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ON MAXIMALITY FOR SOME KINDS OF CODES OVER TWO-LETTER ALPHABETS

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Abstract. Superinfix codes, p-superinfix codes and s-superinfix codes have been introduced and considered by D. L. Van, P. T. Huy and the author in earlier papers. The embedding problem for these classes of codes has been proved to have positive solution in both the finite and regular case. Also, these kinds of codes can be characterized by means of variants of Parikh vectors. In this paper we consider these codes in the case of two-letter alphabets. Based on the mentioned above vector characterizations, it is shown that, for each of the classes of codes under consideration, there exists a procedure to generate all finite maximal codes in the class, starting from anyone among them. Embedding algorithms, other than those obtained earlier, for these classes of codes are also exhibited.

1. Introduction and Preliminaries

Prefix codes and suffix codes are among the simplest codes, but most of the important problems in the theory of codes may arise for them. Superinfix codes, p-superinfix codes and s-superinfix codes, which are particular cases of prefix codes and suffix codes, have been introduced and considered in [3, 4, 12, 13, 14]. All these classes of codes can be defined by length-increasing transitive binary relations. The embedding problem for these kinds of codes has been solved positively for both the finite and regular case [3, 12, 14]. It turns out that variants of Parikh vectors are adequate tools to characterize these codes, especially maximal codes in corresponding classes. These codes have also some other interesting properties, e.g. every maximal p-superinfix (s-superinfix) code is maximal as a code.

For a given class C of codes, it is interesting to find out a way to generate all the finite maximal codes in C. To our knowledge, this problem was solved only for some classes of codes, namely those of bifix codes (Y. Césari [2]), infix codes, solid codes, hypercodes (N. H. Lam [6, 7, 8]) and supercodes (D. L. Van [11]). Binary codes, i.e. codes over two-letter alphabets, being used in most of practical applications, are subject of many research works. In this paper we consider the mentioned above kinds of codes in the case of two-letter alphabets. Based on vector characterizations, it is shown that, for each of the classes of codes under

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consideration, there exists a procedure to generate all finite maximal codes in the class, starting from anyone among them. Embedding algorithms, other than those obtained earlier, for these classes of codes are also exhibited. This work is motivated by the idea of D. L. Van who characterizes supercodes as independent sets of Parikh vector with respect to an appropriate binary and generates all the maximal supercodes starting from an arbitrary given maximal supercode [11].

We now recall some notions, notations and facts. Let A be a finite alphabet and A^* the set of all the words over A. The empty word is denoted by 1 and A^+ stands for $A^* - \{1\}$. The number of all occurrences of letters in a word u is the *length* of u, denoted by $|u|$. A word u is called an *infix* (a *prefix*, a *suffix*) of a word v if there exist words $x, y \in A^*$ such that $v = xuy$ (resp., $v = uy$, $v = xu$). The infix (prefix, suffix) is *proper* if $xy \neq 1$ (resp., $y \neq 1$, $x \neq 1$), denote by $u \prec_i v$ (resp., $u \prec_p v$, $u \prec_s v$). A word u is a *subword* of a word v if, for some $n \geq 1, u = u_1 \dots u_n, v = x_0 u_1 x_1 \dots u_n x_n$ with $u_1, \dots, u_n, x_0, \dots, x_n \in A^*$. If $x_0 \dots x_n \neq 1$ then u is called a *proper subword* of v, denote by $u \prec_h v$. And a word u is called a *permutation* of a word v if $|u|_a = |v|_a$ for all $a \in A$, where $|u|_a$ denotes the number of occurrences of a in u.

A subset X of A^+ is a *code* over A if any word w in X^+ has exactly one factorization into words of X. A code X is *maximal* over A if X is not properly contained in any other code over A. As has been observed by several authors, many codes can be defined by a binary relation (see [9, 10, 12, 14]). Given a binary relation $\prec$ on A^*. A subset X in A^* is an *independent set* with respect to $\prec$ ($\prec$*-independent set*, for short) if any two elements of X are not in this relation. We say that a class C of codes is *defined by* $\prec$ if these codes are exactly the independent sets with respect to $\prec$. For further details of the theory of codes we refer to [1, 5, 9].

Definition 1.1. A subset X of A^+ is a *superinfix* (*p-superinfix, s-superinfix*) *code*, $X \in C_{spi}$ (resp., $X \in C_{p.spi}, X \in C_{s.spi}$), if no word in X is a **su**bword of a **per**mutation of a proper **infix** (resp., **p**refix, **s**uffix) of another word in X. In other words, the classes C_{spi}, $C_{p.spi}$ and $C_{s.spi}$ of these codes have as defining relations the following, respectively

$$u \prec_{spi} v \Leftrightarrow (\exists v' : v' \prec_i v)(\exists v'' \in \pi(v')) : u \preceq_h v'';$$
$$u \prec_{p.spi} v \Leftrightarrow (\exists v' : v' \prec_p v)(\exists v'' \in \pi(v')) : u \preceq_h v'';$$
$$u \prec_{s.spi} v \Leftrightarrow (\exists v' : v' \prec_s v)(\exists v'' \in \pi(v')) : u \preceq_h v'';$$

where $\pi(v')$ denotes the set of all permutations of v', and $\preceq_h$ denotes the reflexive closure of $\prec_h$.

A superinfix (p-superinfix, s-superinfix) code X over A is said to be *maximal* if X is not properly contained in another one over A.

Example 1.1. Consider the sets $X = \{ab, b^3a\}$, $X^R = \{ba, ab^3\}$ and $Y = ab^*a$ over $A = \{a, b\}$. It is easy to check that $X \in C_{p.spi} - C_{spi}$, $X^R \in C_{s.spi} - C_{spi}$ and $Y \in C_{spi}$. Moreover, Y is a maximal superinfix code over A.

Let $A = \{a_1, a_2, \ldots, a_k\}$ and $K = \{1, 2, \ldots, k\}$. For every $u \in A^*$, we denote by $p(u)$ the *Parikh vector* of u, namely

$$p(u) = (|u|_{a_1}, |u|_{a_2}, \ldots, |u|_{a_k}),$$

where $|u|_{a_i}$ denotes the number of occurrences of a_i in u. Thus p is a mapping from A^* into the set V^k of all the k-vectors of non-negative integers. Now, to every $u \in A^+$ we associate two elements of the cartesian product $V^k \times K$, denoted by $p_L(u)$ and $p_F(u)$, which are defined as follows

$$p_L(u) = (p(u), l); \quad p_F(u) = (p(u), f);$$

where l and f are the indices of the last and the first letter in u, respectively. Thus p_L and p_F are mappings from A^+ into $V^k \times K$. These mappings are then extended to sets in a standard way: $p_L(X) = \{p_L(u) \mid u \in X\}$ and $p_F(X) = \{p_F(u) \mid u \in X\}$.

Put $U = \{(\xi, i) \in V^k \times K \mid p_i(\xi) \neq 0\}$. On U we associate a binary relation $\prec$ defined by

$$(\xi, i) \prec (\eta, j) \Leftrightarrow (\xi \leq \eta) \wedge (p_j(\xi) < p_j(\eta)),$$

where $p_i(\xi), 1 \leq i \leq k$, denotes the i-th component of ξ. The relation $\prec$ on U, as easily verified, is transitive. Notice that for all set $X \subseteq A^+$, we have $p_L(X)$ and $p_F(X)$ are subsets of U.

To every subset X of A^+, we associate the sets

$$E_X = \{x \in X \mid \exists y \in X : p(y) < p(x)\}; \quad O_X = X - E_X.$$

Let u be a word in A^+, we define the following operations

$$\pi_L(u) = \pi(u')b, \text{ with } u = u'b, b \in A;$$
$$\pi_F(u) = a\pi(u'), \text{ with } u = au', a \in A;$$
$$\pi_{LF}(u) = \begin{cases} a\pi(u')b, & \text{if } |u| \geq 2 \text{ and } u = au'b \text{ with } a, b \in A; \\ u, & \text{if } u \in A; \end{cases}$$

which are extended to sets in a normal way: $\pi_L(X) = \bigcup_{u \in X} \pi_L(u)$, $\pi_F(X) = \bigcup_{u \in X} \pi_F(u)$ and $\pi_{LF}(X) = \bigcup_{u \in X} \pi_{LF}(u)$. Also, we put $\pi(X) = \bigcup_{u \in X} \pi(u)$.

The following two results have been proved in [3].

Theorem 1.1. *For any $X \subseteq A^+$, the following assertions are equivalent*

(i) *X is a p-superinfix code (resp., a s-superinfix code, a superinfix code);*
(ii) *$\pi(O_X) \cup \pi_L(E_X)$ is a p-superinfix code (resp., $\pi(O_X) \cup \pi_F(E_X)$ is a s-superinfix code, $\pi(O_X) \cup \pi_{LF}(E_X)$ is a superinfix code);*
(iii) *$p_L(X)$ is a $\prec$-independent set (resp., $p_F(X)$ is a $\prec$-independent set, both $p_L(X)$ and $p_F(X)$ are $\prec$-independent sets) on U.*

Theorem 1.2. *For any subset X of A^+, X is a maximal p-superinfix (s-superinfix) code iff $p_L(X)$ (resp., $p_F(X)$) is a maximal $\prec$-independent set on U and $\pi(O_X) \cup \pi_L(E_X) = X$ (resp., $\pi(O_X) \cup \pi_F(E_X) = X$).*

2. P-Superinfix (S-Superinfix) Codes

From now on, unless otherwise specified, fix $A = \{a, b\}$. We always understand a has index 1 and b has index 2. Put

$$U^2 = \{(\xi, i) \in V^2 \times \{1,2\} \mid p_i(\xi) \neq 0\}.$$

On U^2 we introduce the relation $\dashv$ defined by

$$(\xi, i) \dashv (\eta, j) \Leftrightarrow (p_1(\xi) \geq p_1(\eta)) \wedge (p_2(\xi) \leq p_2(\eta)) \\ \wedge (p_1(\xi) = p_1(\eta) \Rightarrow j = 1) \wedge (p_2(\xi) = p_2(\eta) \Rightarrow i = 2),$$

where $p_i(\xi)$ denotes the i-th component of ξ. For simplicity, in the sequel we write $(p_1(\xi), p_2(\xi), i)$ instead of $((p_1(\xi), p_2(\xi)), i)$.

Lemma 2.1. *We have the following assertions*

(i) *The relation $\dashv$ is transitive.*
(ii) *If $(\xi, i) \dashv (\eta, j)$ then $\{(\xi, i), (\eta, j)\}$ is a $\prec$-independent set. Conversely, if $\{(\xi, i), (\eta, j)\}$ is a $\prec$-independent set, and $p_1(\xi) = p_1(\eta)$, $\xi \leq \eta$, $j = 1$ or $p_1(\xi) > p_1(\eta)$ then $(\xi, i) \dashv (\eta, j)$.*

Proof. It follows immediately from definitions of $\dashv$ and $\prec$. □

A finite sequence (may be empty) S: $(\xi_1, i_1), (\xi_2, i_2), \ldots, (\xi_n, i_n)$ of elements in U^2 is a *chain* if

$$(\xi_1, i_1) \dashv (\xi_2, i_2) \dashv \cdots \dashv (\xi_n, i_n).$$

The chain S is *full* if

$$\forall k, 1 \leq k \leq n-1, \nexists (\eta, j) : (\xi_k, i_k) \dashv (\eta, j) \dashv (\xi_{k+1}, i_{k+1}).$$

If the full chain S satisfies moreover the condition

$$p_2(\xi_1) = p_1(\xi_n) = 0,$$

then it is said to be *complete*. A finite subset Z of U^2 is *complete* if it can be arranged to become a complete chain. For $1 \leq s < t \leq n$ we denote by $[(\xi_s, i_s), (\xi_t, i_t)]$ the subsequence $(\xi_s, i_s), (\xi_{s+1}, i_{s+1}), \ldots, (\xi_t, i_t)$ of the sequence S.

We give now characterizations of maximal p-superinfix and s-superinfix codes over two-letter alphabets.

Theorem 2.1. *For any finite subset X of A^+, X is a maximal p-superinfix (s-superinfix) code iff $p_L(X)$ (resp., $p_F(X)$) is complete and $X = \pi(O_X) \cup \pi_L(E_X)$ (resp., $X = \pi(O_X) \cup \pi_F(E_X)$).*

Proof. We treat only the case of p-superinfix codes. For the case of s-superinfix codes the argument is similar. Let X be a finite maximal p-superinfix code. By Theorem 1.2, we have $p_L(X)$ is a maximal $\prec$-independent set on U^2 and $X = \pi(O_X) \cup \pi_L(E_X)$. Let $|p_L(X)| = n$. Arrange $p_L(X)$ to become a sequence S' : $(\xi'_1, i'_1), \ldots, (\xi'_n, i'_n)$ such that $p_1(\xi'_1) \geq \cdots \geq p_1(\xi'_n)$. If for any different $(\xi, i), (\eta, j)$ in $p_L(X)$ with $p_1(\xi) = p_1(\eta)$, $p_2(\xi) \leq p_2(\eta)$ then either $p_2(\xi) = p_2(\eta)$ or $p_2(\xi) < p_2(\eta), j = 1$ because $p_L(X)$ is a $\prec$-independent set. Therefore, for every subsequence $[(\xi'_s, i'_s), (\xi'_t, i'_t)]$ of S', $1 \le s < t \le n$, with $p_1(\xi'_s) = \cdots = p_1(\xi'_t)$, we can arrange it to become a sequence $[(\xi_s, i_s), (\xi_t, i_t)]$ with $p_1(\xi_s) = \cdots = p_1(\xi_t)$ such that $\xi_k \le \xi_{k+1}$ and $i_{k+1} = 1$, for all k, $s \le k \le t-1$. Then, by Lemma 2.1(ii), $[(\xi_s, i_s), (\xi_t, i_t)]$ is a chain. By this, we obtain a sequence $S: (\xi_1, i_1), \ldots, (\xi_n, i_n)$ with $p_1(\xi_1) \ge \cdots \ge p_1(\xi_n)$ and every subsequence $[(\xi_s, i_s), (\xi_t, i_t)]$ with $p_1(\xi_s) = \cdots = p_1(\xi_t)$, is a chain. Again by Lemma 2.1(ii), for any $(\xi_k, i_k), (\xi_{k+1}, i_{k+1})$ in S such that $p_1(\xi_k) > p_1(\xi_{k+1})$, $1 \le k \le n-1$, we have $(\xi_k, i_k) \dashv (\xi_{k+1}, i_{k+1})$. Thus, the sequence S is a chain. If $p_2(\xi_1) \ne 0$ then, choosing (ξ, i) in U^2 with $p_1(\xi) > p_1(\xi_1), p_2(\xi) = 0$ and $i = 1$. Then, $p_L(X) \cup \{(\xi, i)\}$ is still a $\prec$-independent set, a contradiction. Thus $p_2(\xi_1) = 0$. Similarly we have $p_1(\xi_n) = 0$. Now if there exists (η, j) such that $(\xi_k, i_k) \dashv (\eta, j) \dashv (\xi_{k+1}, i_{k+1})$ for some $k, 1 \le k \le n-1$, then by Lemma 2.1(i) and (ii), $p_L(X) \cup \{(\eta, j)\}$ is a $\prec$-independent set, which contradicts again the maximality of $p_L(X)$. So, the sequence S is a complete chain and, therefore, the set $p_L(X)$ is complete.

Conversely, suppose $p_L(X)$ is complete and $X = \pi(O_X) \cup \pi_L(E_X)$. Since, as it is easily verified by Lemma 2.1, every complete set is a maximal $\prec$-independent set, again by Theorem 1.2, it follows that X is a maximal p-superinfix code. □

Example 2.1. It is easy to check that, for any $m, n \ge 1$, the sequence

$$(m,0,1),(m,1,1),\ldots,(m,n-1,1),(m-1,n,2),\ldots,(1,n,2),(0,n,2)$$

is a complete chain. Therefore, the set

$$U_{m,n} = \{(m,0,1),\ldots,(m,n-1,1),(m-1,n,2),\ldots,(0,n,2)\}$$

is complete. With $m = 2, n = 4$, for example

$$U_{2,4} = \{(2,0,1),(2,1,1),(2,2,1),(2,3,1),(1,4,2),(0,4,2)\}.$$

By Theorem 2.1, $X = \pi(\{a^2, b^4\}) \cup \pi_L(\{aba, ab^2a, ab^3a, ab^4\}) = \{a^2, b^4, aba, ba^2, ab^2a, baba, b^2a^2, ab^3a, bab^2a, b^2aba, b^3a^2, ab^4, bab^3, b^2ab^2, b^3ab\}$ is a maximal p-superinfix code. The set $Y = \pi(\{a^2, b^4\}) \cup \pi_F(\{aba, ab^2a, ab^3a, b^4a\})$ is a maximal s-superinfix code.

By Theorem 2.1, in order to characterize the finite maximal p-superinfix (s-superinfix) codes over $A = \{a, b\}$ we may characterize the complete sets instead. For this we first consider some transformations on complete chains. Let S : $(\xi_1, i_1), (\xi_2, i_2), \ldots, (\xi_n, i_n)$ be a complete chain.

(T1) (*extension*). It consists in doing consecutively the following:

- Add on the left of S an element (ξ, i) with $p_1(\xi) > p_1(\xi_1)$;
- If $i = 1$ then delete from S all elements (ξ_k, i_k) with $p_2(\xi_k) \le p_2(\xi)$, else delete all elements (ξ_k, i_k) with $p_2(\xi_k) < p_2(\xi)$;
- If (ξ_{k_0}, i_{k_0}) is the first among the (ξ_k, i_k) remained, then insert between (ξ, i) and (ξ_{k_0}, i_{k_0}) any chain such that $[(\xi, i), (\xi_{k_0}, i_{k_0})]$ is a full chain;
- If there is no such a (ξ_{k_0}, i_{k_0}), then add on the right of (ξ, i) any chain ending with a $(\eta, 2)$, $p_1(\eta) = 0$, and such that $[(\xi, i), (\eta, 2)]$ is a full chain;

• Add on the left of (ξ, i) any chain begining with a $(\theta, 1)$, $p_2(\theta) = 0$, and such that $[(\theta, 1), (\xi, i)]$ is a full chain.

(T2) (*insertion*). This consists of the following successive steps:

• For some k, insert in the middle of (ξ_k, i_k) and (ξ_{k+1}, i_{k+1}), $1 \leq k \leq n-1$, an element (ξ, i) with $p_1(\xi_k) \geq p_1(\xi) \geq p_1(\xi_{k+1})$;

• Delete all elements (ξ_t, i_t) on the left of (ξ, i) which satisfy one of the following conditions

– or $p_2(\xi_t) > p_2(\xi)$, or $p_2(\xi_t) = p_2(\xi)$ and $i_t = 1$, or $p_1(\xi_t) = p_1(\xi)$ and $i = 2$, in the case $p_2(\xi) \leq p_2(\xi_k)$;

– or $p_1(\xi_t) = p_1(\xi)$ and $i = 2$, if $p_2(\xi) > p_2(\xi_k)$;

• If (ξ_{r_0}, i_{r_0}) is the last among the (ξ_r, i_r) remained, then insert between (ξ_{r_0}, i_{r_0}) and (ξ, i) any chain such that $[(\xi_{r_0}, i_{r_0}), (\xi, i)]$ is a full chain (by convention, $(\xi_{r_0}, i_{r_0}) = (\xi_k, i_k)$ if no elements deleted);

• If there is no such a (ξ_{r_0}, i_{r_0}), then add on the left of (ξ, i) any chain commencing with a $(\theta, 1)$, $p_2(\theta) = 0$, and such that $[(\theta, 1), (\xi, i)]$ is a full chain;

• Delete all elements (ξ_t, i_t) on the right of (ξ, i) which satisfy one of the following conditions

– or $p_1(\xi_t) = p_1(\xi)$ and $i_t = 2$, or $p_2(\xi_t) = p_2(\xi)$ and $i = 1$, in the case $p_2(\xi) \leq p_2(\xi_k)$;

– or $p_2(\xi_t) < p_2(\xi)$, or $p_2(\xi_t) = p_2(\xi)$ and $i = 1$, or $p_1(\xi_t) = p_1(\xi)$ and $i_t = 2$, in the case $p_2(\xi) > p_2(\xi_k)$;

• If (ξ_{r_0}, i_{r_0}) is the first among the (ξ_r, i_r) remained, then insert between (ξ, i) and (ξ_{r_0}, i_{r_0}) any chain such that $[(\xi, i), (\xi_{r_0}, i_{r_0})]$ is a full chain (by convention, $(\xi_{r_0}, i_{r_0}) = (\xi_{k+1}, i_{k+1})$ if no elements deleted);

• If there is no such a (ξ_{r_0}, i_{r_0}), then add on the right of (ξ, i) any sequence ending with a $(\eta, 2)$, $p_1(\eta) = 0$, and such that $[(\xi, i), (\eta, 2)]$ is a full chain.

Theorem 2.2. *The following assertions hold true*

(i) *The transformations (T1) and (T2) preserve the completeness of a chain.*
(ii) *Any complete chain can be obtained from another one by a finite number of applications of the transformations (T1) and (T2).*
(iii) *Every chain S can be embedded in a complete chain by a finite number of applications of the transformations (T1) and (T2).*

Proof. (i) Easily seen by the definitions of (T1) and (T2).

(ii) Let S: $(\xi_1, i_1), \ldots, (\xi_n, i_n)$ and S': $(\eta_1, j_1), \ldots, (\eta_m, j_m)$ be two complete chains. To obtain S' from S we can do as follows. According as $p_1(\eta_1) > p_1(\xi_1)$ or $p_1(\eta_1) \leq p_1(\xi_1)$ we apply to S the transformations (T1) or (T2) with $(\xi, i) = (\eta_1, j_1)$. In any case we obtain a complete chain $S^{(1)}$ commencing with (η_1, j_1). Suppose $S^{(k)}$, $1 \leq k \leq m-1$, have been constructed, which is a complete chain commencing with $(\eta_1, j_1), \ldots, (\eta_k, j_k)$. Let $S^{(k)} : (\eta_1, j_1), \ldots, (\eta_k, j_k), (\theta_{k+1}, t_{k+1}), \ldots, (\theta_r, t_r)$. We construct $S^{(k+1)}$ as follows. If $p_1(\eta_{k+1}) \geq p_1(\theta_{k+1})$ then, since

$p_1(\eta_k) \geq p_1(\eta_{k+1})$, we may apply (T2) to insert (η_{k+1}, j_{k+1}) in the middle of (η_k, j_k) and (θ_{k+1}, t_{k+1}). Because S' is complete, in the chain obtained, (η_{k+1}, j_{k+1}) must be next to (η_k, j_k). If $p_1(\eta_{k+1}) < p_1(\theta_{k+1})$ then, since $p_1(\theta_{k+1}) \geq \cdots \geq p_1(\eta_{k+s+1}) \geq p_1(\theta_r)$, there exists $s \geq 1$ such that $p_1(\eta_{k+1}) = p_1(\theta_{k+s+1})$. Let $(\theta_{k+s+1}, t_{k+s+1})$ be the leftmost element in $S^{(k)}$ such that $p_1(\eta_{k+1}) = p_1(\theta_{k+s+1})$. Then, $p_1(\theta_{k+s}) > p_1(\eta_{k+1}) = p_1(\theta_{k+s+1})$. If either $p_2(\eta_{k+1}) > p_2(\theta_{k+1})$ or $p_2(\eta_{k+1}) = p_2(\theta_{k+1})$ and $t_{k+1} = 2$ then it follows that $(\eta_k, j_k) \dashv (\theta_{k+1}, t_{k+1}) \dashv (\eta_{k+1}, j_{k+1})$, a contradiction with the completeness of S'. So we must have either $p_2(\eta_{k+1}) < p_2(\theta_{k+1})$ or $p_2(\eta_{k+1}) = p_2(\theta_{k+1})$ and $t_{k+1} = 1$. Since $p_1(\theta_{k+s}) > p_1(\eta_{k+1}) = p_1(\theta_{k+s+1})$, we may apply (T2) to insert (η_{k+1}, j_{k+1}) in the middle of (θ_{k+s}, t_{k+s}) and $(\theta_{k+s+1}, t_{k+s+1})$. Because either $p_2(\eta_{k+1}) < p_2(\theta_{k+1})$ or $p_2(\eta_{k+1}) = p_2(\theta_{k+1})$ and $t_{k+1} = 1$, it follows that (θ_{k+1}, t_{k+1}) will be deleted and in the chain obtained, (η_{k+1}, j_{k+1}) must be next to (η_k, j_k). Thus, in any case, the chain obtained is complete and commences with $(\eta_1, j_1), \ldots, (\eta_{k+1}, j_{k+1})$. We take this chain to be $S^{(k+1)}$. As $p_1(\eta_m) = 0$, $S^{(m)}$ must coincide with S'.

(iii) Given a chain $S : (\theta_1, t_1), \ldots, (\theta_k, t_k)$. Choose S' to be any complete chain. Similarly as above, we may apply to S' appropriate transformations (T1) and (T2), to "enter" $S : (\theta_1, t_1), \ldots, (\theta_k, t_k)$ consecutively. Notice that, always entering (θ_{i+1}, t_{i+1}), $i \geq 1$, on the right of (θ_i, t_i) for does not delete any of $(\theta_1, t_1), \ldots, (\theta_i, t_i)$ which have been entered previously. □

Example 2.2. Consider the chain $S : (4,1,1), (2,3,2), (2,5,1)$. We try to embed S in a complete chain by using (T1) and (T2). For this, we choose an arbitrary complete chain S', say $S' : (3,0,1), (3,1,1), (2,2,2), (1,2,2), (0,2,2)$ ($S' = U_{3,2}$, see Example 2.1, and manipulate like this:

• Applying (T1) to S' with $(\xi, i) = (4,1,1)$ we obtain from step to step the following sequences, where underline indicates the elements added in every step.

$\underline{(4,1,1)}, (3,0,1), (3,1,1), (2,2,2), (1,2,2), (0,2,2);$

$(4,1,1), (2,2,2), (1,2,2), (0,2,2);$

$(4,1,1), \underline{(3,2,2)}, (2,2,2), (1,2,2), (0,2,2);$

$\underline{(5,0,1)}, \underline{(4,1,2)}, (4,1,1), (3,2,2), (2,2,2), (1,2,2), (0,2,2).$

• Applying (T2) to the last chain with $(\xi, i) = (2,3,2)$ we obtain successively

$(5,0,1), (4,1,2), (4,1,1), (3,2,2), \underline{(2,3,2)}, (2,2,2), (1,2,2), (0,2,2);$

$(5,0,1), (4,1,2), (4,1,1), (3,2,2), \underline{(3,2,1)}, (2,3,2);$

$(5,0,1), (4,1,2), (4,1,1), (3,2,2), (3,2,1), (2,3,2), \underline{(1,3,2)}, \underline{(0,3,2)}.$

• Applying (T2) to the last chain with $(\xi, i) = (2,5,1)$ such that (ξ, i) is to the right of $(2,3,2)$, we obtain

$(5,0,1), (4,1,2), (4,1,1), (3,2,2), (3,2,1), (2,3,2), \underline{(2,5,1)}, (1,3,2), (0,3,2);$

$(5,0,1), (4,1,2), (4,1,1), (3,2,2), (3,2,1), (2,3,2), \underline{(2,3,1)}, \underline{(2,4,1)}, (2,5,1);$

$(5,0,1), (4,1,2), (4,1,1), (3,2,2), (3,2,1), (2,3,2), (2,3,1), (2,4,1), (2,5,1),$
$\underline{(1,6,2)}, \underline{(0,6,2)}.$

The last chain $(5,0,1),\dots,(0,6,2)$ is a complete chain containing S.

As a consequence of Theorem 2.2 we have

Theorem 2.3. *Let A be a two-letter alphabet. Then, we have*

- (i) *There exists a procedure to generate all the finite maximal p-superinfix (s-superinfix) codes over A starting from an arbitrary given finite maximal p-superinfix (s-superinfix, resp.) code.*
- (ii) *There is an algorithm allowing to construct, for every finite p-superinfix (s-superinfix) code X over A, a finite maximal p-superinfix (s-superinfix, resp.) code Y containing X.*

Proof. We treat only the case of p-superinfix codes. The reasonements for the case of s-superinfix codes is similar.

(i) Let X be a given finite maximal p-superinfix code. Compute first $p_L(X)$, which is a complete set. Arrange $p_L(X)$ to become a complete chain S. By Theorem 2.2(ii), every possible complete chain, hence every complete set, can be obtained from S by a finite number of applications of the transformations (T1) and (T2). The inverse images of all such sets with respect to the mapping p_L give all the possible finite maximal p-superinfix codes.

(ii) Let X be a finite p-superinfix code. By Theorem 1.1, $p_L(X)$ is a $\prec$-independent set on U^2. So it can be arranged to become a chain S. By Theorem 2.2(iii), we can construct a complete chain S' containing S. Let T be the complete set corresponding to S'. Put $Y = p_L^{-1}(T)$. Evidently $Y = \pi(O_Y) \cup \pi_L(E_Y)$, Y contains X and $p_L(Y) = T$. Thus, by Theorem 2.1, Y is a finite maximal p-superinfix code. □

Example 2.3. Let $X = \{ab^5a, a^2b^3, ba^4\}$. Since

$$p_L(X) = \{(2,5,1),\ (2,3,2),\ (4,1,1)\}$$

is a $\prec$-independent set on U^2, by Theorem 1.1, X is a p-superinfix code over A. The corresponding chain of $p_L(X)$ is S : $(4,1,1)$, $(2,3,2)$, $(2,5,1)$. By Example 2.2, the sequence S': $(5,0,1)$, $(4,1,2)$, $(4,1,1)$, $(3,2,2)$, $(3,2,1)$, $(2,3,2)$, $(2,3,1)$, $(2,4,1)$, $(2,5,1)$, $(1,6,2)$, $(0,6,2)$ is a complete chain containing S. The corresponding complete set of S' is

$$T = \{(5,0,1),(4,1,2),(4,1,1),(3,2,2),(3,2,1),(2,3,2),\\ (2,3,1),(2,4,1),(2,5,1),(1,6,2),(0,6,2)\}.$$

So $Y = p_L^{-1}(T)$ is a finite maximal p-superinfix code containing X. More explicitly, $Y = \pi(Z) \cup \pi_L(Z')$ with $Z = \{a^5,\ a^4b,\ a^3b^2,\ b^2a^3,\ b^6\}$ and $Z' = \{b^4a^2,\ b^5a^2,\ ab^6\}$.

3. Superinfix Codes

Put $W^2 = \{(\xi, i) \in U^2 \mid p_i(\xi) \geq 2\} \cup \{(1,0,1),(0,1,2)\}$ and $G = A \cup aA^*a \cup bA^*b$. A $\prec$-independent set Z on U^2 is called a *weakly maximal $\prec$-independent* set if $Z \cup \{w\}$ is not a $\prec$-independent set, for all $w \in W^2$.

In order to characterize maximal superinfix codes by means of weakly maximal $\prec$-independent sets on U^2. We need first some lemmas.

Lemma 3.1. *If X is a maximal superinfix code over a finite alphabet A then $p_L(X) = p_F(X)$.*

Proof. Let X be a maximal superinfix code over A. By Theorem 1.1, it follows that $X = \pi(O_X) \cup \pi_{LF}(E_X)$. Therefore, $p_L(X) = p_L(\pi(O_X)) \cup p_L(\pi_{LF}(E_X))$ and $p_F(X) = p_F(\pi(O_X)) \cup p_F(\pi_{LF}(E_X))$. Since $p_L(\pi(O_X)) = p_F(\pi(O_X))$ it suffices to show that $p_L(\pi_{LF}(E_X)) = p_F(\pi_{LF}(E_X))$. Assume $q \in p_L(\pi_{LF}(E_X))$, it means $q = p_L(x)$ for some $x \in \pi_{LF}(E_X)$. Suppose $x = a_i w a_j$ and $x' = a_j w a_i$ with $a_i, a_j \in A, w \in A^*$. Since $p(x') = p(x)$, it follows that $x' \notin \pi(O_X)$. If $x' \notin \pi_{LF}(E_X)$ then $X \cup \{x'\}$ is not a superinfix code because X is a maximal superinfix code. Then either $x' \prec_{spi} z$ or $z \prec_{spi} x'$ for some $z \in X$. By definition of $\prec_{spi}$, it follows that $x \prec_{spi} z$ or $z \prec_{spi} x$, which contradicts the fact that X is a superinfix code. Thus $x' \in \pi_{LF}(E_X)$. Hence $q = p_F(x') \in p_F(\pi_{LF}(E_X))$ and therefore $p_L(\pi_{LF}(E_X)) \subseteq p_F(\pi_{LF}(E_X))$. Similarly we have $p_F(\pi_{LF}(E_X)) \subseteq p_L(\pi_{LF}(E_X))$. So $p_L(\pi_{LF}(E_X)) = p_F(\pi_{LF}(E_X))$ and hence $p_L(X) = p_F(X)$. □

Lemma 3.2. *If X is a superinfix code over A then $E_X \subseteq G$.*

Proof. Let X be a superinfix code over A and let $x \in E_X$ but $x \notin G$. As $x \in E_X$, it follows that there is $y \in O_X$ with $p(y) < p(x)$. Since $x \notin G$, we have $x \in aA^*b \cup bA^*a$, and hence there exists x', $x' \prec_i x$ such that $p(y) \leq p(x')$. Therefore, $y \prec_{spi} x$, i. e. X is not a superinfix code, a contradiction. So $x \in G$ and hence $E_X \subseteq G$. □

Lemma 3.3. *If $X \subseteq G$ then $\pi_L(X) \cap \pi_F(X) = \pi_{LF}(X)$.*

Proof. For any $Y \subseteq A^*$, by definitions of $\pi_L(Y)$, $\pi_F(Y)$ and $\pi_{LF}(Y)$, we have $\pi_{LF}(Y) \subseteq \pi_L(Y)$ and $\pi_{LF}(Y) \subseteq \pi_F(Y)$. Therefore, $\pi_{LF}(Y) \subseteq \pi_L(Y) \cap \pi_F(Y)$. Let $X \subseteq G$. By the above, it suffices to show that $\pi_L(X) \cap \pi_F(X) \subseteq \pi_{LF}(X)$. The assertion is true if for all $g \in G$, we have $\pi_L(g) \cap \pi_F(g) \subseteq \pi_{LF}(g)$. If $g \in A$ then the inclusion is trivial. Suppose $g = aua$ with $u \in A^*$, and let $x \in \pi_L(g) \cap \pi_F(g)$. Then $x = ava$ with $av \in \pi(au)$. We shall prove that $v \in \pi(u)$. Note that for any $w \in A^*$, by definitions of $p(w)$ and $\pi(w)$, we have $t \in \pi(w)$ iff $p(t) = p(w)$. If now $v \notin \pi(u)$ then, by the above, $p(v) \neq p(u)$. Therefore, $p(av) \neq p(au)$. Again by the above, $av \notin \pi(au)$, which contradicts with $av \in \pi(au)$. Thus $v \in \pi(u)$ and hence $x \in \pi_{LF}(g)$. □

Theorem 3.1. *For any subset X of A^+, X is a maximal superinfix code iff $p_L(X) = p_F(X)$ and it is a weakly maximal $\prec$-independent set on U^2, and $\pi(O_X) \cup \pi_{LF}(E_X) = X$.*

Proof. Let X be a maximal superinfix code. By Lemma 3.1, $p_L(X) = p_F(X)$. By Theorem 1.1, $\pi(O_X) \cup \pi_{LF}(E_X) = X$ and $p_L(X)$ is a $\prec$-independent set on U^2. If $p_L(X)$ were not a weakly maximal then $\exists t \in W^2 - p_L(X)$ such that $p_L(X) \cup \{t\}$ is still a $\prec$-independent set. Let $t = (\xi, i)$. According as $t \in \{(1,0,1),(0,1,2)\}$ or

$t \in U^2$ with $p_i(\xi) \geq 2$, we can choose a word u such that $u \in \{a, b\}$, or $p(u) = \xi$ and i is the index of the last and the first letter in u. Thus, in any case, we have $p_L(u) = p_F(u) = t$. Since $t = p_L(u) \notin p_L(X)$, it follows that $u \notin X$. We have $p_L(X \cup \{u\}) = p_L(X) \cup \{t\} = p_F(X) \cup \{t\} = p_F(X \cup \{u\})$. Again by Theorem 1.1, $X \cup \{u\}$ is both a p-superinfix code and a s-superinfix code, i.e. $X \cup \{u\}$ is a superinfix code, a contradiction with the maximality of X.

Conversely, let $p_L(X) = p_F(X)$ and it is a weakly maximal $\prec$-independent set on U^2, and $\pi(O_X) \cup \pi_{LF}(E_X) = X$. By Theorem 1.1, X is a superinfix code. Assume that X is not maximal as a superinfix code. Then, there exists y, $1 \neq y \notin X$ such that $Y = X \cup \{y\}$ is still a superinfix code. We consider the following two cases.

- *Case 1:* $y \in G$. Then $p_L(y) = p_F(y) = q \in W^2$. As X is a superinfix code, by Lemma 3.2, we have $E_X \subseteq G$. By Lemma 3.3, $\pi_L(E_X) \cap \pi_F(E_X) = \pi_{LF}(E_X)$. Therefore, if $q \in p_L(X) = p_F(X)$ then $y \in \pi_L(X) \cap \pi_F(X) \subseteq \pi(O_X) \cup \pi_{LF}(E_X) = X$, a contradiction to $y \notin X$. Thus $q \notin p_L(X)$. Again by Theorem 1.1, $p_L(X \cup \{y\}) = p_L(X) \cup \{q\}$ is still a $\prec$-independent set, which contradicts the weakly maximality of $p_L(X)$.

- *Case 2:* $y \notin G$. Then, since $y \neq 1$, either $y \in \{ab, ba\}$ or $|y| > 2$. If $y \in \{ab, ba\}$ then, by Y is a superinfix code, $Y \subseteq \{a^m, ab, ba, b^n \mid \text{for some } m, n \geq 2\}$. Therefore, $p_L(X) = p_L(Y - \{y\})$ is not a weakly maximal $\prec$-independent set on U^2, a contradiction. Thus $|y| > 2$, and hence either $|y|_a \geq 2$ or $|y|_b \geq 2$. Without loss of generality we may assume that $|y|_a \geq 2$. Then, there exists $y' \in \pi(y)$ such that $y' = aua$ with $u \in A^+$, i.e. $y' \in G$. Suppose that $X \cup \{y'\}$ is not a superinfix code. By definition, there is $x \in X$ such that either $x \prec_{spi} y'$ or $y' \prec_{spi} x$. If $x \prec_{spi} y'$ then $p(x) < p(y') = p(y)$. Consequently $y \in E_Y$, and hence $y \in G$ by Lemma 3.2, a contradiction with $y \notin G$. If $y' \prec_{spi} x$ then, evident, $y \prec_{spi} x$, which contradicts the Y is a superinfix code. Thus, $X \cup \{y'\}$ is also a superinfix code. By Case 1, $p_L(X \cup \{y'\}) = p_L(X) \cup \{q'\}$ with $q' \in W^2 - p_L(X)$, is a $\prec$-independent set, a contradiction.

So X must be maximal as a superinfix code. □

Example 3.1. Let $m, n \geq 2$ and $k \geq 1$. Consider the sets $X = \{a^m, ab, ba, b^n\}$, $Y = \{a^k, b\}$ and $Z = \{a, b^k\}$ over A. We have evidently $E_X = E_Y = E_Z = \emptyset$. A simple verification leads to $\pi(O_X) \cup \pi_{LF}(E_X) = X$, $\pi(O_Y) \cup \pi_{LF}(E_Y) = Y$, $\pi(O_Z) \cup \pi_{LF}(E_Z) = Z$, and

$$p_L(X) = p_F(X) = \{(m, 0, 1), (1, 1, 2), (1, 1, 1), (0, n, 2)\},$$
$$p_L(Y) = p_F(Y) = \{(k, 0, 1), (0, 1, 2)\},$$
$$p_L(Z) = p_F(Z) = \{(1, 0, 1), (0, k, 2)\}.$$

It is easy to check that $p_L(X)$, $p_L(Y)$ and $p_L(Z)$ are weakly maximal $\prec$-independent sets on U^2. By virtue of Theorem 3.1, we may conclude that X, Y and Z are maximal superinfix codes over A.

In the rest of this section, we shall give a procedure to generate all the finite maximal superinfix codes, starting from an arbitrary given full uniform code, and an algorithm to embed a finite superinfix code in a finite maximal one. For this, we need the following lemma.

Lemma 3.4. *Let* $S : (u_1, u_2, s), (v_{11}, v_{12}, i_1), \ldots, (v_{k1}, v_{k2}, i_k), (w_1, w_2, t),\ k \geq 1$ *be a full chain with* $(u_1, u_2, s), (w_1, w_2, t) \in W^2$ *and* $(v_{j1}, v_{j2}, i_j) \in U^2 - W^2$ *for all* $j \in \{1, \ldots, k\}$. *Then* S *has one of the following forms*

(i) $S^0_{m,n} : (m, 0, 1), (m-1, 1, 2), \ldots, (1, 1, 2), (1, 1, 1), (1, 2, 1), \ldots, (1, n-1, 1), (0, n, 2)$, *with* $m, n \geq 2$.
(ii) $S^0_{m,1} : (m, 0, 1), (m-1, 1, 2), \ldots, (m-i, 1, 2), \ldots, (1, 1, 2), (0, 1, 2)$, *with* $m \geq 2$.
(iii) $S^0_{1,n} : (1, 0, 1), (1, 1, 1), \ldots, (1, n-i, 1), \ldots, (1, n-1, 1), (0, n, 2)$, *with* $n \geq 2$.
(iv) $S^1_{m,n} : (m, 0, 1), (m-1, 1, 2), \ldots, (n, 1, 2), (n, 1, 1)$, *with* $m > n \geq 2$.
(v) $S^2_{m,n} : (1, m, 2), (1, m, 1), (1, m+1, 1), \ldots, (1, n-1, 1), (0, n, 2)$, *with* $n > m \geq 2$.

Proof. Since $s, t \in \{1, 2\}$, only the following cases are possible.

- *Case 1:* $s = t = 1$. The full chain S has the form

$$S : (u_1, u_2, 1), \ldots, (v_{j1}, 1, 2), \ldots, (w_1, w_2, 1),\ 1 \leq j \leq k,$$

with $0 \leq u_2 \leq 1 \leq w_2$ and $u_1 \geq v_{j1} \geq w_1 \geq 2$. If $w_2 > 1$ then, by S is a chain, $v_{k1} = w_1$, but we have $(v_{k1}, 1, 2) \dashv (w_1, w_2, 2) \dashv (w_1, w_2, 1)$, which contradicts the fullness of S. Thus $w_2 = 1$. Also, we have $u_2 = 0$ by S is a chain. Therefore, S has the form

$$S : (u_1, 0, 1), \ldots, (v_{j1}, 1, 2), \ldots, (w_1, 1, 1).$$

Since S is a full chain, we must have $S = S^1_{m,n}$ with $m > n \geq 2$.

- *Case 2:* $s = 1, t = 2$. If $u_1, w_2 \geq 2$ then S has the form

$$S : (u_1, u_2, 1), \ldots, (v_{i1}, 1, 2), \ldots, (1, v_{j2}, 1), \ldots, (w_1, w_2, 2),$$

with $0 \leq u_2 \leq 1$ and $1 \geq w_1 \geq 0$. By definition of S, it is easy to verify that $u_2 = w_1 = 0$. Therefore, since S is a full chain, we must have $S = S^0_{m,n}$ with $m, n \geq 2$. If $u_1 \geq 2$ and $(w_1, w_2, 2) = (0, 1, 2)$ then S has the form

$$S : (u_1, u_2, 1), \ldots, (v_{i1}, 1, 2), \ldots, (0, 1, 2),$$

with $0 \leq u_2 \leq 1$. Clearly, $u_2 = 0$. As S is a full chain, it follows that $S = S^0_{m,1}$ with $m \geq 2$. Similarly, if $(u_1, u_2, 1) = (1, 0, 1)$ and $w_2 \geq 2$ then $S = S^0_{1,n}$ with $n \geq 2$.

- *Case 3:* $s = 2, t = 1$. Then, we have $u_1 \geq w_2 \geq 2$ and $w_2 \geq u_2 \geq 2$. Since every element in $U^2 - W^2$ has the form either $(r, 1, 2)$ or $(1, r', 1)$, therefore, we cannot insert between $(u_1, u_2, 2)$ and $(w_1, w_2, 1)$ any elements in $U^2 - W^2$.

- *Case 4:* $s = t = 2$. In the same way as in Case 1, we obtain $S = S^2_{m,n}$. ☐

To every subset T of U^2, we set

$$E_T = \{(\xi, i) \in T \mid \exists (\eta, j) \in T : \eta < \xi\};\quad O_T = T - E_T.$$

The following result shows relationship between complete chains and maximal superinfix codes.

Theorem 3.2. *Every complete chain S always contains a weakly maximal $\prec$-independent set T on U^2 such that $S_R = p^{-1}(O'_T) \cup (p_L^{-1}(E_T) \cap p_F^{-1}(E_T))$ is a maximal superinfix code over A, where $O'_T = \{\xi \mid (\xi, i) \in O_T\}$.*

Proof. Since S is a complete chain, only the following cases possible.

- *Case 1: $S = S^0$, with $S^0 : (1,0,1),(0,1,2)$.* Clearly, $T = S$ is a weakly maximal $\prec$-independent set, and $S_R = A$ is a maximal superinfix code over A.
- *Case 2: S has one of the forms $S^0_{m,n}$, $S^0_{m,1}$ and $S^0_{1,n}$.* The assertion is obvious by Example 3.1.
- *Case 3: $S : (m,0,1),\dots,(0,n,2)$ with $m,n \geq 2$, and S has not one of the forms in Cases 1 and 2.* Put $T = S - \{(\theta,t) \mid (\theta,t) \in U^2 - W^2\}$. Then $T \neq \emptyset$ and hence $S_R \neq \emptyset$. By definition, $O_{S_R} = p^{-1}(O'_T) = \pi(O_{S_R})$ and

$$p_L^{-1}(E_T) \cap p_F^{-1}(E_T) = \pi_L(E_{S_R}) \cap \pi_F(E_{S_R}).$$

Therefore, by Lemma 3.3, $S_R = \pi(O_{S_R}) \cup \pi_{LF}(E_{S_R})$. If $T = S$ then, evident, $p_L(S_R) = p_F(S_R) = T$ and it is a weakly maximal $\prec$-independent set. Thus, by Theorem 3.1. S_R is a maximal superinfix code. Suppose now $T \subset S$. Then, always there exist sub-chains of S such that every sub-chain S' of S, S' has the form $(\xi,i),(\theta_1,t_1),\dots,(\theta_k,t_k),(\eta,j)$ with $k \geq 1$, $(\xi,i),(\eta,j) \in W^2$ and (θ_1,t_1), $\dots$, $(\theta_k,t_k) \in U^2 - W^2$. Clearly, S' has not one of the forms S^0, $S^0_{m,n}$, $S^0_{m,1}$ and $S^0_{1,n}$. By Lemma 3.4, we may check that $\nexists(\theta',t') \in W^2$ such that $(\xi,i) \dashv (\theta',t') \dashv (\eta,j)$. Therefore, T must be a weakly maximal $\prec$-independent set on U^2. A direct verification shows that $T \subseteq p_L(S_R) = p_F(S_R) \subseteq S$. Hence, $p_L(S_R) = p_F(S_R)$ and it is a weakly maximal $\prec$-independent set on U^2. Again by Theorem 3.1, S_R must be a maximal superinfix code. □

Recall that the set A^k, $k \geq 1$, is called a *full uniform code.* As a consequence of Theorems 2.2 and 3.2 we obtain

Theorem 3.3. *Let A be a two-letter alphabet. Then, we have*

(i) *There exists a procedure to generate all the finite maximal superinfix codes over A starting from an arbitrary given full uniform code.*
(ii) *There is an algorithm allowing to construct, for every finite superinfix code X over A, a finite maximal superinfix code Y containing X.*

Proof. (i) Let $X = A^k$ for $k \geq 1$. Compute first $p_L(X)$, which is a complete set. Arrange $p_L(X)$ to become a complete chain S. By Theorem 2.2(ii), every possible complete chain S', can be obtained from S by a finite number of applications of the transformations (T1) and (T2). According to Theorem 3.2, S'_R is a finite maximal superinfix code. By this, we can obtain all the possible finite maximal superinfix codes.

(ii) Let X be a finite superinfix code. Then, by Theorem 1.1, $p_L(X)$ is a $\prec$-independent set on U^2. So it can be arranged to become a chain S. By

Theorem 2.2(iii), we can construct a complete chain S' containing S. Thus, by Theorem 3.2, S'_R is a finite maximal superinfix code which contains X. □

Example 3.2. Let $X = \{ab^5a, ba^2b^2, aba^3\}$. Since

$$p_L(X) = \{(2,5,1), (2,3,2), (4,1,1)\} \ = p_F(X),$$

and it is a $\prec$-independent set on U^2, by Theorem 1.1, X is a superinfix code over A. The corresponding chain of $p_L(X)$ is S : $(4,1,1)$, $(2,3,2)$, $(2,5,1)$. As has been shown in Example 2.2, the sequence S': $(5,0,1)$, $(4,1,2)$, $(4,1,1)$, $(3,2,2)$, $(3,2,1)$, $(2,3,2)$, $(2,3,1)$, $(2,4,1)$, $(2,5,1)$, $(1,6,2)$, $(0,6,2)$ is a complete chain containing S. The weakly maximal $\prec$-independent set on U^2 is

$$T = \{(5,0,1), (4,1,1), (3,2,2), (3,2,1), (2,3,2), \\ (2,3,1), (2,4,1), (2,5,1), (1,6,2), (0,6,2)\}.$$

So $S'_R = p^{-1}(O'_T) \cup (p_L^{-1}(E_T) \cap p_F^{-1}(E_T))$ is a finite maximal superinfix code containing X, where $O'_T = \{(5,0)$, $(4,1)$, $(3,2)$, $(2,3)$, $(0,6)\}$ and $E_T = \{(2,4,1)$, $(2,5,1)$, $(1,6,2)\}$. More explicitly, $S'_R = \pi(Y) \cup \pi_{LF}(Z)$ with $Y = \{a^5$, a^4b, a^3b^2, b^2a^3, $b^6\}$ and $Z = \{ab^4a$, ab^5a, $bab^5\}$.

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