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CONVERGENCE OF ADAPTED SEQUENCES IN BANACH SPACES WITHOUT THE RADON-NIKODYM PROPERTY

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Abstract. An adapted sequence (X_n) of Pettis integrable functions is said to be a game fairer with time iff for every $\varepsilon > 0$ there exists $p \in \mathbb{N}$ such that for all $n \geqslant q \geqslant p$ we have $P(\|E_q(X_n) - X_q\| > \varepsilon) < \varepsilon$. We prove some Pettis mean and almost sure convergence results for such games in Banach spaces without the Radon-Nikodym property.

1. Introduction

Recently, several martingale generalizations of Pettis integrable functions in general Banach spaces have been considered by Uhl (1977), Musial (1980), Egghe (1984), Maraffa (1988), Davis et al. (1990), Edgar and Sucheston (1992), Krupa and Zieba (1996), Luu (1997), Bouzar (2001) and others. The main aim of this paper is to extend some convergence results of these authors to (weak) games fairer with time of Pettis integrable functions in Banach spaces without the Radon-Nikodym property. Namely, after giving some fundamental notations and definitions in the next section, using a general Vitali convergence results for Pettis integrals we shall prove in Section 3 some Pettis mean convergence theorems for games fairer with time. Almost sure (a.s.) convergence of mils, a class of games fairer with time, is the subject of Section 4, where as a consequence, some important versions of the Ito-Nisio theorem (cf. [6]) are given.

2. Notations and definitions

Throughout the paper, let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $(\mathcal{F}_n)$ a nondecreasing sequence of complete subσ-fields of $\mathcal{F}$ with $\mathcal{F}_n \uparrow \mathcal{F}$ and $\mathbb{T}$ the set of all bounded stopping times for $(\mathcal{F}_n)$. Given a (real) Banach space $\mathbb{E}$, we denote by $M(\mathbb{E})$ the collection of all strongly $\mathcal{F}$-measurable functions $X : \Omega \to \mathbb{E}$. Such an X is said to be *Bochner integrable*, write $X \in L^1(\mathbb{E})$, or *Pettis integrable*, write $X \in P^1(\mathbb{E})$, respectively, if its L^1-norm

$$E(\|X\|) = \int_\Omega \|X\| dP < \infty$$

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or its Pettis norm

$$F\big(\|X\|\big) = \sup\{E(|\langle x^*, X\rangle|): \quad x^* \in B(\mathbb{E}^*)\} < \infty,$$

where $B(\mathbb{E}^*)$ denotes the closed unit ball of the topological dual $\mathbb{E}^*$ of $\mathbb{E}$.

It is known that the Banach space $L^1(\mathbb{E})$ coincides with $P^1(\mathbb{E})$ if and only if $\dim\mathbb{E} < \infty$ and for every $X \in P^1(\mathbb{E})$ we have

$$\sup_{A\in\mathcal{F}}\{\|v[1_A X]\|\} \leqslant F(\|X\|) \leqslant 2\sup_{A\in\mathcal{F}}\{\|v[1_A X]\|\}, \tag{2.1}$$

where 1_A is the characteristic function of $A \in \mathcal{F}$ and $v[Y]$ is the Pettis integral of $Y \in P^1(\mathbb{E})$. Unless otherwise stated, from now on we shall consider only the sequences (X_n) in $P^1(\mathbb{E})$ for which each X_n is strongly $\mathcal{F}_n$-measurable and the Pettis $\mathcal{F}_q$-conditional expectation $E_q(X_n)$ exists for every $1 \leqslant q \leqslant n$. Thus by the Pettis measurability theorem, we can assume that $\mathbb{E}$ is separable. However, it should be noted that an $X \in P^1(\mathbb{E})$ would fail to have the Pettis $\mathcal{A}$-conditional expectation for some subσ-field $\mathcal{A}$ of $\mathcal{F}$, if it is not bounded enough. For more information of $P^1(\mathbb{E})$, the interested reader is refered to [16].

Now let us recall the following notions.

Definition 2.1. A sequence (X_n) in $P^1(\mathbb{E})$ is said to be

(a) *an amart* (cf. [4]) if the net $\big(v[X_\tau]\big)$ converges in norm, where $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$ for every $\omega \in \Omega$ and $\tau \in \mathbb{T}$,

(b) *a mil* (cf. [7]) if for every $\varepsilon > 0$ there exists $p \in \mathbb{N}$ such that for all $n > p$ we have

$$P\big(\sup_{p\leqslant q\leqslant n} \|E_q(X_n) - X_q\| > \varepsilon\big) < \varepsilon, \tag{2.2}$$

where $E_q(X_n)$ denotes the Pettis $\mathcal{F}_q$-conditional expectation of X_n. More generally, if for every $x^* \in \mathbb{E}^*$ the sequence $\big(\langle x^*, X_n\rangle\big)$ is a mil, then (X_n) is called *a weak mil* (cf. [10]).

It is easy to construct examples to show that both an amart and a mil are weak mils, but any of the converse implications fails. We will extend some Pettis mean and a.s. convergence results of Uhl [18], Musial [13], Davis et al. [3], Edgar and Sucheston [4], Krupa and Zieba [8] and Bouzar [1] for amarts to the following class of (weak) games fairer with time.

Definition 2.2. A sequence (X_n) in $P^1(\mathbb{E})$ is said to be *a game which becomes fairer with time* (cf. [9]) if for every $\varepsilon > 0$ there exists $p \in \mathbb{N}$ such that for all $n \geqslant q \geqslant p$ we have

$$P\big(\|E_q(X_n) - X_q\| > \varepsilon\big) < \varepsilon. \tag{2.3}$$

More generally, if for every $x^* \in \mathbb{E}^*$ the sequence $\big(\langle x^*, X_n\rangle\big)$ is a real-valued game fairer with time, then (X_n) is called *a weak game fairer with time.*

It is easily checked that, by (2.2) and (2.3), every (weak) mil is a (weak) game fairer with time, but the converse fails.

3. Pettis mean convergence of games fairer with time

Recall that a sequence (X_n) in $M(\mathbb{E})$ is said to be *converging scalarly a.s. (resp. scalarly in probability)* to some $X \in M(\mathbb{E})$ iff for every $x^* \in \mathbb{E}^*$ the sequence $\big(\langle x^*, X_n\rangle\big)$ converges a.s. (resp. in probability) to $\langle x^*, X\rangle$. Related to the scalar a.s. and Pettis mean convergence in $P^1(\mathbb{E})$, Edgar and Sucheston proved in ([4], Proposition 5.3.6, p. 199) that *if (X_n) is a uniformly Pettis continuous, i.e.*

$$\lim_{P(A)\to 0} \sup_{n\in\mathbb{N}} F\big(\|1_A X_n\|\big) = 0, \tag{3.1}$$

and it converges scalarly a.s. to some $X \in M(\mathbb{E})$, then (X_n) converges also to X in the Pettis norm. Unfortunately, the result is wrong even for an L_∞-bounded sequential amart (X_n) in ℓ_2, i.e. for every increasing sequence (τ_n) of $\mathbb{T}$ the sequence $\big(E(X_{\tau_n})\big)$ converges weakly in $\mathbb{E}$. Indeed, as a counter-example, let $X_n = e_n$, where (e_n) is the usual basis of ℓ_2. It is easily seen that (X_n) is an L_∞-bounded sequential amart in ℓ_2 which converges weakly to zero everywhere. However, it never converges to zero in the Pettis norm, since $E\big(\,\|X_n\|\,\big) = 1$ for every $n \in \mathbb{N}$. The following general Vitali convergence result is not only a correction for the above statement of Edgar and Sucheston, but also a starting point of our investigation.

Proposition 3.1 (Vitali convergence theorem for Pettis integrals)**.** *Let (X_n) be a uniformly Pettis continuous sequence in $P^1(\mathbb{E})$. Suppose that (X_n) converges scalarly in probability to some $X \in M(\mathbb{E})$. Then $X \in P^1(\mathbb{E})$. Consequently, the sequence $(v[1_A X_n])$ converges weakly to $v[1_A X]$ uniformly $A \in \mathcal{F}$, i.e. for every $x^* \in \mathbb{E}^*$ the sequence $\big(\langle x^*, v[1_A X_n]\rangle\big)$ converges to $\langle x^*, v[1_A X]\rangle$ uniformly in $A \in \mathcal{F}$.*

Suppose more that (X_n) converges to X in probability. Then it also converges to X in the Pettis norm, hence the sequence $(v[1_A X_n])$ converges in norm to $v[1_A X]$ uniformly in $A \in \mathcal{F}$.

Proof. Let (X_n) and X be as assumed in the proposition. Then by the usual Vitali convergence theorem for Lebesgue integrals, the sequence $\big(\langle x^*, X_n\rangle\big)$ converges in $L^1(\mathbb{R})$ to $\langle x^*, X\rangle$ for every $x^* \in \mathbb{E}^*$. This with the uniform Pettis continuity of (X_n) guarantees the P-absolute continuity of the family $\{\langle x^*, X\rangle : x^* \in B(\mathbb{E}^*)\}$ of scalar functions, i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \in \mathcal{F}$ with $P(A) < \delta$, then

$$\sup_{x^*\in B(\mathbb{E}^*)} \{E(|\langle x^*, 1_A X\rangle|)\} < \varepsilon. \tag{3.2}$$

Now define $A_n = \{\|X\| \leqslant n\}, n \in \mathbb{N}$. Then each Pettis integral $v[1_{A_n} X]$ is equal to the Bochner integral $E(1_{A_n} X)$ and $P(\Omega \setminus A_n) \downarrow 0$ as $n \uparrow \infty$. We shall show that the sequence $\big(E(1_{A_n} X)\big)$ is Cauchy in norm. To see that, let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that if $A \in \mathcal{F}$ with $P(A) < \delta$, then A satisfies (3.2). Thus if we choose $n_0 \in \mathbb{N}$ such that $P(\Omega \setminus A_{n_0}) < \delta$ then, by (3.2), for all

$n > m \geqslant n_0$ we have

$$\begin{aligned}\|E(1_{A_n}X) - E(1_{A_m}X)\| &= \sup_{x^*\in B(\mathbb{E}^*)} \left\{ |\langle x^*, E(1_{A_n\setminus A_m}X)\rangle| \right\} \\ &\leqslant \sup_{x^*\in B(\mathbb{E}^*)} \left\{ \int_{(A_n\setminus A_m)} |\langle x^*, X\rangle| \right\} \\ &\leqslant \sup_{x^*\in B(\mathbb{E}^*)} \left\{ \int_{(\Omega\setminus A_{n_0})} |\langle x^*, X\rangle| \right\} < \varepsilon.\end{aligned}$$

It means that the sequence $E(1_{A_n}X)$ is Cauchy in norm. Similarly, for every $A \in \mathcal{F}$ the sequence $E(1_{A\cap A_n}X)$ is also Cauchy in norm, hence it converges strongly to some $x_A \in \mathbb{E}$. It is easily checked that in the case, for every $x^* \in \mathbb{E}^*$ we have

$$\langle x^*, x_A\rangle = \int_A \langle x^*, X\rangle dP.$$

In other words, $X \in P^1(\mathbb{E})$ and for every $A \in \mathcal{F}$ one obtains $v[1_A X] = x_A$. Consequently, by (1.1) the sequence $(v[1_A X_n])$ converges weakly to $v[1_A X]$ uniformly in $A \in \mathcal{F}$, since for every $x^* \in \mathbb{E}^*$ we have

$$\sup_{A\in\mathcal{F}} |\langle x^*, v[1_A X_n]\rangle - \langle x^*, v[1_A X]\rangle| \leqslant E\big(|\langle x^*, X_n - X\rangle|\big).$$

This proves the first part of the proposition. Now suppose more that (X_n) converges to X in probability. We shall show that (X_n) also converges to X in the Pettis norm. To see this, let $\varepsilon > 0$ be given. By the uniform Pettis continuity of (X_n), there exists $\delta > 0$ such that if $A \in \mathcal{F}$ with $P(A) < \delta$, then

$$\sup_{n\in\mathbb{N}} F(\|1_{A_n}X\|) < \frac{\varepsilon}{3}. \tag{3.3}$$

Further, choose a subsequence (n_k) of $\mathbb{N}$ such that the subsequence (X_{n_k}) converges a.s. to X. By the Fatou lemma and (3.3), for every $x^* \in B(\mathbb{E}^*)$ we have

$$\begin{aligned}E\big(|\langle x^*, 1_A X\rangle|\big) &\leqslant \liminf_k E\big(|\langle x^*, 1_A X_{n_k}\rangle|\big) \\ &\leqslant \sup_{n\in\mathbb{N}} F\big(\|1_A X_n\|\big) < \frac{\varepsilon}{3}.\end{aligned}$$

Thus, by taking the supremum over $x^* \in B(\mathbb{E}^*)$ we obtain

$$F(\|1_A X\|) < \frac{\varepsilon}{3}. \tag{3.4}$$

On the other hand, if for every $n \in \mathbb{N}$ we set

$$A_n = \left\{ \|X_n - X\| > \frac{\varepsilon}{3} \right\},$$

then also by the convergence in probability of (X_n) to X, it follows that there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ we have $P(A_n) < \delta$. This with (3.3) and

(3.4) implies that for every $x^* \in B(\mathbb{E}^*)$ and $n \geqslant n_0$ the following estimate holds

$$\begin{aligned} E\big(|\langle x^*, X_n - X\rangle|\big) &= E\big(|\langle x^*, X_n - X\rangle|1_{A_n}\big) + E\big(|\langle x^*, X_n - X\rangle|1_{\Omega\setminus A_n}\big) \\ &\leqslant E\big(|\langle x^*, X_n\rangle|1_{A_n}\big) + E\big(|\langle x^*, X\rangle|1_{A_n}\big) + \frac{\varepsilon}{3} \\ &\leqslant F\big(\|1_{A_n}X_n\|\big) + F\big(\|1_{A_n}X\|\big) + \frac{\varepsilon}{3} < \varepsilon. \end{aligned}$$

Therefore, by taking the supremum over $x^* \in B(\mathbb{E}^*)$ we get

$$F\big(\|X_n - X\|\big) < \varepsilon.$$

This shows that the sequence (X_n) converges in the Pettis norm to X. The proof is completed. □

Note that the first conclusion of the proposition is a stronger version of Theorem 1 of Musial [14], where the author used a deep theorem of James [7] to show that for every $A \in \mathcal{F}$ the sequence $(v[1_A X_n])$ is relatively weakly compact, hence its weak cluster point defines the Pettis integral of X over A. Further, as we have seen that the Pettis infinite integrals are only σ-bounded. Thus to establish the next convergence results for (weak) games fairer with time (X_n) in $P^1(\mathbb{E})$, it is reasonable to impose on them some weaker conditions.

Definition 3.1. A sequence (X_n) in $P^1(\mathbb{E})$ is said to be *σ-bounded* iff there exists a nondecreasing sequence (B_n) of events adapted to $(\mathcal{F}_n)$ with $\lim\limits_{n\to\infty} P(B_n) = 1$ and such that restricted to each B_k, the sequence (X_n) is L^1-bounded. More generally, if for every $x^* \in \mathbb{E}^*$ the sequence $\big(\langle x^*, X_n\rangle\big)$ is *σ-bounded*, then (X_n) is called *scalarly σ-bounded.*

Corollary 3.1. *Let (X_n) be a scalarly σ-bounded weak game fairer with time. Suppose that the sequence $(X_n(\omega))$ is relatively weakly compact a.s. Then (X_n) converges scalarly in probability to some $X \in M(\mathbb{E})$. Consequently, if (X_n) is uniformly Pettis continuous, then $X \in P^1(\mathbb{E})$ and the sequence $(v[1_A X_n])$ converges weakly to $v[1_A X]$ uniformly in $A \in \mathcal{F}$.*

Proof. Let (X_n) be as given in the corollary. Then by Theorem 2.2 [8], for each $x^* \in \mathbb{E}^*$ the sequence $\big(\langle x^*, X_n\rangle\big)$ converges in probability to some $\Phi(x^*) \in M(\mathbb{R})$. Next, since $\mathbb{E}^*$ is total and $\mathbb{E}$ is separable, by Lemma III.31 and III.32 in ([2], p. 81), there exists a sequence (y_n^*) of $\mathbb{E}^*$ such that the countable collection D of all linear combinations of rational coefficients of elements of (y_n^*) is dense in $\mathbb{E}^*$ for the Mackey topology $\tau(\mathbb{E}^*, \mathbb{E})$, i.e. the topology of uniform convergence on convex cirled weakly compact subsets of $\mathbb{E}$ (cf. [15], IV. 3.3, p. 132).

Further, suppose that (X_n) is relatively weakly compact a.s. Then by the Krein-Smulian theorem, there exist a subset Ω_0 of Ω with $P(\Omega_0) = 1$ and an $X \in M(\mathbb{E})$ such that for every $\omega \in \Omega_0$ we have

(a) the closed cirled convex hull $\widehat{co}\{X_n(\omega)\}$ of $(X_n(\omega))$ is a weakly compact subset of $\mathbb{E}$.

(b) $X(\omega)$ is a weak cluster point of $(X_n(\omega))$.

Hence, by (b) and the countability of D, for every $\omega \in \Omega_0$ there exists a subsequence (n_p) of $\mathbb{N}$, which may depend on ω, such that the sequence

$$\big(\langle e, X_{n_p}(\omega)\rangle\big)$$

converges to $\langle e, X(\omega)\rangle$ for every $e \in D$. This with the $\tau(\mathbb{E}^*, \mathbb{E})$-density of D in $\mathbb{E}^*$ and (a) guarantees that $(X_{n_p}(\omega))$ also converges weakly to $X(\omega)$ for every $\omega \in \Omega_0$. But as we have noted at the beginning of the proof that for each $x^* \in \mathbb{E}^*$ the sequence $\big(\langle x^*, X_n\rangle\big)$ converges in probability to $\Phi(x^*) \in M(\mathbb{R})$, so $\Phi(x^*) = \langle x^*, X\rangle$ a.s. It means that (X_n) converges scalarly in probability to X. This proves the main assertion of the corollary. The second assertion follows immediately from the first part of Proposition 3.1. □

Before going to the next convergence result for games fairer with time, it is worth noting that the above proof allows one to conclude that if a sequence (X_n) in $M(\mathbb{E})$ converges scalarly to both X and X' of $M(\mathbb{E})$ simultaneously, then $X = X'$ a.s. Indeed, by the countability of D there exists a subset Ω_0 with $P(\Omega_0) = 1$ such that $\langle e, X(\omega)\rangle = \langle e, X'(\omega)\rangle$ for every $e \in D$ and $\omega \in \Omega_0$. But on the other hand, as D is $\tau(\mathbb{E}^*, \mathbb{E})$-dense in $\mathbb{E}^*$, it follows that $\langle x^*, X(\omega)\rangle = \langle x^*, X'(\omega)\rangle$ for every $x^* \in \mathbb{E}^*$ and $\omega \in \Omega_0$. Consequently, $X(\omega) = X'(\omega)$ for every $\omega \in \Omega_0$.

Proposition 3.2. *Let (X_n) be a σ-bounded game fairer with time in $P^1(\mathbb{E})$ and $X \in M(\mathbb{E})$. Then the following conditions are equivalent:*

(a) *(X_n) converges in probability to X;*

(b) *(X_n) converges scalarly in probability to some X;*

(c) *There exists a total subset S of $\mathbb{E}^*$ such that the sequence $(\langle x^*, X_n\rangle)$ converges in probability to $\langle x^*, X\rangle$ for every $x^* \in S$.*

Consequently, if (X_n) is uniformly Pettis continuous and one of the above conditions holds, then (X_n) converges to X in the Pettis norm.

Proof. Let (X_n) and X be as given in the proposition. Since the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are true in general, we have to prove only that (c) implies (a). To see this, let S be given as in (c) and (B_n) nondecreasing sequence of events adapted to $(\mathcal{F}_n)$ with $\lim\limits_{n\to\infty} P(B_n) = 1$ and such that restricted to each B_m, the sequence (X_n) is L^1-bounded. Now fix any $m \in \mathbb{N}$. Applying Theorem 2.5 in [9] to the L^1-bounded game fairer with time (X_n^m), given by

$$X_n^m = \begin{cases} 1_{B_n} X_n, & n < m, \\ 1_{B_m} X_n, & n \geqslant m, \end{cases}$$

(X_n^m) can be written in a unique form: $X_n^m = M_n^m + P_n^m$, where (M_n^m) is a uniformly integrable martingale and (P_n^m) goes to zero in probability. Consequently, the real-valued uniformly integrable martingale $\big(\langle x^*, M_n^m\rangle\big)$ converges in probability, hence also a.s. to $\langle x^*, 1_{B_m} X\rangle$ for every $x^* \in S$. This with an a.s. convergence result of Davis et al. [3] (see also [4], Theorem 5.3.27, p. 209) implies that (M_n^m) converges itself a.s. to $1_{B_m} X$. Taking the pieces, it is easily checked that the sequence (X_n) converges in probability to the resulting random element just equal to X. This completes the proof of the main part, hence also the whole proposition, since the final consequence follows immediately from Proposition 3.1. □

4. Almost sure convergence of mils

The notion of cluster point in a linear topological space is well-known. Here an $x \in \mathbb{E}$ is said to be *a scalar cluster point of a sequence* (x_n) *of* $\mathbb{E}$ iff for every $x^* \in \mathbb{E}^*$, the real number $\langle x^*, x\rangle$ is a cluster point of $\big(\langle x^*, x_n\rangle\big)$. Obviously, if x is a weak cluster point of (x_n), then it is a scalar cluster point of (x_n). Further, a sequence (X_n) in $M(\mathbb{E})$ is said to be *essentially tight* (cf. [8]) if for every $\varepsilon > 0$ there exists a compact subset K of $\mathbb{E}$ such that $P\big(\bigcap_{n=1}^{\infty}[X_n \in K]\big) \geqslant 1-\varepsilon$. Thus it is clear that if (X_n) is essentially tight, then the sequence $(X_n(\omega))$ is relatively compact a.s. Therefore, by the remark given after Definition 2.1, the main a.s. convergence result of Krupa and Zieba in [8] for amarts is an easy consequence of the following proposition.

Proposition 4.1. *Let (X_n) be a scalarly σ-bounded weak mil and $X \in M(\mathbb{E})$ its scalar cluster point a.s. Then (X_n) converges scalarly a.s. to X. Consequently, (X_n) converges a.s. if and only if $(X_n(\omega))$ is relatively compact a.s.*

Proof. Let (X_n) and X be as given in the proposition. By Theorem 2.4 of [10], for every $x^* \in \mathbb{E}^*$ the σ-bounded real-valued mil $\big(\langle x^*, X_n\rangle\big)$ converges a.s. to some $\Phi(x^*) \in M(\mathbb{R})$. On the other hand, as $\langle x^*, X\rangle$ is a cluster point of $\big(\langle x^*, X_n\rangle\big)$ a.s., by the cluster point approximation theorem 1.2.4 in ([4], p. 11) there exists a sequence $\big(\tau_n(x^*)\big)$ of $\mathbb{T}$ with each $\tau_n(x^*) \geqslant n$ such that the sequence $\big(\langle x^*, X_{\tau_n(x^*)}\rangle\big)$ converges to $\langle x^*, X\rangle$. By Lemma 3 of [11], the optional sequence

$$\big(\langle x^*, X_{\tau_n(x^*)}\rangle\big)$$

converges also a.s. to $\Phi(x^*)$, so it follows that $\Phi(x^*) = \langle x^*, X\rangle$ a.s. This means that the weak mil (X_n) converges scalarly a.s to X, which proves the first conclusion of the proposition. To see its consequence, suppose now that the sequence (X_n) is relatively compact a.s. Then by the Mazur theorem and the countability of the subset D of $\mathbb{E}^*$, given in the proof of Corollary 3.1, there exists a subset Ω_0 of Ω with $P(\Omega_0) = 1$ such that

(a) the sequence $\big(\langle e, X_n(\omega)\rangle\big)$ converges to $\langle e, X_n(\omega)\rangle$ for every $\omega \in \Omega_0$ and $e \in D$.

(b) the closed cirled convex hull $\widehat{co}\{X_n(\omega)\}$ of $(X_n(\omega))$ is compact for every $\omega \in \Omega_0$.

Therefore, by the density of D in $\mathbb{E}^*$ for the Mackey topology, the sequence $(X_n(\omega))$ converges weakly to $X(\omega)$ for every $\omega \in \Omega_0$. But on every convex compact subset K of $\mathbb{E}$ the weak and the norm topology coincide, it follows by (b) that $(X_n(\omega))$ converges even in norm to $X(\omega)$ for every $\omega \in \Omega_0$. This proves that (X_n) converges a.s. to X and completes the proof, since the necessity of the consequence is trivial. □

Finally, let us recall that a sequence (X_n) in $L^1(\mathbb{E})$ is said to be *a uniform amart* (cf. [4]) if for every $\varepsilon > 0$ there exists $p \in \mathbb{N}$ such that for all $\sigma, \tau \in \mathbb{T}$ with $\tau \geqslant \sigma \geqslant p$ we have

$$E\big(\|E_\sigma(X_\tau) - X_\sigma\|\big) < \varepsilon,$$

where $\mathcal{F}_\sigma$ is the subσ-field of $\mathcal{F}$ given by

$$\mathcal{F}_\sigma = \big\{A \in \mathcal{F} : \quad A \cap \{\sigma = n\} \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}\big\}$$

and $E_\sigma(X)$ is the $\mathcal{F}_\sigma$-conditional expectation of an $X \in L^1(\mathbb{E})$. By Definition 2.1, every uniform amart is a mil. Therefore, the following proposition gives several important versions of the Ito-Nisio theorem (cf. [6]) for mils of Pettis integrable functions which contain the main convergence results of Davis et al. [3] and Bouzar [1] for L^1-bounded uniform amarts.

Proposition 4.2. *Let (X_n) be a σ-bounded mil in $P^1(\mathbb{E})$. Then the following conditions are equivalent:*

(a) *(X_n) converges a.s.;*

(b) *(X_n) converges scalarly a.s.;*

(c) *There exist a total subset S of $\mathbb{E}^*$ and element $X \in M(\mathbb{E})$ such that the scalar sequence $\big(\langle x^*, X_n\rangle\big)$ converges in probability to $\langle x^*, X\rangle$ for every $x^* \in S$;*

(d) *There exist S and X as in* (c) *such that $\langle x^*, X\rangle$ is a cluster point of $\big(\langle x^*, X_n\rangle\big)$ a.s. for every $x^* \in S$.*

Proof. Since the first implications (a) ⇒ (b) ⇒ (c) ⇒ (d) are true in general, so it remains to prove only that (d) implies (a). For this purpose, let (X_n) be as given in the proposition. Then there exists a nondecreasing sequence (B_n) of events adapted to $(\mathcal{F}_n)$ with $\lim_{n\to\infty} P(B_n) = 1$ and such that restricted to each B_m, the sequence (X_n) is an L^1-bounded mil in $L^1(\mathbb{E})$. Thus if we fix an $m \in \mathbb{N}$ and define the sequence (X_n^m) as given in the proof of Proposition 3.2, then by Theorem 8 [17], (X_n^m) can be written in a unique form: $X_n^m = M_n^m + P_n^m, n \in \mathbb{N}$, where (M_n^m) is a uniformly integrable martingale and (P_n^m) goes to zero a.s. Now let S and X be as mentioned in (d). Then for any but fixed $x^* \in S$, $\langle x^*, 1_{B_m} X\rangle$ is a cluster point of $(\langle x^*, X_n^m\rangle)$ a.s. Consequently, by the cluster point approximation theorem 1.2.4 in ([4], p. 11), there exists a sequence (τ_n) of $\mathbb{T}$ (which may depend on x^* and m) with each $\tau_n \geqslant n$ such that the sequence $\big(\langle x^*, X_{\tau_n}^m\rangle\big)$ converges a.s. to $\langle x^*, 1_{B_m} X\rangle$. But as $\big(\langle x^*, P_n^m\rangle\big)$ converges to zero a.s., by Lemma 3 of [11], so does the sequence $\big(\langle x^*, P_{\tau_n}^m\rangle\big)$. Consequently, the sequence $\big(\langle x^*, M_{\tau_n}^m\rangle\big)$ converges a.s. to $\langle x^*, 1_{B_m} X\rangle$. On the other hand, as a uniformly integrable real-valued martingale, the sequence $\big(\langle x^*, M_n^m\rangle\big)$ must converge a.s. Therefore, it should converges a.s. to the same limit $\langle x^*, 1_{B_m} X\rangle$. It follows from the recent martingale a.s. convergence result of Davis et al. [3] (see also [4], Theorem 5.3.27, p. 209) that the martingale (M_n^m) converges itself a.s. to $1_{B_m} X$, hence so does the sequence (X_n^m). Since $X_n^m = 1_{B_m} X_n$ for all $n \geqslant m$, this shows that the sequence (X_n), restricted to B_m, converges also a.s. to $1_{B_m} X$. Taking the pieces, it is clear that the mil (X_n) converges a.s. to X. This completes the proof. □

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