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WELL-POSEDNESS OF A COMMON FIXED POINT PROBLEM FOR WEAKLY TANGENTIAL MAPPINGS

MOHAMED AKKOUCHI

Abstract. In this paper, we introduce the concept of weakly tangential mappings. This concept is a generalization of the property (E.A.) introduced by Aamri and El Moutawakil in a paper published in 2002. We study the well-posedness of the common fixed point problem for two weakly tangential and occasionally weakly compatible self-mappings of a metric space (X, d) which satisfy a variant of a contractive condition considered by Ćirić in a paper published in 2005. The results established here provide natural extensions and continuations to some results obtained by Ćirić, Sharma and Yuel and Guay and Singh.

1. Introduction

The notion of contractive mapping has been introduced by Banach in [5]. In the last four decades many generalizations of this concept have appeared. The connection between them has been studied in different papers (see [15], [20], [25] and [28]).

Browder and Petryshyn (see [7]) defined the following notion.

Definition 1.1. A selfmapping T on a metric space (X, d) is said to be asymptotically regular at a point x in X if

$$\lim_{n\to\infty} d(T^n x, T^n T x) = 0, \tag{1.1}$$

where $T^n x$ denotes the n-th iterate of T at x.

Almost all of the contractive conditions ensuring the existence of fixed points and generalizing the Banach principle imply asymptotic regularity of the mappings under consideration. So the investigation of asymptotically regular maps plays an important role in fixed point theory.

Ćirić (see [10]) pointed out that Sharma and Yuel [27] and Guay and Singh [11] were among the first who used the concept of asymptotic regularity to prove fixed point theorems.

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In [10], Ćirić generalized the results of Sharma and Yuel [27] and Guay and Singh [11] and studied a wide class of asymptotically regular mappings which possess fixed points in complete metric spaces and proved the following theorem.

Theorem 1.2. *Let $\mathbb{R}^+$ be the set of nonnegative reals and let $F_i : \mathbb{R}^+ \to \mathbb{R}^+$ be functions such that $F_i(0) = 0$ and that F_i is continuous at 0 for $i = 1, 2$.*

Let (X, d) be a complete metric space and let T be a selfmapping on X satisfying the following condition:

$$d(Tx, Ty) \leq a_1 F_1(\min\{d(x, Tx), d(y, Ty)\}) + a_2 F_2(d(x, Tx).d(y, Ty))+$$

$$+a_3 d(x, y) + a_4[d(x, Tx) + d(y, Ty)] + a_5[d(x, Ty) + d(y, Tx)] \qquad (1.2)$$

for all x, y in X, where $a_i = a_i(x, y)$ $(i = 1, 2, 3, 4, 5)$ are nonnegative functions for which there exist three constants $K > 0$ and $\lambda_1, \lambda_2 \in (0, 1)$, such that the inequalities

$$a_1(x, y), a_2(x, y) \leq K, \qquad (1.3)$$

$$a_4(x, y) + a_5(x, y) \leq \lambda_1, \qquad (1.4)$$

$$a_3(x, y) + 2a_5(x, y) \leq \lambda_2 \qquad (1.5)$$

are satisfied for all x, y in X.

If T is asymptotically regular at some x_0 in X, then T has a unique fixed point in X and is continuous at this point.

Let (X, d) be a metric space and let $T, S : X \to X$ be two self-mappings of X. For these mappings, we consider the condition

$$d(Tx, Ty) \leq a_0 F(d(Sx, Tx), d(Sy, Ty))$$

$$+a_1 d(Sx, Sy) + a_2[d(Sx, Tx) + d(Sy, Ty)] + a_3[d(Sx, Ty) + d(Sy, Tx)] \qquad (1.6)$$

for all x, y in X, where $a_i = a_i(x, y)$ $(i = 0, 1, 2, 3)$ are non-negative functions and $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying some conditions similar to the ones considered in the theorem above.

The aim of this paper is threefold.

First, we introduce the concept of weakly tangential mappings (see Definition 3.1). This concept is a generalization of the property (E.A.) introduced by Aamri and El Moutawakil in a paper published in 2002 (see [1]). Our concept is also a generalization of the concept of asymptotically regular mapping introduced by Browder and Petryshyn (see [7]). An example is given in the third section to show that the concept of weakly tangential mappings is actually different from those two concepts.

Second, we investigate conditions on the functions involved in the inequality (1.6) ensuring the existence and uniqueness of a common fixed point for two weakly tangential and occasionally weakly compatible self-mappings S, T of a metric space (X, d). This study is the subject of the third section. The main result of that section is Theorem 3.2.

Third, in the sixth section, under the considerations of Theorem 3.2, we study the well-posedness of the common fixed point problem of the two weakly tangential and occasionally weakly compatible self-mappings S, T of a metric space (X, d). The main result of the sixth section is Theorem 6.2.

In the second section we recall some definitions used in fixed point theory and collect some preliminaries.

In Section 4 we gather some corollaries and consequences of our main results.

In Section 5 we provide a related general result in compact metric spaces.

Our main results extend the results of Ćirić [10], Sharma and Yuel [27] and Guay and Singh [11] to the general setting of two mappings and provide some complements to them.

We recall that the notion of well-posedness is usually considered only for one self-mapping.

Definition 1.3. Let (X, d) be a metric space and let $T : (X, d) \to (X, d)$ be a mapping. The fixed point problem of T is said to be well posed if

(a) T has a unique fixed point z in X;

(b) for any sequence $\{x_n\}$ of points in X such that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ we have $\lim_{n\to\infty} d(x_n, z) = 0$.

The notion of well-posednes of a fixed point problem has evoked much interest to several mathematicians, for example F.S. De Blassi and J. Myjak (see [6]), S. Reich and A. J. Zaslavski (see [24]), B.K. Lahiri and P. Das (see [16]), V. Popa (see [22] and [23]) and M. Akkouchi and V. Popa (see [2] and [3]).

2. Definitions and preliminaries

Before introducing the concept of weakly tangential mappings, we make a brief recall of some concepts used in metric fixed point theory.

Let (X, d) be a metric space and let S, T be two self-mappings of X. In [12], Jungck defined S and T to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$. This concept has been used to prove existence theorems in common fixed point theory. The study on common fixed point theory for noncompatible mappings is also interesting. A study in this direction has been initiated by Pant [17], [18], [19].

In 2002, Aamri and Moutawakil [1] introduced a generalization of the concept of noncompatible mappings.

Definition 2.1. Let S and T be two self mappings of a metric space (X,d). We say that S and T satisfy property (E.A.) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$.

Remark 2.1. It is clear that two self-mappings of a metric space (X,d) will be noncompatible if there exists at least one sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$ but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either non-zero or does not exist.

Therefore, two noncompatible self-mappings of a metric space (X,d) satisfy property (E.A.).

Definition 2.2. [13]. Two self mappings S and T of a metric space (X,d) are said to be weakly compatible if $Tu = Su$, for some $u \in X$, then $STu = TSu$.

Two compatible mappings are weakly compatible.

Recently, Al-Thagafi and Naseer Shahzad [4] introduced the concept of occasionally weakly compatible mappings.

Definition 2.3. Let X be a nonempty set and let T, S be self-mappings on X.
A point $x \in X$ is called a coincidence point of T and S if $Tx = Sx$.
A point $w \in X$ is called a point of coincidence of T and S if there exists a coincidence point $x \in X$ of T and S such that $w = Tx = Sx$.

Definition 2.4. Two self-maps T and S of a nonempty set X are called occasionally weakly compatible maps (shortly owc) [4] if there exists a point x in X which is a coincidence point for T and S at which T and S commute.
We say also that the pair (T,S) is occasionally weakly compatible.

Remark 2.2. Two weakly compatible mappings having coincidence points are occasionally weakly compatible. In [4] it was shown that the converse is not true.

An arbitrary mapping $T : X \to X$ and id, the identity map of X, are weakly compatible, while T and id are occasionally weakly compatible if and only if T has a fixed point in X. Thus, weak compatibility does not imply occasionally weak compatibility

The following lemma will be used later.

Lemma 2.5 (Jungck and Rhoades [14]). *Let X be a nonempty set and let T and S be two occasionally weakly compatible self-mappings of X. If T and S have a unique point of coincidence $w = Tx = Sx$, then w is the unique common fixed point of T and S.*

3. A common fixed point problem

We start by introducing the notion of weakly tangential mappings.

Definition 3.1. Let (X, d) be a metric space and let $T, S : (X, d) \to (X, d)$ be two self-mappings. S and T are said to be weakly tangential mappings if there exists a sequence $\{x_n\}$ of points in X such that

$$\lim_{n\to\infty} d(Sx_n, Tx_n) = 0.$$

Remark 3.1.

(1) Let $T : X \to X$ be a self-mapping of a metric space (X, d). For each point $x \in X$, we set $x_n := T^n x$ for every non-negative integer n. We denote by I the identity mapping. We observe that if T is asymptotically regular at a point x, then the mappings T and I are weakly tangential.

(2) If the mappings T and S satisfy property (E.A.), then S and T are weakly tangential.

Thus, the notion of weakly tangential mappings generalizes and unifies the property (E.A.) for two mappings and the notion of asymptotic regularity of one mapping.

To show that the concept of weakly tangential mappings is different from property E.A., we give below an example of two mappings which are weakly tangential without satisfying property (E.A.).

Example 3.1. Let $X := [1, +\infty)$ be endowed with its usual metric. Let $T, S : X \to X$ be defined by

$$Tx = x + \frac{1}{2x+1} \quad \text{and} \quad Sx = x + \frac{1}{x} \quad \forall x \in [1, +\infty).$$

For all $x \in [1, +\infty)$ we have

$$|Tx - Sx| = \frac{x+1}{x(2x+1)},$$

which implies that $\lim_{n\to\infty} |Tx_n - Sx_n| = 0$ for all sequences $\{x_n\}$ converging to $+\infty$. Thus, the mappings T and S are weakly tangential.

Suppose now that there exist a number $t \in [1, +\infty)$ and a sequence $\{x_n\}$ in $[1, +\infty)$ such that

$$\lim_{n\to\infty} x_n + \frac{1}{2x_n+1} = \lim_{n\to\infty} x_n + \frac{1}{x_n} = t. \tag{$*$}$$

$(*)$ shows that the sequence $\{x_n\}$ is bounded. Thus we can find a subsequence $\{x_{\phi(n)}\}$ which converges to some $x \in [1, +\infty)$. By $(*)$ and continuity of the mappings involved in $(*)$, we obtain the identity

$$x + \frac{1}{2x+1} = x + \frac{1}{x} = t,$$

which implies that $2x + 1 = x$. Hence, we get $x = -1$, which is a contradiction. We deduce that the mappings T and S do not satisfy property (E.A.).

Our first main result is the following.

Theorem 3.2. *Let* $\mathbb{R}^+$ *be the set of nonnegative reals and let* $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *be a continuous function such that* $F(t, 0) = 0 = F(0, t)$ *for all* $t \in \mathbb{R}^+$.

Let (X, d) *be a complete metric space and let* T, S *be two self-mappings of* X *satisfying the condition*

$$d(Tx, Ty) \leq a_0 F(d(Sx, Tx), d(Sy, Ty))$$

$$+a_1 d(Sx, Sy) + a_2[d(Sx, Tx) + d(Sy, Ty)] + a_3[d(Sx, Ty) + d(Sy, Tx)] \quad (3.1)$$

for all x, y *in* X, *where* $a_i = a_i(x, y)$ $(i = 0, 1, 2, 3)$ *are nonnegative functions for which there exist three constants* $K > 0$ *and* $\lambda_1, \lambda_2 \in (0, 1)$, *such that the inequalities*

$$a_0(x, y) \leq K, \quad (3.2)$$

$$a_2(x, y) + a_3(x, y) \leq \lambda_1, \quad (3.3)$$

$$a_1(x, y) + 2a_3(x, y) \leq \lambda_2 \quad (3.4)$$

are satisfied for all x, y *in* X.

We suppose that

(A1) *the mappings* S *and* T *are weakly tangential and occasionally weakly compatible.*

(A2) $S(X)$ *is a complete subspace of* X.

Then, the mappings T *and* S *have a unique common fixed point in* X. *Moreover, if* S *is continuous at the unique common fixed point, then* T *is continuous at the unique common fixed point.*

Proof. 1) Since T and S are weakly tangential then there exists a sequence in X such that

$$\lim_{n \to \infty} d(Sx_n, Tx_n) = 0. \quad (3.5)$$

For each n we set $y_n := T(x_n)$ and $z_n := S(x_n)$. We shall prove that both $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences. By virtue of (3.5) it suffices to show that $\{y_n\}$ is a Cauchy sequence.

To simplify notations we set

$$d_n := d(Sx_n, Tx_n). \quad (3.6)$$

By using the inequality (3.1) we have

$$\begin{aligned} d(y_n, y_m) =& d(Tx_n, Tx_m) \\ &\leq a_0 F(d_n, d_m) + a_1 d(Sx_n, Sx_m) + a_2(d_n + d_m) \\ &\quad + a_3[d(Sx_n, Tx_m) + d(Sx_m, Tx_n)], \end{aligned}$$

where $a_i = a_i(x_n, x_m)$ for $i = 0, 1, 2, 3$.

Using the triangle inequality we get

$$d(y_n, y_m) \leq (a_1 + 2a_3)d(y_n, y_m) + (a_2 + a_3)(d_n + d_m) + a_0 F(d_n, d_m).$$

Hence, because of (3.2), (3.3) and (3.4) we obtain

$$(1 - \lambda_2)d(y_n, y_m) \leq \lambda_1(d_n + d_m) + KF(d_n, d_m). \quad (3.7)$$

Since $\lim_{n\to\infty} d(Sx_n, Tx_n) = 0$ and F is continuous at $(0, 0)$, by taking the limit as n, m tend to infinity we obtain

$$(1 - \lambda_2) \lim_{n,\, m\to\infty} d(y_n, y_m) = 0, \tag{3.8}$$

which implies that $\{y_n\}$ is a Cauchy sequence.

As $S(X)$ is complete, the sequences $\{y_n\}$ and $\{z_n\}$ are convergent to a common limit y in $S(X)$. So, there exists $v \in X$ such that $y = Sv$. Thus, we have obtained

$$Sv = y = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n. \tag{3.9}$$

Now we show that y is a common fixed point of T and S. We start by proving that $Tv = Sv$. Let us suppose that $d(Sv, Tv) > 0$. Then, by using the inequality (3.1) we get

$$\begin{aligned} d(Tx_n, Tv) \;\; \leq \;\; & a_0 F(d_n, d(Sv, Tv)) + a_1 d(Sx_n, Sv) + a_2[d_n + d(Sv, Tv)] \\ & + a_3[d(Sx_n, Tv) + d(Sv, Tx_n)], \end{aligned}$$

where $a_i = a_i(x_n, v)$ for $i = 1, 2, 3$.

Using the triangle inequality and the above inequality we get

$$\begin{aligned} d(Tx_n, Tv) \;\; \leq \;\; & KF(d_n, d(Su, Tv)) + a_1 d(Sx_n, Sv) + a_2[d_n + d(Sv, Tv)] \\ & + a_3[d(Sx_n, Sv) + d(Sv, Tv) + d(Sv, Sx_n) + d(Sx_n, Tx_n)]. \end{aligned}$$

We deduce that

$$\begin{aligned} d(Tx_n, Tv) \;\; \leq \;\; & KF(d_n, d(Su, Tv)) + [a_1 + 2a_3] d(Sx_n, Sv) \\ & + [a_2 + a_3] d_n + [a_2 + a_3] d(Sv, Tv) \\ \leq \;\; & KF(d_n, d(Su, Tv)) + \lambda_2 d(Sx_n, Sv) + \lambda_1 d_n + \lambda_1 d(Sv, Tv). \end{aligned}$$

Taking the limit and using the properties of F we get

$$d(Sv, Tv) \leq \lambda_1 d(Sv, Tv) < d(Sv, Tv),$$

a contradiction. Therefore $d(Sv, Tv) = 0$, that is $Tv = Sv$.

Hence $y = Tv = Sv$ is a point of coincidence of S and T.

We prove now that y is the unique point of coincidence. Suppose that $z = Tu = Su$ is an other point of coincidence. Then, by (3.4) we obtain

$$\begin{aligned} d(y, z) \;\; = \;\; & d(Tv, Tu) \\ \leq \;\; & a_1(y, z) d(Sv, Su) + a_3(y, z)[d(Sv, Tu) + d(Su, Tv)] \\ = \;\; & [a_1(y, z) + 2a_3(y, z)] d(y, z) \\ \leq \;\; & \lambda_2 d(y, z), \end{aligned}$$

which implies by (3.4) that $d(y, z) = 0$. Hence, we get $y = Sv = Tv = Su = Tu = z$. Thus, y is the unique point of coincidence of the mappings S and T.

Since S and T are occasionally weakly compatible, we conclude by Lemma 2.5 (see [14]) that y is the unique common fixed point of the mappings S and T.

3) Suppose that S is continuous at the common fixed point y of S and T. To prove that T is continuous at y, suppose that $u_n \to y = Ty$. Then, by (3.1),

$$\begin{aligned} d(Tu_n, y) =& d(Tu_n, Ty) \\ \leq& a_0 F[(d(Su_n, Tu_n), 0)] \\ &+ a_1 d(Su_n, y) + a_2 d(Su_n, Tu_n) + a_3[d(Su_n, y) + d(Tu_n, y)] \\ =& (a_1 + a_2 + a_3) d(Su_n, y) + (a_2 + a_3) d(Tu_n, y), \end{aligned}$$

where $a_i = a_i(u_n, y)$ for $i = 1, 2, 3$.

Hence, using (3.3) and (3.4) we get

$$(1 - \lambda_1) d(Tu_n, y) \leq (\lambda_1 + \lambda_2) d(Su_n, y). \tag{3.10}$$

Letting n go to infinity and using continuity of S at y we obtain

$$(1 - \lambda_1) \limsup_n d(Tu_n, y) \leq 0,$$

which implies that $\lim_{n\to\infty} Tu_n = y$. This completes the proof. □

4. Consequences and Applications

We have the following corollaries:

Corollary 4.1. *Let $\mathbb{R}^+$ be the set of nonnegative reals and let $F_i : \mathbb{R}^+ \to \mathbb{R}^+$ be functions such that $F_i(0) = 0$ and that F_i is continuous at 0 for $i = 1, 2$.*

Let (X, d) be a metric space and let T, S be two self-mappings of X satisfying the condition

$$d(Tx, Ty) \leq b_1 F_1(\min\{d(Sx, Tx), d(Sy, Ty)\}) + b_2 F_2(d(Sx, Tx).d(Sy, Ty)) +$$

$$+ b_3 d(Sx, Sy) + b_4[d(Sx, Tx) + d(Sy, Ty)] + b_5[d(Sx, Ty) + d(Sy, Tx)] \tag{4.1}$$

for all x, y in X, where $b_i = b_i(x, y)$ $(i = 1, 2, 3, 4, 5)$ are nonnegative functions for which there exist three constants $K > 0$ and $\lambda_1, \lambda_2 \in (0, 1)$, such that the inequalities

$$b_1(x, y), b_2(x, y) \leq K, \tag{4.2}$$

$$b_4(x, y) + b_5(x, y) \leq \lambda_1, \tag{4.3}$$

$$b_3(x, y) + 2b_5(x, y) \leq \lambda_2 \tag{4.4}$$

are satisfied for all x, y in X.

We suppose that

(A1) *the mappings S and T are weakly tangential and occasionally weakly compatible.*

(A2) *$S(X)$ is a complete subspace of X.*

Then, the mappings T and S have a unique common fixed point in X. Moreover, if S is continuous at the unique common fixed point, then T is continuous at the unique common fixed point.

The proof follows from Theorem 3.2 by considering the functions

$$F(s,t) := F_1(\min\{s,t\}) + F_2(st),$$

$$a_0(x,y) := \max\{b_1(x,y), b_2(x,y)\}, \quad \text{and} \quad a_1 := b_3,\ a_2 := b_4,\ a_3 := b_5.$$

Corollary 4.2. *Let $\alpha \geq 0$ and $\beta \in [0,1)$.*

Let (X,d) be a metric space and let T,S be two self-mappings of X satisfying the condition

$$d(Tx,Ty) \leq \alpha \frac{\min\{d(Sx,Tx), d(Sy,Ty) + d(Sx,Tx)d(Sy,Ty)\}}{1 + d(x,y)} + \beta d(Sx,Sy) \tag{4.5}$$

for all x,y in X.

We suppose that

(A1) *the mappings S and T are weakly tangential and occasionally weakly compatible.*

(A2) *$S(X)$ is a complete subspace of X.*

Then, the mappings T and S have a unique common fixed point in X. Moreover, if S is continuous at the unique common fixed point, then T is continuous at the unique common fixed point.

The proof follows from Theorem 3.2, by considering the functions

$$F(s,t) := \alpha[(\min\{s,t\}) + st],$$

$$a_0(x,y) := \frac{1}{1 + d(x,y)}, \quad \text{and} \quad a_1 := \beta,\ a_2 := 0,\ a_3 := 0.$$

Beside these considerations, we can take $K = 1$, $\lambda_1 = \lambda_2 := \beta$.

We observe that the contractive condition (4.5) is more general than the one considered by Sharma and Yuel in [27].

Corollary 4.3. *Let (X,d) be a metric space and let T,S be two self-mappings of X satisfying the condition*

$$d(Tx,Ty) \leq pd(Sx,Sy) + q[d(Sx,Tx) + d(Sy,Ty)] + r[d(Sx,Ty) + d(Sy,Tx)] \tag{4.6}$$

for all $x,y \in X$, where p,q and r are fixed nonnegative real numbers such that $q + r < 1$ and $p + 2r < 1$.

We suppose that

(A1) *the mappings S and T are weakly tangential and occasionally weakly compatible.*

(A2) *$S(X)$ is a complete subspace of X.*

Then, the mappings T and S have a unique common fixed point in X. Moreover, if S is continuous at the unique common fixed point, then T is continuous at the unique common fixed point.

The proof follows from 3.2 by considering the functions

$$F(s,t) := 0,\ a_0(x,y) := 0, \quad \text{and} \quad a_1 := p,\ a_2 := q,\ a_3 := r.$$

Beside these considerations, we can take $K = 0$, $\lambda_1 = q + r$ and $\lambda_2 := p + 2r$.

(4.6) is connected to the contractive condition, introduced and considered by Guay and Singh in [11].

5. General results in compact metric spaces

In the case where the metric space (X, d) is compact we have the following general result.

Theorem 5.1. *Let $\mathbb{R}^+$ be the set of nonnegative reals and let $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that $F(t, 0) = 0 = F(0, t)$ for all $t \in \mathbb{R}^+$.*

Let (X, d) be a compact metric space and let T, S be two self-mappings of X satisfying the condition

$$d(Tx, Ty) \leq a_0 F(d(Sx, Tx), d(Sy, Ty))$$

$$+a_1 d(Sx, Sy) + a_2[d(Sx, Tx) + d(Sy, Ty)] + a_3[d(Sx, Ty) + d(Sy, Tx)] \quad (5.1)$$

for all x, y in X, where $a_i = a_i(x, y)$ $(i = 0, 1, 2, 3)$ are nonnegative functions for which there exist three constants $K > 0$ and $\lambda_1, \lambda_2 \in (0, 1)$, such that the following inequalities:

$$a_0(x, y) \leq K, \quad (5.2)$$

$$a_2(x, y) + a_3(x, y) \leq \lambda_1, \quad (5.3)$$

$$a_1(x, y) + 2a_3(x, y) \leq \lambda_2 \quad (5.4)$$

are satisfied for all x, y in X.

We suppose that

(A1) *the mappings S and T are weakly tangential and occasionally weakly compatible.*

(A2) *$S(X)$ is a closed subspace of X.*

Then

(i) *T and S have a unique common fixed point in X.*

(ii) *If S is continuous at the common fixed point, then T is continuous at the common fixed point.*

Proof. Since T and S are weakly tangential there exists a sequence in X such that

$$\lim_{n\to\infty} d(Sx_n, Tx_n) = 0.$$

As before, for each n we set $y_n := T(x_n)$ and $z_n := S(x_n)$. As in the proof of Theorem 3.2 one can prove that both $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences. Since the set $S(X)$ is compact, the sequence $\{z_n\}$ contains a subsequence which converges to a point x, and we conclude that

$$\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} S(x_n) = x.$$

The rest of proof is based on arguments similar to the ones used in the proof of Theorem 3.2. This completes the proof. □

6. Well-posedness of common fixed point problem

Let (X, d) be a metric space. Let $\mathcal{A}$ be a set of self-mappings of X. We suppose that $\mathcal{A}$ contains at least two mappings. We denote by $Fix(\mathcal{A})$ the set of common fixed points of $\mathcal{A}$, hence

$$Fix(\mathcal{A}) = \{x \in X : Tx = x, \quad \forall T \in \mathcal{A}\}.$$

The following definition extends Definition 1.2 to the case of an arbitrary set of mappings.

Definition 6.1. Let (X, d) be a metric space. Let $\mathcal{A}$ be a set of self-mappings of X. The common fixed point problem of the set $\mathcal{A}$ is said to be well-posed if:
(i) $\mathcal{A}$ has a unique common fixed point x in X, that is, there is a unique point $x \in X$ such that $Fix(\mathcal{A}) = \{x\}$.
(ii) for every sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0 \quad \forall T \in \mathcal{A},$$

we have

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

Concerning well-posedness, we have the following theorem.

Theorem 6.2. *Let $\mathbb{R}^+$ be the set of nonnegative reals and let $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that $F(t, 0) = 0 = F(0, t)$ for all $t \in \mathbb{R}^+$.*

Let (X, d) be a complete metric space and let T, S be two self-mappings of X satisfying the condition

$$d(Tx, Ty) \le a_0 F(d(Sx, Tx), d(Sy, Ty))$$

$$+a_1 d(Sx, Sy) + a_2[d(Sx, Tx) + d(Sy, Ty)] + a_3[d(Sx, Ty) + d(Sy, Tx)] \quad (6.1)$$

for all x, y in X, where $a_i = a_i(x, y)$ $(i = 0, 1, 2, 3)$ are nonnegative functions for which there exist three constants $K > 0$ and $\lambda_1, \lambda_2 \in (0, 1)$, such that the inequalities

$$a_0(x, y) \le K, \quad (6.2)$$

$$a_2(x, y) + a_3(x, y) \le \lambda_1, \quad (6.3)$$

$$a_1(x, y) + 2a_3(x, y) \le \lambda_2 \quad (6.4)$$

are satisfied for all x, y in X.

We suppose that
(A1) *the mappings S and T are weakly tangential and occasionally weakly compatible.*
(A2) *$S(X)$ is a complete subspace of X.*

Then, the common fixed point problem of the pair $\{T, S\}$ is well-posed. Moreover, if S is continuous at the unique common fixed point, then T is continuous at the unique common fixed point.

Proof. By Theorem 3.2, T and S have a unique common fixed point in X. Now let $\{w_n\}$ be a sequence of points in X such that

$$\lim_{n\to\infty} d(w_n, Tw_n) = \lim_{n\to\infty} d(w_n, Sw_n) = 0.$$

Then, the sequence $\{w_n\}$ satisfies

$$\lim_{n\to\infty} d(Sw_n, Tw_n) = 0.$$

As in the proof of Theorem 3.2, for each n we set $y_n := T(w_n)$ and $z_n := S(w_n)$. Then, as in the proof of Theorem 3.2 one can prove that both $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences. Since the set $S(X)$ is complete, the sequence $\{z_n\}$ converges to a point $x = Sv$.

We conclude that

$$\lim_{n\to\infty} T(w_n) = \lim_{n\to\infty} S(w_n) = x.$$

As in the proof of Theorem 3.2, x must be the unique common fixed point of T and S. Therefore, the sequence $\{w_n\}$ must converge to the unique common fixed point of T and S. This completes the proof. □

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Université Cadi Ayyad
Faculté des Sciences-Semlalia
Département de Mathématiques
Av. Prince My Abdellah,
BP. 2390, Marrakech, Maroc, Morocco.
E-mail address: akkouchimo@yahoo.fr