, we apply the condition (i) of Theorem 1 with $u = \pi$. Since $\operatorname{supp}(\hat{\varphi}) = [\sqrt{2}-2, \sqrt{2}]$, there exists $y = \sqrt{2} \in [0, 2\pi/u) = [0, 2)$ such that $\hat{\varphi}(y + 2\pi s/u) = \hat{\varphi}(\sqrt{2} + 2s) = 0$ for all $s \in \mathbb{Z}$. This means that the sequence $\{\hat{\varphi}(\sqrt{2} + 2s)\}_{s\in\mathbb{Z}}$ is linearly dependent. By Theorem 1, π-step integer translates of φ are L_2-unstable. We verify (5). For all k, we have,

$$\widehat{\pi_{2^k}(\varphi)}(n) = \frac{1}{\sqrt{2\pi}} 2^{-k} \hat{\varphi}(n/2^k).$$

Hence, for $\gamma = 2$ and $s = 0, \ldots, 2 \times 2^k - 1$,

$$\sum_{l\in\mathbb{Z}} |\widehat{\pi_{2^k}(\varphi)}(s + 2l2^k)|^2 = \frac{1}{2\pi} 2^{-2k} \sum_{l\in\mathbb{Z}} |\hat{\varphi}(s/2^k + 2l)|^2 \geqslant \frac{1}{2\pi} 2^{-2k}.$$

Since $\operatorname{supp}(\hat{\varphi}) = [\sqrt{2}-2, \sqrt{2}]$ and $0 \leqslant s/2^k < 2$, we have

$$\frac{1}{2\pi} 2^{-2k} \sum_{l\in\mathbb{Z}} |\hat{\varphi}(s/2^k + 2l)|^2 = \frac{1}{2\pi} 2^{-2k} \sum_{p=-1}^{1} |\hat{\varphi}(s/2^k + 2l)|^2 \leqslant \frac{3}{2\pi} 2^{-2k}.$$

Thus,

$$\frac{1}{2\pi} 2^{-2k} \leqslant \sum_{l\in\mathbb{Z}} |\widehat{\pi_{2^k}(\varphi)}(s + 2l2^k)|^2 \leqslant \frac{3}{2\pi} 2^{-2k}.$$

By Proposition 3.1 in [3], this is equivalent to (5).

4. The following theorem give us a necessary and sufficient condition for the L_p-stability of periodic multi-wavelet decompositions (2) for the case when the scaling functions are an orthogonal set.

Theorem 4. *Let $1 \leqslant p < \infty$ and φ_k^i, $i = 1, \ldots, n$, $k \in \mathbb{Z}_+$, be functions defined on $\mathbb{T}$, $h = 2\pi/2^k$. Assume that*

(i) *For each $k \in \mathbb{Z}_+$, $\{\varphi_k^i(\cdot - 2\pi s/2^k), s = 0, \ldots, 2^k - 1, i = 1, \ldots, n\}$ are an orthogonal set.*
(ii) *$|\varphi_k^i|_{\infty,h} \leqslant C$, for all $k \in \mathbb{Z}_+$.*

Then the multi-wavelet decomposition (2) is L_p-stable if and only if

$$\|\varphi_k^i\|_2^2 \geqslant C_1 2^{-k}, \textit{for all } i = 1, \ldots, n, k \in \mathbb{Z}_+.$$

Here C, C_1 are positive constants not depending on k.

Proof. Let $1 \leqslant p_1 < p_2$, $0 \leqslant \alpha \leqslant 2\pi$, and $f \in L_{p_2}(\mathbb{T})$. Then

$$\int_o^\alpha |f(x)|^{p_1} dx \leqslant \Big(\int_0^\alpha dx\Big)^{(p_2-p_1)/p_2} \Big(\int_0^\alpha |f(x)|^{p_2} dx\Big)^{p_1/p_2}.$$

Hence

$$\Big(\int_o^\alpha |f(x)|^{p_1}dx\Big)^{1/p_1} \leqslant (\alpha)^{1/p_1-1/p_2}\Big(\int_0^\alpha |f(x)|^{p_2}dx\Big)^{1/p_2}. \tag{7}$$

We prove the sufficient condition. By the condition (ii) we have

(8)

$$|\varphi_k^i|_{p,h} = \Big(\int_0^h |\varphi_{k,h}^i|^p dx\Big)^{1/p} \leqslant C(2\pi)^{1/p}2^{-k/p} = C_2 2^{-k/p},\ \text{for all } 1 \leqslant p \leqslant \infty,$$

where $C_2 := C(2\pi)^{1/p}$. For any $f = \sum_{i=1}^n \varphi_k^i * a^i$, by (4) we have

$$\|f\|_p \leqslant \sum_{i=1}^n \|\varphi_k^i * a^i\|_p \leqslant \sum_{i=1}^n |\varphi_k^i|_{p,h}\|a^i\|_p \leqslant C_2 2^{-k/p}\sum_{i=1}^n \|a^i\|_p \tag{9}$$

On the other hand, using condition (i), we have

$$c(f,\varphi_k^i)_h(s) = a^i(s)\|\varphi_k^i\|_2^2.$$

Thus

$$a^i(s) = \frac{c(f,\varphi_k^i)_h(s)}{\|\varphi_k^i\|_2^2}, \quad s = 0,\ldots,2^k-1,\ i = 1,\ldots,n.$$

Using (3) and (8), we imply that

$$\begin{aligned}\|a^i\|_p &\leqslant \frac{1}{\|\varphi_k^i\|_2^2}\|f\|_p|\varphi_k^i|_{p',h}\\ &\leqslant \frac{1}{\|\varphi_k^i\|_2^2}\|f\|_p C_2 2^{-k/p'}, \quad i = 1,\ldots,n.\\ &\leqslant \frac{C_2}{C_1}2^{k/p}\|f\|_p. \qquad (10)\end{aligned}$$

From (9) and (10), we see that the multi-wavelet decomposition (2) is L_p-stable. Finally, we prove the necessary conditions. If $2 \leqslant p < \infty$, then we have, for $a(s) = 1$, $s = 0,\ldots,2^k-1$,

$$\begin{aligned}\|\varphi_k^i * a\|_p^p &= \int_0^{2\pi} |(\varphi_k^i * a)(x)|^p dx\\ &= \int_0^{2\pi} |(\varphi_k^i * a)(x)|^{p-2}|(\varphi_k^i * a)(x)|^2 dx\\ &\leqslant 2\pi C^{p-2}\|\varphi_k^i * a\|_2^2\\ &= 2\pi C^{p-2}\|\varphi_k^i\|_2^2\|a\|_2^2.\end{aligned}$$

But $\|a\|_2^2 = \|a\|_p^p = 2^k$, and L_p-stability of multi-wavelet decomposition (2), we can write

$$2\pi C^{p-2}2^k\|\varphi_k^i\|_2^2 \geqslant C'2^{-k}2^k = C'.$$

Thus

$$\|\varphi_k^i\|_2^2 \geqslant \frac{C'}{C^{p-2}}2^{-k} = C_2'2^{-k}.$$

If $1 \leqslant p \leqslant 2$, then Using (7) with $p_1 = p$, $p_2 = 2$ and $\alpha = 2\pi$, we abtain

$$\|\varphi_k^i * a\|_p \leqslant (2\pi)^{1/p-1/2}\|\varphi_k^i * a\|_2.$$

Hence,

$$(11)\quad (2\pi)^{1/p-1/2}\|\varphi_k^i\|_2\|a\|_2 = (2\pi)^{1/p-1/2}\|\varphi_k^i * a\|_2 \geqslant \|\varphi_k^i * a\|_p \geqslant C' 2^{-k/p}\|a\|_p.$$

By taking the sequence $a = \{a(s)\}_{s=0}^{2^k-1}$ with $a(s) = 1$ for all $s = 0, \dots, 2^k - 1$, we have $\|a\|_2 = 2^{k/2}$, $\|a\|_p = 2^{k/p}$. Then (11) tells that

$$\|\varphi_k^i\|_2 \geqslant C'(2\pi)^{1/2-1/p}2^{-k/2}.$$

In the orther words $\|\varphi_k^i\|_2^2 \geqslant C_3 2^{-k}$, where $C_3 := C'^2(2\pi)^{1-2/p}$. We set $C_1 = \min\{C_3, C_2'\}$, then

$$\|\varphi_k^i\|_2^2 \geqslant C_1 2^{-k}.$$

□

When $1 \leqslant p \leqslant 2$, Theorem 4 can be sharpened as follows.

Theorem 5. *Let $1 \leqslant p \leqslant 2$ and φ_k^i, $i = 1, \dots, n$, $k \in \mathbb{Z}_+$, be functions defined on $\mathbb{T}$, $h = 2\pi/2^k$. Assume that*

(i) *For each $k \in \mathbb{Z}_+$, $\{\varphi_k^i(\cdot - 2\pi s/2^k), s = 0, \dots, 2^k - 1, i = 1, \dots, n\}$ are an orthogonal set.*
(ii) $|\varphi_k^i|_{p',h} \leqslant C2^{-k/p'}$, $i = 1, \dots, n$, $k \in \mathbb{Z}_+$, $1/p + 1/p' = 1$.

Then the multi-wavelet decomposition (2) is L_p-stable if and only if

$$\|\varphi_k^i\|_2^2 \geqslant C_1 2^{-k}, \quad \text{for all } i = 1, \dots, n, k \in \mathbb{Z}_+.$$

Here C, C_1 are positive constants not depending on k.

Proof. The necessary condition has already proven in Theorem 4. We prove the sufficient condition. Using (7) with $p_1 = p$, $p_2 = p'$, $\alpha = h$, and (ii), we obtain

$$|\varphi_k^i|_{p,h} \leqslant (2\pi/2^k)^{1/p-1/p'}|\varphi_k^i|_{p,h} \leqslant C_0 2^{-k/p}, C_0 := (2\pi)^{1/p-1/p'}.$$

Now we suppose that

$$f = \sum_{i=1}^{n} \varphi_k^i * a^i.$$

Then similarly to the proof of Theorem 4, we have

$$\|f\|_p \leqslant C_0 2^{-k/p} \sum_{i=1}^{n} \|a^i\|_p$$

and

$$\|f\|_p \geqslant \frac{C_1}{nC} 2^{-k/p} \sum_{i=1}^{n} \|a^i\|_p.$$

Therefore, the multi-wavelet decompostion (2) is L_p-stable. □

Finally, in the case $p = \infty$, we have

Theorem 6. *Let φ_k^i, $i = 1, \dots, n$, $k \in \mathbb{Z}_+$, be functions defined on $\mathbb{T}$, $h = 2\pi/2^k$. Assume that*

(i) *For each $k \in \mathbb{Z}_+$, $\{\varphi_k^i(\cdot - 2\pi s/2^k), s = 0, \dots, 2^k - 1, i = 1, \dots, n\}$ are an orthogonal set.*
(ii) $\|\varphi_k^i\|_2^2 \geqslant C_1 2^{-k}$, *for all $k \in \mathbb{Z}_+$.*

Then the multi-wavelet decomposition (2) is L_∞-stable if and only if

$$|\varphi_k^i|_{\infty,h} \leqslant C, \quad \text{for all } k \in \mathbb{Z}_+. \tag{12}$$

Here C, C_1 are positive constants not depending on k.

Proof. Assume that the multi-wavelet decomposition (2) is L_∞-stable. Then for all $a = \{a(s)\}_{s=0}^{2^k-1}$, we have

$$\|\varphi_k^i * a\|_\infty \leqslant C\|a\|_\infty. \tag{13}$$

For any $x \in \mathbb{T}$, we put

$$a(s) = \begin{cases} |\varphi_k^i(x - 2\pi s/2^k)|/\varphi_k^i(x - 2\pi s/2^k) & \text{if } \varphi_k^i(x - 2\pi s/2^k) \neq 0; \\ 0 & \text{if } \varphi_k^i(x - 2\pi s/2^k) = 0. \end{cases}$$

Then

$$\sum_{s=0}^{2^k-1} |\varphi_k^i(x - 2\pi s/2^k)| = (\varphi_k^i * a)(x) \leqslant C\|a\|_\infty = C.$$

This inequality implies that $|\varphi_k^i|_{\infty,h} \leqslant C$. Finally, if (13) holds, then similary to the proof of Theorem 4, we can see that the multi-wavelet decomposition (2) is L_∞-stable. ☐

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Information Technology Institute,
Vietnam National University, Hanoi,
E3, 144 Xuan Thuy Rd., Cau Giay, Hanoi, Vietnam

E-mail address: dinhdung@vnu.edu.vn

E-mail address: bangvc@vnu.edu.vn