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THE BERTRAND OFFSETS OF RULED SURFACES IN $\mathbb{R}^3_1$

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Abstract. The problem of finding a curve whose principal normals are also the principal normals of another curve was apparently first proposed by Saint-Venant but solved by J. Bertrand. Such curves were referred to as 'Bertrand offsets'. In this paper, a generalization of the theory of Bertrand curves is presented for ruled surfaces in Minkowski space $\mathbb{R}^3_1$. Using lines instead of points, two ruled surfaces which are offset in the sense of Bertrand are defined. The obtained results are illustrated by computer-aided examples.

1. Preliminaries

Let us consider the Minkowski 3-space $\mathbb{R}^3_1 = [\,\mathbb{R}^3, (+,+,-)\,]$. The Lorentzian inner product of $\mathbf{X} = (x_1, x_2, x_3)$ and $\mathbf{Y} = (y_1, y_2, y_3) \in \mathbb{R}^3_1$ is defined by

$$\langle X\, ,\, Y\rangle = x_1y_1 + x_2y_2 - x_3y_3.$$

A vector $X \in \mathbb{R}^3_1$ is called a *spacelike, timelike* and *null (lightlike) vector* if $\langle X\, ,\, X\rangle > 0$ or $X = 0$, $\langle X\, ,\, X\rangle < 0$, and $\langle X\, ,\, X\rangle = 0$ for $X \neq 0$, respectively [6].

For a regular curve in $\mathbb{R}^3_1$, if its tangent vector at every point is a spacelike, timelike or null vector, then the curve is called a *spacelike*, *timelike* and *null curve*, respectively [6].

A surface in $\mathbb{R}^3_1$ is called a *timelike* surface if the induced metric on the surface is a Lorentz metric [2]. The normal vector on the timelike surface is a spacelike vector [7]. A surface in $\mathbb{R}^3_1$ is called a *spacelike surface* if the induced metric on the surface is a positive definite Riemannian metric [2]. The normal vector on the spacelike surface is a timelike vector [8].

The *pseudo-hyperbolic space* of radius $r > 0$ in $\mathbb{R}^3_1$ is the hyperquadric

$$H^2_0 = \{X \in \mathbb{R}^3_1 \big| \langle X, X\rangle = -r^2\}$$

with dimension 2 [6].

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Let $\mathbf{X} = (x_1, x_2, x_3)$ and $\mathbf{Y} = (y_1, y_2, y_3)$ be the vectors in $\mathbb{R}^3_1$. The *cross product* of $\mathbf{X}$ and $\mathbf{Y}$ is defined by

$$\mathbf{X} \times \mathbf{Y} = (x_3 y_2 - x_2 y_3\,,\, x_1 y_3 - x_3 y_1\,,\, x_1 y_2 - x_2 y_1) \tag{1.1}$$

where $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$, $\mathbf{e_i} = (\delta_{i1}, \delta_{i2}, \delta_{i3})$, $1 \leqslant i \leqslant 3$, is the standart basis of $\mathbb{R}^3_1$. This definition shows that

$$\mathbf{e_1} \times \mathbf{e_2} = \mathbf{e_3}, \quad \mathbf{e_2} \times \mathbf{e_3} = -\mathbf{e_1} \quad \text{and} \quad \mathbf{e_3} \times \mathbf{e_1} = \mathbf{e_2}. \tag{1.2}$$

Theorem 1.1. [1] *Let* $\mathbf{X}\ \mathbf{Y} \in \mathbb{R}^3_1$. *We have*

(i) *If* $\mathbf{X}$ *and* $\mathbf{Y}$ *are the space-like vectors,* $\mathbf{X} \times \mathbf{Y}$ *is a time-like vector.*

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(iii) *If* $\mathbf{X}$ *is the space-like vector and* $\mathbf{Y}$ *is the time-like vector,* $\mathbf{X} \times \mathbf{Y}$ *is a space-like vector.*

A *ruled surface* φ in $\mathbb{R}^3_1$ is a surface swept out by a straight line e along a curve α and has the parametric representation

$$\varphi(s, v) = \boldsymbol{\alpha}(s) + v\mathbf{e}(s), \quad \|\mathbf{e}\| = 1. \tag{1.3}$$

The curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ is called *base curve* and the various positions of the generating line $\mathbf{e}(s)$ are called the *rulings* of the surface φ.

2. The Frenet equations of a timelike ruled surface according to H^2_0

Let φ be a ruled surface in $\mathbb{R}^3_1$. The unit normal U of φ along a general generator $\mathbf{l} = \varphi(s_0, v)$ approaches a limiting direction as v infinitely decreases. This direction is called the *asymptotic normal direction* and denoted by $\mathbf{g(s)}|_{s=s_o}$. Thus, we can write

$$\mathbf{g(s)}|_{s=s_o} = \lim_{v \to -\infty} \mathbf{U(s, v)}.$$

The point at which $\mathbf{U}$ is perpendicular to $\mathbf{g}$ is called *striction point* on $l = \varphi(s_o, v)$. The direction of $\mathbf{U}$ at this point is called the *central normal* of the ruled surface and is denoted by $\mathbf{t}$. Thus, we have the orthonormal system $\{\mathbf{e}, \mathbf{t}, \mathbf{g}\}$. This system is called the *Frenet trihedron* of the ruled surface φ and each vectors in this system is called the *Frenet vector* for the ruled surface φ. Since $\mathbf{t}$ and $\mathbf{g}$ have the same type, the Frenet trihedron $\{\mathbf{e}, \mathbf{t}, \mathbf{g}\}$ is not established on a spacelike ruled surface φ. For this reason, in this paper, the surface φ will be taken as a timelike ruled surface. In this case, $\mathbf{e}$ is timelike, $\mathbf{t}$ and $\mathbf{g}$ are spacelike vectors.

Because $\mathbf{e}$ is timelike, $\mathbf{e}$ traces a general space curve on the pseudo-hyperbolic space H^2_0. This curve is denoted by $(\mathbf{e})$ and is called *indicatrix curve of* φ *according to pseudo-hyperbolic space* H^2_0. Also, $\mathbf{e}$ is called *indicatrix vector.*

For the ruled surface given in (1.3), we can write

$$\mathbf{U}(s, v) = \frac{\left(\dfrac{d\boldsymbol{\alpha}}{ds} + v\dfrac{d\mathbf{e}}{ds}\right) \times \mathbf{e}}{\left[\left(\left\langle \dfrac{d\boldsymbol{\alpha}}{ds}, \mathbf{e}\right\rangle\right)^2 + \left\|\dfrac{d\boldsymbol{\alpha}}{ds} + v\dfrac{d\mathbf{e}}{ds}\right\|^2\right]^{1/2}} \tag{2.1}$$

If we calculates limit of the relation (2.1) as v infinetely decreases, we obtain

$$\mathbf{g}(s)|_{s=s_0} = \frac{\mathbf{e}\times\mathbf{e_s}}{\|\mathbf{e_s}\|}\Big|_{s=s_0}, \quad \mathbf{e_s} = \frac{d\mathbf{e}}{ds}\,. \tag{2.2}$$

From the definition of the central normal $\mathbf{t}$ we get

$$\mathbf{t} = -\frac{\mathbf{e_s}}{\|\mathbf{e_s}\|}\,. \tag{2.3}$$

Let $\mathbf{T_o}$ be the unit tangent vector of the indicatrix curve $(\mathbf{e})$ and $\mathbf{N_o}$ be the unit normal vector field of the pseudo-hyperbolic space H_0^2. Let

$$\gamma := \left\langle \mathbf{N_o}\times\mathbf{T_o}, \frac{d\mathbf{T_o}}{dq}\right\rangle$$

where q is the arc-length of the indicatrix curve $(\mathbf{e})$ of φ. For the equation of $(\mathbf{e})$, we can write

$$\boldsymbol{\alpha}_{\mathbf{e}}(q) = \mathbf{e}(s),$$

where s is the arc-length of the base curve α of φ. If we take the derivative of the last equation with respect to the arc q of $(\mathbf{e})$, we obtain $T_o = -\mathbf{t}$.

If ψ_o is the hyperbolic angle between N_o and g. Then, we can write

$$N_o = \cosh\psi_o\, e + \sinh\psi_o\, g.$$

Thus, we obtain

$$\gamma = -(\cosh\psi_o\langle\mathbf{t_q},\mathbf{g}\rangle + \sinh\psi_o\langle\mathbf{t_q},\mathbf{e}\rangle).$$

Assuming that

$$\langle\mathbf{t_q},\mathbf{g}\rangle = \overline{\gamma}. \tag{2.4}$$

We get

$$\overline{\gamma} = -\frac{1}{\cosh\psi_o}(\gamma + \sinh\psi_o).$$

Thus, we can give the following formulas

$$\begin{cases}\mathbf{e_q} = -\mathbf{t}\\ \mathbf{t_q} = \overline{\gamma}\mathbf{g} - \mathbf{e}\\ \mathbf{g_q} = -\overline{\gamma}\mathbf{t}.\end{cases} \tag{2.5}$$

These equations are called the *Frenet equations of the timelike ruled surface* φ *according to pseudo-hyperbolic space* H_0^2 .

The matrix form of (2.5) is

$$\begin{bmatrix}\mathbf{e_q}\\ \mathbf{t_q}\\ \mathbf{g_q}\end{bmatrix} = \begin{bmatrix}0 & -1 & 0\\ -1 & 0 & \overline{\gamma}\\ 0 & -\overline{\gamma} & 0\end{bmatrix}\begin{bmatrix}\mathbf{e}\\ \mathbf{t}\\ \mathbf{g}\end{bmatrix}. \tag{2.6}$$

The 3×3 matrix given above is a skew-adjoint matrix. From the equations (2.5) we obtain

$$\overline{\gamma} = -\frac{\langle\mathbf{e},\mathbf{e_s}\times\mathbf{e_{ss}}\rangle}{\|\mathbf{e_s}\|^3}\,. \tag{2.7}$$

The curve which is drawn by the striction points of the timelike ruled surface φ is called *striction curve* and is denoted by (c).

For the striction curve of φ we have

$$\mathbf{c}(s)=\boldsymbol{\alpha}(s)-\frac{\langle\boldsymbol{\alpha}_{\mathbf{s}},\mathbf{e}_{\mathbf{s}}\rangle}{\langle\mathbf{e}_{\mathbf{s}},\mathbf{e}_{\mathbf{s}}\rangle}\mathbf{e}_{\mathbf{s}}.$$

If the consecutive rulings of the timelike ruled surface φ intersect, then φ is said to be *developable*. For the developable timelike ruled surface φ, we can write

$$\langle\boldsymbol{\alpha}_{\mathbf{s}},\mathbf{e}\times\mathbf{e}_{\mathbf{s}}\rangle=0.$$

Thus, we can give the following theorem without proof.

Theorem 2.1. *Let the striction curve (c) is taken as the base curve of the timelike ruled surface φ in $\mathbb{R}^3_1$. In this case, φ is developable if and only if the indicatrix vector $\mathbf{e}$ is the tangent of the striction curve of φ.*

Let φ be a timelike ruled surface in $\mathbb{R}^3_1$. If φ is developable, then from Theorem 2.1 and the Frenet formulas for a timelike curve in $\mathbb{R}^3_1$ [8] we obtain

$$\mathbf{e}=\mathbf{T}\ \ \mathbf{t}=-\mathbf{N}\,,\ \mathbf{g}=-\mathbf{B}, \tag{2.8}$$

where $\{\mathbf{T}\,,\,\mathbf{N}\,,\,\mathbf{B}\}$ is the Frenet trihedron of the striction curve (c).

From (2.7), (2.8) and [5] we can give the following result.

Corollary 2.1. *Let φ be a developable timelike ruled surface in $\mathbb{R}^3_1$. Then $\overline{\gamma}$ is constant if and only if the striction curve $(\mathbf{c})$ of φ is a helix.*

In this paper, the striction curve of the timelike ruled surface φ will be taken as the base curve. In this case, for the parametric equation of φ, we can write

$$\varphi(s,v)=\mathbf{c}(s)+v\mathbf{e}(s).$$

3. Bertrand offsets of the timelike ruled surfaces in $\mathbb{R}^3_1$

Let φ and φ^* be two timelike ruled surfaces in $\mathbb{R}^3_1$. φ^* is said to be Bertrand offset of φ if there exists a one-to-one correspondence between their rulings such that both surfaces have a common central normal at the striction points of their corresponding rulings.

The base timelike ruled surface $\varphi(s,v)$ can be expressed as

$$\varphi(s,v)=\mathbf{c}(s)+v\mathbf{e}(s)$$

where $(\mathbf{c})$ is its striction curve and s is arc-length along $(\mathbf{c})$. If $\mathbf{e}$, $\mathbf{t}$ and $\mathbf{g}$ are the Frenet vectors of φ, then from [4] the Frenet vectors of the timelike Bertrand offset φ^* of φ is given by

$$\begin{cases}\mathbf{e}^*=\cosh\theta\,\mathbf{e}+\sinh\theta\,\mathbf{g}\\ \mathbf{t}^*=\mathbf{t}\\ \mathbf{g}^*=\sinh\theta\,\mathbf{e}+\cosh\theta\,\mathbf{g}\end{cases} \tag{3.1}$$

where θ is the hyperbolic angle between the vectors $\mathbf{e}$ and $\mathbf{e}^*$.

Therefore, the equation of φ^* in terms of φ can be written as

$$\varphi^*(s\,,\,v) = [\mathbf{c}(\mathrm{s}) + \mathrm{R}\mathbf{t}(\mathrm{s})] + \mathrm{v}\,[\cosh\theta\,\mathbf{e}(\mathrm{s}) + \sinh\theta\,\mathbf{g}(\mathrm{s})] \tag{3.2}$$

where R is the Lorentzian distance between the corresponding central normals of the timelike ruled surfaces φ and φ^*.

If $\theta = 0$, then two timelike ruled surfaces will be referred to as 'oriented timelike offsets'.

Theorem 3.1. *If φ^* is the Bertrand offset of φ, then $R = \text{const}$ and $\theta = \text{const}$.*

Proof. By definition, the central normal $\mathbf{t}^*$ of φ^* is the same as the central normal $\mathbf{t}$ of φ, i.e. $\mathbf{t} = \mathbf{t}^*$. From the equality (2.3), we get

$$\mathbf{t}^* = -\frac{\mathbf{e}^*_\mathbf{s}}{\|\mathbf{e}^*_\mathbf{s}\|}\,.$$

Because of the last two equalities, we have

$$\mathbf{e}^*_\mathbf{s} = \lambda\mathbf{t}\ \ (\lambda \text{ a scalar}).$$

Let q be the arc-length of the indicatrix curve $(\mathbf{e})$ of φ, then using equations (3.1) and the chain rule of differentiation we can write

$$\sinh\theta\theta_s\mathbf{e} + \cosh\theta\theta_s\mathbf{g} - (\cosh\theta + \overline{\gamma}\sinh\theta)q_s\mathbf{t} = \lambda\,\mathbf{t}.$$

Therefore, it is clear that

$$\theta_s(\sinh\theta\,\mathbf{e} + \cosh\theta\mathbf{g}) = 0$$

or

$$\theta_s = \frac{d\theta}{ds} = 0.$$

This implies that $\theta = \text{const}$. Since the base curve of φ^* is its striction curve, we get $\langle \mathbf{c}^*_\mathbf{s}, \mathbf{e}^*_\mathbf{s}\rangle = 0$.

From the equality $\mathbf{e}^*_\mathbf{s} = \lambda\mathbf{t}$ it follows that

$$\langle(\mathbf{c} + R\mathbf{t})_s, \mathbf{t}\rangle = 0.$$

This last equation simplifies to

$$\langle \mathbf{c}_s + R_s\mathbf{t} + Rq_s(\overline{\gamma}\mathbf{g} - \mathbf{e}), \mathbf{t}\rangle = 0$$

which implies that $R_s = 0$ or $R = \text{const}$. □

Theorem 3.2. *Let φ^* be the Bertrand offset of the developable timelike ruled surface φ. Then φ^* is developable if and only if the following relationship can be written between the curvature κ and the torsion τ of the striction curve (c) of φ:*

$$(1 - \kappa R)\sinh\theta - \tau R\cosh\theta = 0.$$

Proof. Because of Theorem 2.1, developable surface φ can be described as

$$\varphi(s, v) = \mathbf{c}(s) + v\mathbf{T}(s)$$

where $\mathbf{T}(s)$ is tangent to $\mathbf{c}(s)$. Therefore, the surface φ^* can be expressed as

$$\varphi^*(s, v) = [\mathbf{c} + R\mathbf{N}] + v\,[\cosh\theta\mathbf{T} + \sinh\theta\mathbf{B}]\,,$$

where $\mathbf{N}(s)$ and $\mathbf{B}(s)$ are the principal normal and the binormal to $\mathbf{c}(s)$, respectively.

If φ^* is developable, then we get

$$\frac{d}{ds}(\mathbf{c}+R\mathbf{N})=\lambda\left[\cosh\theta\,\mathbf{T}+\sinh\theta\,\mathbf{B}\right](\lambda \text{ a scalar})$$

or

$$\mathbf{c}_s+R\mathbf{N}_s=\lambda\left[\cosh\theta\,\mathbf{T}+\sinh\theta\,\mathbf{B}\right].$$

By equation (2.8),

$$(1-\kappa\,\mathrm{R})\mathbf{T}\text{ - }\tau\,\mathrm{R}\mathbf{B}=\lambda\left[\cosh\theta\,\mathbf{T}-\sinh\theta\,\mathbf{B}\right]$$

The last equation implies that

$$(1-\kappa\,\mathrm{R})\sinh\theta\text{ - }\tau\,\mathrm{R}\cosh\theta=0.$$

Conversely, suppose that the equality

$$(1-\kappa\,\mathrm{R})\sinh\theta\text{ - }\tau\,\mathrm{R}\cosh\theta=0$$

is satisfied. Then we can write

$$\frac{d}{ds}(\mathbf{c}+R\mathbf{N})=\lambda\left[\cosh\theta\,\mathbf{T}+\sinh\theta\,\mathbf{B}\right].$$

Because of Theorem 2.1, φ^* is developable. □

Let φ^* be the developable Bertrand offset of the developable timelike ruled surface φ.

From (2.8) and (3.1), we have

$$\begin{cases}\mathbf{T}^*=\cosh\theta\,\mathbf{T}-\sinh\theta\,\mathbf{B}\\ \mathbf{N}^*=\mathbf{N}\\ \mathbf{B}^*=-\sinh\theta\,\mathbf{T}+\cosh\theta\,\mathbf{B}\end{cases}\tag{3.3}$$

where $\{\mathbf{T}^*, \mathbf{N}^*, \mathbf{B}^*\}$ is the Frenet triply at the striction point of the striction curve of φ^*.

Because of the Frenet Formulas for a time-like curve in $\mathbb{R}^3_1$, we get

$$\mathbf{T}^*_{s^*}=\kappa^*\,\mathbf{N}^*\tag{3.4}$$

where s^* and κ^* are the arc-parameter and the curvature of the striction curve of φ^*, respectively.

Taking the derivative of $\mathbf{T}^*$ in the relation (3.3) with respect to the arc s^* and using the Frenet Formulas yields

$$\kappa^*=\frac{\mathrm{ds}}{\mathrm{ds}^*}(\kappa\cosh\theta-\tau\sinh\theta).\tag{3.5}$$

Similarly, for the torsion τ^* of the striction curve of φ^*, we can write,

$$\tau^*=\frac{\mathrm{ds}}{\mathrm{ds}^*}(\tau\cosh\theta-\kappa\sinh\theta).\tag{3.6}$$

Since φ^* is the Bertrand offset of φ, we get $\langle \mathbf{t}\,,\,\mathbf{t}^*\rangle = \langle \mathbf{t}\,,\,\mathbf{t}\rangle = 1 \;>\; 0$. Thus

$$\langle \mathbf{t}\,,\,\mathbf{e}^*_s\rangle \;<0.$$

If φ and φ^* are developable, we have $\langle \mathbf{N}\,,\,\mathbf{T}^*_s\rangle \;>\; 0$. Then

$$\kappa\mathrm{cosh}\theta - \tau\mathrm{sinh}\theta \;>\; 0.$$

Thus, if φ and φ^* are developable, we can give the following results.

Corollary 3.1. *There is the following relation between the curvatures and the torsions of the striction curves of φ and φ^*:*

$$\frac{\tau^*}{\kappa^*} = \frac{\tau\mathrm{cosh}\theta - \kappa\mathrm{sinh}\theta}{\kappa\mathrm{cosh}\theta - \tau\mathrm{sinh}\theta}\,.$$

Corollary 3.2. *The striction curve of φ is a helix if and only if the striction curve of φ^* is a helix.*

Example 1. For the timelike ruled surface

$$\varphi(s,v) = \big(\mathrm{cosh}(s) + v\sqrt{2}\mathrm{sinh}(s), 2s+v, \mathrm{sinh}(s) + v\sqrt{2}\mathrm{cosh}(s)\big).$$

(i) A Bertrand offset with spacelike base curve of φ is

$$\begin{aligned}\varphi^*_1(s,v) = \big(&-\mathrm{cosh}(s) + v(2\sqrt{2}+\sqrt{3})\mathrm{sinh}(s), 2s + v(2+\sqrt{6}),\\ &-\mathrm{sinh}(s) + v(2\sqrt{2}+\sqrt{3})\,\mathrm{cosh}(s)\big) \quad \text{(Fig. 1)}.\end{aligned}$$

(ii) A Bertrand offset with timelike base curve of φ is

$$\begin{aligned}\varphi^*_2(s,v) = \big(&-4\,\mathrm{cosh}(s) + v\frac{1}{2}(3\sqrt{2}+\sqrt{5})\mathrm{sinh}(s), 2s + v\frac{1}{2}(3+\sqrt{10}),\\ &-4\mathrm{sinh}(s) + v\frac{1}{2}(3\sqrt{2}+\sqrt{5})\mathrm{cosh}(s)\big) \quad \text{(Fig. 2)}.\end{aligned}$$

(iii) A Bertrand offset with null base curve of φ is

$$\begin{aligned}\varphi^*_3(s,v) = (&-2\mathrm{cosh}(s) + v\sqrt{2}\mathrm{sinh}(s), 2s+v,\\ &-2\mathrm{sinh}(s) + v\sqrt{2}\mathrm{cosh}(s)) \quad \text{(Fig. 3)}.\end{aligned}$$

Example 2. The surface

$$\eta(s,v) = \big(\mathrm{cosh}(s) + v\mathrm{sinh}(s), 0, \mathrm{sinh}(s) + v\mathrm{cosh}(s)\big)$$

is a timelike ruled surface. All of the Bertrand offsets of η have timelike base curve. One of the offsets of η is

$$\eta^*(s,v) = \big(-\mathrm{cosh}(s) + 2v\mathrm{sinh}(s), \sqrt{3}v, -\mathrm{sinh}(s) + 2v\mathrm{cosh}(s)\big) \quad \text{(Fig. 4)}.$$

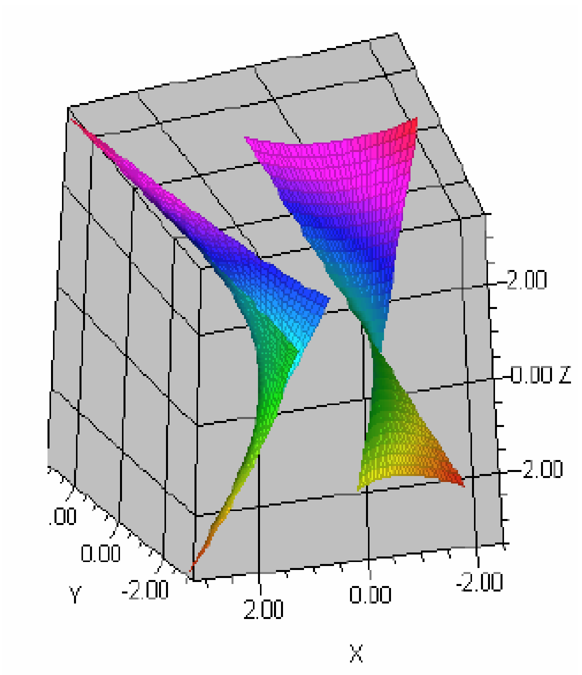

Fig. 1. Ruled surface φ and its Bertrand offset with spacelike base curve

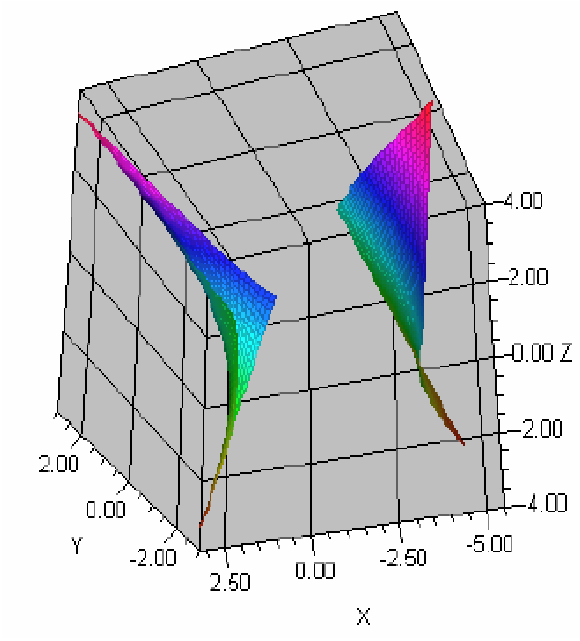

Fig. 2. Ruled surface φ and its Bertrand offset with timelike base curve

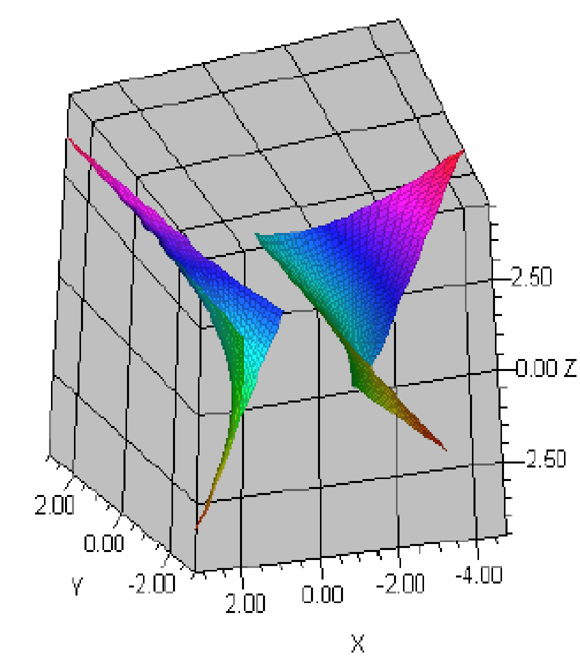

Fig. 3. Ruled surface φ and its Bertrand offset with null base curve

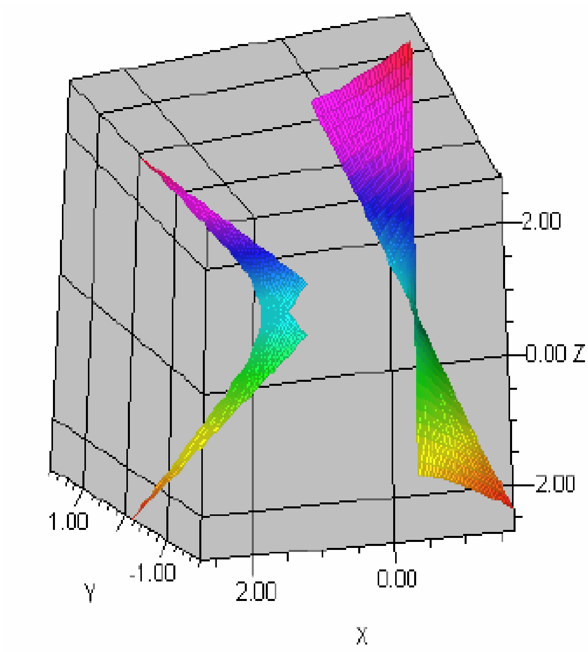

Fig. 4. Ruled surface η and its Bertrand offset

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