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HOPF-LAX-OLEINIK TYPE FORMULA FOR MULTI-TIME HAMILTON-JACOBI EQUATIONS

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Abstract. We consider the Cauchy problem for multi-time Hamilton-Jacobi equations whose Hamiltonians depend on the unknown function and its spacial gradient. We obtain an explicit formula for viscosity solution in a special case.

1. Introduction

Consider the Cauchy problem for multi-time Hamilton-Jacobi equations of the form:

$$\frac{\partial u}{\partial t} + H_1(t, s, x, u, Du) = 0 \quad \text{in } U_T := (0, T]^2 \times \mathbb{R}^n \tag{1.1}$$

$$\frac{\partial u}{\partial s} + H_2(t, s, x, u, Du) = 0 \quad \text{in } U_T := (0, T]^2 \times \mathbb{R}^n \tag{1.2}$$

$$u(0, 0, x) = u_0(x) \quad \text{on } \{t = 0,\ s = 0,\ x \in \mathbb{R}^n\}. \tag{1.3}$$

Here, the Hamiltonians $H_i = H_i(t, s, x, \gamma, p)$, $i = 1, 2$, and initial data $u_0 = u_0(x)$ are given functions, $u = u(t, s, x)$ is unknown,

$$Du = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n).$$

Though it is known that this kind of problem appears in Mathematical Economics, we will not mention the underlying models. According to our knowledge, the works where these kinds of problems are studied from a mathematical point of view are the articles of P. L. Lions and J-C. Rochet [5], G. Barles and A. Tourin [3], S. Plaskacz and M. Quincampoix [6], T. D. Van and M. D. Thanh [9].

In [5], the case where H_i, $i = 1, 2$, depend only on Du is completely solved. The arguments rely on the use of explicit formulas such as the Hopf and Oleinik-Lax formulas. Using commutation properties of the semigroups for the standard equation, P. L. Lions and J-C. Rochet proposed a generalization of the formula that gives explicit solutions of these equations.

G. Barles and A. Tourin [3] proved, under rather natural assumptions, the existence and uniqueness of multi-time viscosity solution to this problem in the case in which Hamitonians depend on the space variable. This is the generalization of the results of P. L. Lions and J-C. Rochet [5].

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T. D. Van and M. D. Thanh [9] considered the Cauchy problem (1.1)-(1.3) in some special cases and sought conditions which guarantee the existence and uniqueness of multi-time viscosity solution. The authors considered multi-time Hamilton-Jacobi equations in two cases:

1. Hamiltonians depend only on spacial gradient of the unknown function.
2. Hamiltonians depend on the unknown function and its spacial gradient.

In [6], S. Plaskacz and M. Quincampoix investigated a system of multi-time Hamilton-Jacobi equations in $(-\infty, 0]^2 \times \mathbb{R}^n$, where Hamiltonians have form

$$H_i(\gamma, p) = \tilde{H}_i(\gamma, p) + \lambda(\gamma), \ i = 1, 2,$$

$\lambda : \mathbb{R} \to \mathbb{R}_+$ is nonincreasing and C^1 and $\tilde{H}_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfy:

1. $\tilde{H}_i(\gamma, \cdot)$ are concave and positively homogeneous of degree one,
2. $\tilde{H}_i(\cdot, p)$ are non increasing and C^1.

Note that, in [3], [5] and [9], the Hamiltonians H_i , $i = 1, 2$, satisfy one of the following two conditions:

(i) $H_i = H_i(p)$, $i = 1, 2$, (independent of γ) are convex in p and have superlinear growth:

$$\lim_{|p| \to +\infty} \frac{H_i(p)}{|p|} = +\infty .$$

(ii) $H_i = H_i(\gamma, p)$, $i = 1, 2$, are nondecreasing in γ for all p, convex and positively homogeneous of degree one in p for all γ.

We will study the question what would happen in the case where $H_i(\gamma, p)$, $i = 1, 2$, are positively homogeneous of degree $m > 1$ in p for all γ?

Definition 1.1. ([7]) A function H is said to be positively homogeneous of degree m, $1 < m < +\infty$, if

$$H(kp) = k^m H(p), \ \forall\, k > 0, \ \forall p.$$

In this paper we will analyze the above question by considering a form of the Cauchy problem for multi-time Hamilton-Jacobi equations (1.1)-(1.3) and give a formula of Hopf-Lax-Oleinik type for multi-time viscosity solution of this problem.

Adimurthi and Veerappa Gowda [1] have studied this subject for the Cauchy problem for an equation.

For the notions of viscosity solution and their Hopf-Lax type formulas we refer to [1]-[6], [8], [9].

Definition 1.2. ([4]) Consider the equation

$$u_t + F(t, x, u, Du) = 0 \ \text{ in } \Omega. \tag{1.4}$$

The upper semicontinuous (u.s.c.) function $u = u(t, x)$ is called a *viscosity subsolution* of (1.4) if $u - \phi$ has local maximum at $(t_0, x_0) \in \Omega$ for any $\phi \in C^1(\Omega)$ then we have

$$\phi_t(t_0, x_0) + F(t_0, x_0, u(t_0, x_0), D\phi(t_0, x_0)) \leq 0.$$

The lower semicontinuous (l.s.c.) function $u = u(t,x)$ is called a *viscosity supersolution* of (1.4) if $u - \phi$ has local minimum at $(t_0, x_0) \in \Omega$ for any $\phi \in C^1(\Omega)$ then we have

$$\phi_t(t_0, x_0) + F(t_0, x_0, u(t_0, x_0), D\phi(t_0, x_0)) \geq 0.$$

A function $u = u(t,x)$ is called *viscosity solution* of (1.4) if it is both viscosity subsolution and viscosity supersolution.

By adding a constant, it is not restrictive to assume that $u - \phi$ has local maximum (minimum) zero at (t_0, x_0).

Definition 1.3. ([9]) A function $u \in C(U_T)$ will be called a *multi-time viscosity solution* of Problem (1.1)-(1.3) if it partially satisfies the equations (1.1), (1.2) in the viscosity sense, i.e., for each $s \in (0,T]$, $u(.,s,.)$ is a viscosity solution of (1.1) and for each $t \in (0,T]$, $u(t,.,.)$ is a viscosity solution of (1.2), and u satisfies the initial condition (1.3) in the sense that

$$\lim_{\substack{(t,s,y)\to(0^+,0^+,x)\\(t,s,y)\in U_T}} u(t,s,y) = u_0(x)\,, \quad x \in \mathbb{R}^n.$$

2. Hopf-Lax-Oleinik type formula for multi-time Hamilton-Jacobi equation

Consider the Cauchy problem in a special form:

$$\frac{\partial u}{\partial t} + f(u)H_1(Du) = 0 \text{ in } U_T, \tag{2.1}$$

$$\frac{\partial u}{\partial s} + f(u)H_2(Du) = 0 \text{ in } U_T, \tag{2.2}$$

$$u(0,0,x) = u_0(x) \text{ on } \mathbb{R}^n. \tag{2.3}$$

In this section the following conditions are assumed:

(I) $H_i : \mathbb{R}^n \to [0, +\infty)$, $i = 1, 2$, are convex, homogeneous functions of degree $m > 1$, satisfying

$$\lim_{|p|\to+\infty} \frac{H_i(p)}{|p|} = +\infty.$$

(II) u_0 is continuous.

(III) $f : \mathbb{R} \to [0, +\infty)$ is a continuous function such that:

(i) $\{\gamma \in \mathbb{R} : f(\gamma) = 0\}$ is of measure zero,

(ii) For each $(t,s,x) \in U_T$,

$$\inf_{y\in\mathbb{R}^n} \Big\{ h(u_0(y)) + (tH_1 + sH_2)^*(x-y) \Big\} \in \mathrm{Im}(h)$$

where

$$h(a) = \int_0^a f(\gamma)^{\frac{1}{m-1}} d\gamma\,.$$

(IV) For each bounded subset V of U_T, there exists a positive number $N(V)$ such that

$$h(u_0(y)) + (tH_1 + sH_2)^*(x-y) > \inf_{|z|\le N(V)} \{h(u_0(z)) + (tH_1 + sH_2)^*(x-z)\}$$

whenever $(t,s,x) \in V,\ \forall y:\ |y| > N(V)$.

Let

$$u(t,s,x) = h^{-1}\Big(\inf_{y\in\mathbb{R}^n} \big\{h(u_0(y)) + (tH_1 + sH_2)^*(x-y)\big\}\Big). \tag{2.4}$$

Recall that the Fenchel conjugate of a function g is given by

$$g^*(p) = \sup_{q\in\mathbb{R}^n} \{\langle p, q\rangle - g(q)\}.$$

Remark 2.1. The conditions (IV) and (ii) in (III) may be considered as a compatible condition between the Hamiltonians, the initial data and the function f for the existence of generalized solution of the Cauchy problem (2.1)- (2.3).

According to the condition (IV) we see that the infimum in (2.4) has to be taken over the ball $B(0; N(V))$ for $(t,s,x) \in V$.

Remark 2.2. We can write, for $(t,s,x) \in U_T$,

$$\begin{aligned}(tH_1 + sH_2)^*(x-y) &= \sup_{z\in\mathbb{R}^n} \big\{\langle x-y, z\rangle - (tH_1 + sH_2)(z)\big\} \\ &= \inf_{z\in\mathbb{R}^n} \Big\{(tH_1)^*(x-z) + (sH_2)^*(z-y)\Big\} \\ &= \inf_{z\in\mathbb{R}^n} \Big\{tH_1^*\Big(\frac{x-z}{t}\Big) + (sH_2)^*(z-y)\Big\} \\ &= \inf_{z\in\mathbb{R}^n} \Big\{(tH_1)^*(x-z) + sH_2^*\Big(\frac{z-y}{s}\Big)\Big\} \\ &= \inf_{z\in\mathbb{R}^n} \Big\{tH_1^*\Big(\frac{x-z}{t}\Big) + sH_2^*\Big(\frac{z-y}{s}\Big)\Big\}.\end{aligned}$$

Remark 2.3. From the assumptions on f we remark that h is a C^1, strictly increasing function, and hence $h^{-1}:\ h(\mathbb{R}) \to \mathbb{R}$ exists, and it is a strictly increasing function too.

We prepare several lemmas for the proof of the main theorem.

Lemma 2.1. 1) ([9]) *The function $(tH_1 + sH_2)^*(z)$, $(t,s,z) \in U_T$ is finite and convex in the open set U_T, therefore locally Lipschitz continuous in U_T. Moreover,*

$$(tH_1 + sH_2)^*(z) = (t+s)\Big(\frac{t}{t+s}H_1 + \frac{s}{t+s}H_2\Big)^*\Big(\frac{z}{t+s}\Big),\ \forall (t,s,z) \in U_T,$$

and

$$\lim_{\frac{|z|}{t+s}\to+\infty} \frac{(tH_1 + sH_2)^*(z)}{|z|} = +\infty.$$

2) $(tH_1 + sH_2)^*(z) \ge 0$ *for all* $z \in \mathbb{R}^n$.

Proof. 1) The proof of the first part is given in [9].

2) From the definition of Fenchel conjugate we have

$$(tH_1 + sH_2)^*(z) = \sup_{x\in\mathbb{R}^n}\{\langle z, x\rangle - (tH_1 + sH_2)(x)\} \geq -(tH_1 + sH_2)(0).$$

Because $H_i,\ i = 1, 2$, are positively homogeneous of degree $m > 1$, $H_i(0) = 0$, $i = 1, 2$. Hence 2) is verified. □

Lemma 2.2 (Dynamic Programming Principle)**.** 1) *Fix* s_0. *For* $0 \leq t_1 < t \leq T$, *we have*

$$u(t, s_0, x) = h^{-1}\Big(\inf_{y\in\mathbb{R}^n}\big\{h(u(t_1, s_0, y)) + (t - t_1)H_1^*(\frac{x - y}{t - t_1})\big\}\Big). \tag{2.5}$$

2) *Fix* t_0. *For* $0 \leq s_1 < s \leq T$, *we have*

$$u(t_0, s, x) = h^{-1}\Big(\inf_{y\in\mathbb{R}^n}\big\{h(u(t_0, s_1, y)) + (s - s_1)H_2^*(\frac{x - y}{s - s_1})\big\}\Big). \tag{2.6}$$

In other words, for fixed s_0, to compute $u(t, s_0, .)$ we can calculate u at time t_1 and then use $u(t_1, s_0, .)$ as the initial condition on the remaining time interval $[t_1, t]$. Arguing analogously, we also have the same result for fixed t_0.

Remark 2.4. The function $u(t, s, x)$ can be rewritten by

$$u(t, s, x) = h^{-1}\Big(\inf_{y\in\mathbb{R}^n}\inf_{z\in\mathbb{R}^n}\big\{h(u_0(y)) + tH_1^*(\frac{x - z}{t}) + (sH_2)^*(z - y)\big\}\Big).$$

Proof of Lemma 2.2. 1) Fix $s = s_0$. Let $v(t, s_0, x)$ denote the right-hand expression of (2.5). Choose $\omega \in \mathbb{R}^n$ such that

$$\begin{aligned} u(t, s_0, x) &= h^{-1}\big\{h(u_0(\omega)) + (tH_1 + s_0H_2)^*(x - \omega)\big\} \\ &= h^{-1}\Big(\inf_{z\in\mathbb{R}^n}\big\{h(u_0(\omega)) + tH_1^*(\frac{x - z}{t}) + (s_0H_2)^*(z - \omega)\big\}\Big). \end{aligned}$$

Set

$$y = \frac{t_1}{t}x + (1 - \frac{t_1}{t})z.$$

Then

$$\frac{y - z}{t_1} = \frac{x - z}{t} = \frac{x - y}{t - t_1}\,. \tag{2.7}$$

Since

$$\begin{aligned} h(u(t_1, s_0, y) &\leq h(u_0(\omega)) + (t_1H_1 + s_0H_2)^*(y - \omega) \\ &= \inf_{z\in\mathbb{R}^n}\big\{h(u_0(\omega)) + t_1H_1^*(\frac{y - z}{t_1}) + (s_0H_2)^*(z - \omega)\big\} \\ &= h(u_0(\omega)) + \inf_{z\in\mathbb{R}^n}\big\{t_1H_1^*(\frac{y - z}{t_1}) + (s_0H_2)^*(z - \omega)\big\}, \end{aligned}$$

we have

$$\begin{aligned} h(v(t,s_0,x)) &\le h(u(t_1,s_0,y)) + (t-t_1)H_1^*(\frac{x-y}{t-t_1}) \\ &\le h(u_0(\omega)) + \inf_{z\in\mathbb{R}^n}\big\{t_1H_1^*(\frac{y-z}{t_1}) + (s_0H_2)^*(z-\omega)\big\} \\ &\qquad + (t-t_1)H_1^*(\frac{x-y}{t-t_1}). \end{aligned}$$

Then, using (2.7) we have

$$\begin{aligned} h(v(t,s_0,x)) &\le h(u_0(\omega)) + \inf_{z\in\mathbb{R}^n}\big\{t_1H_1^*(\frac{y-z}{t_1}) + (s_0H_2)^*(z-\omega) \\ &\qquad + (t-t_1)H_1^*(\frac{x-y}{t-t_1})\big\} \\ &= h(u_0(\omega)) + \inf_{z\in\mathbb{R}^n}\big\{t_1H_1^*(\frac{x-z}{t}) + (s_0H_2)^*(z-\omega) \\ &\qquad + (t-t_1)H_1^*(\frac{x-z}{t})\big\} \\ &= h(u_0(\omega)) + \inf_{z\in\mathbb{R}^n}\big\{tH_1^*(\frac{x-z}{t}) + (s_0H_2)^*(z-\omega)\big\} \\ &= \inf_{z\in\mathbb{R}^n}\big\{h(u_0(\omega)) + tH_1^*(\frac{x-z}{t}) + (s_0H_2)^*(z-\omega)\big\} \\ &= h(u(t,s_0,x)). \end{aligned}$$

Because h is increasing and continuous, this implies

$$v(t,s_0,x) \le u(t,s_0,x). \tag{2.8}$$

Since

$$v(t,s_0,x) = h^{-1}\Big(\inf_{y\in\mathbb{R}^n}\big\{h(u(t_1,s_0,y)) + (t-t_1)H_1^*(\frac{x-y}{t-t_1})\big\}\Big),$$

$$h(v(t,s_0,x)) = \inf_{y\in\mathbb{R}^n}\big\{h(u(t_1,s_0,y)) + (t-t_1)H_1^*(\frac{x-y}{t-t_1})\big\},$$

we can choose $y\in\mathbb{R}^n$ such that

$$h(u(t_1,s_0,y)) + (t-t_1)H_1^*(\frac{x-y}{t-t_1}) \le h(v(t,s_0,x)) + \varepsilon,$$

for $\varepsilon > 0$. Let us choose $\omega\in\mathbb{R}^n$ such that

$$h(u(t_1,s_0,y)) = h(u_0(\omega)) + (t_1H_1 + s_0H_2)^*(y-\omega).$$

Then

$$h(u(t_1,s_0,y)) = h(u_0(\omega)) + \inf_{z\in\mathbb{R}^n}\big\{t_1H_1^*(\frac{y-z}{t_1}) + (s_0H_2)^*(z-\omega)\big\}.$$

Now from the convexity of H_1^* and

$$\frac{x-z}{t} = \big(1-\frac{t_1}{t}\big)\frac{x-y}{t-t_1} + \frac{t_1}{t}\,\frac{y-z}{t_1}$$

we have

$$H_1^*(\frac{x-z}{t}) \leq \big(1-\frac{t_1}{t}\big)H_1^*\big(\frac{x-y}{t-t_1}\big)+\frac{t_1}{t}H_1^*\big(\frac{y-z}{t_1}\big),$$

hence

$$\begin{aligned} h(u(t,s_0,x)) &\leq h(u_0(\omega)) + (tH_1+s_0H_2)^*(x-\omega) \\ &= \inf_{z\in\mathbb{R}^n}\big\{h(u_0(\omega)) + tH_1^*(\frac{x-z}{t}) + (s_0H_2)^*(z-\omega)\big\} \\ &= h(u_0(\omega)) + \inf_{z\in\mathbb{R}^n}\big\{tH_1^*(\frac{x-z}{t}) + (s_0H_2)^*(z-\omega)\big\} \\ &\leq h(u_0(\omega)) + \inf_{z\in\mathbb{R}^n}\big\{(t-t_1)H_1^*(\frac{x-y}{t-t_1}) \\ &\qquad + t_1H_1^*(\frac{y-z}{t_1}) + (s_0H_2)^*(z-\omega)\big\} \\ &= h(u_0(\omega)) + \inf_{z\in\mathbb{R}^n}\big\{t_1H_1^*(\frac{y-z}{t_1}) + (s_0H_2)^*(z-\omega)\big\} \\ &\qquad + (t-t_1)H_1^*(\frac{x-y}{t-t_1}) \\ &= h(u_0(\omega)) + (t_1H_1+s_0H_2)^*(y-\omega) + (t-t_1)H_1^*(\frac{x-y}{t-t_1}) \\ &= h(u(t_1,s_0,y)) + (t-t_1)H_1^*(\frac{x-y}{t-t_1}) \\ &\leq h(v(t,s_0,x)) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$h(u(t,s_0,x)) \leq h(v(t,s_0,x)).$$

Consequently,

$$u(t,s_0,x) \leq v(t,s_0,x). \tag{2.9}$$

From (2.8) and (2.9) we obtain 1). By a similar argument we also obtain 2). □

The next theorem is the main result of this paper.

Theorem 2.3. *Assume the conditions* (I)-(IV) *hold. Then the function* $u = u(t,s,x)$ *given by* (2.4) *is a multi-time viscosity solution of problem* (2.1)-(2.3).

Proof. Step 1. Define

$$\begin{aligned} L(t,s,x) &:= \operatorname*{Argmin}_{y\in\mathbb{R}^n} h^{-1}\{h(u_0(y)) + (tH_1+sH_2)^*(x-y)\} \\ &= \Big\{y_0\in\mathbb{R}^n \,:\, h^{-1}\{h(u_0(y_0)) + (tH_1+sH_2)^*(x-y_0)\} \\ &\qquad = h^{-1}\Big(\min_{y\in\mathbb{R}^n}\{h(u_0(y)) + (tH_1+sH_2)^*(x-y)\}\Big)\Big\}. \end{aligned}$$

By Remark 2.1, $L(t,s,x)$ is nonempty and locally bounded, that means

$$||L(t,s,x)|| = \sup_{y\in L(t,s,x)} |y| \leq N\,,\ \forall (t,s)\in[0,T]^2,\ x\in B(0,r),$$

where $B(0,r)$ is a ball with radius r, which is a neighbourhood of x. Let K be the Lipschitz constant of the function

$$(t,s,x,y) \longmapsto (tH_1+sH_2)^*(x-y),\ t,s\in(0,T],\ x,y\in B(0,r).$$

Then for all $y\in L(t,s,x), t,s,t',s'\in(0,T]$,

$$\begin{aligned} h(u(t,s,x)) &= h(u_0(y)) + (tH_1+sH_2)^*(x-y)\\ h(u(t',s',x')) &\le h(u_0(y)) + (t'H_1+s'H_2)^*(x'-y). \end{aligned}$$

Hence

$$\begin{aligned} h(u(t',s',x')) - h(u(t,s,x)) &\le (t'H_1+s'H_2)^*(x'-y) - (tH_1+sH_2)^*(x-y)\\ &\le K\big(|t-t'|+|s-s'|+||x-x'||\big). \end{aligned}$$

Interchanging (t,s,x) and (t',s',x') implies that

$$|h(u(t',s',x')) - h(u(t,s,x))| \le K\big(|t-t'|+|s-s'|+||x-x'||\big). \tag{2.10}$$

Let $\{t_k,s_k,x_k\}$ be a sequence converging to (t_0,s_0,x_0) as $k\to+\infty$ and $y_k\in L(t_k,s_k,x_k)$. Assume that

$$\lim_{k\to+\infty} u(t_k,s_k,x_k) = l.$$

We have

$$\begin{aligned} h(u(t_k,s_k,x_k)) &= h(u_0(y_k)) + (t_kH_1+s_kH_2)^*(x_k-y_k)\\ &\le h(u_0(y)) + (t_kH_1+s_kH_2)^*(x_k-y), \end{aligned}$$

hence, when $k\to+\infty$,

$$\begin{aligned} &h(l) \le h(u_0(y)) + (t_0H_1+s_0H_2)^*(x_0-y) \text{ for all } y\in\mathbb{R}^n\\ &h(l) \le \inf_{y\in\mathbb{R}^n}\{h(u_0(y)) + (t_0H_1+s_0H_2)^*(x_0-y)\}\\ &h(l) \le h(u(t_0,s_0,x_0)). \end{aligned}$$

Since h is a strictly monotone function, we conclude that

$$\lim_{k\to+\infty} u(t_k,s_k,x_k) = l \le u(t_0,s_0,x_0). \tag{2.11}$$

On the other hand, from (2.10) we get

$$h(u(t_0,s_0,x_0)) \le K\big(|t_k-t_0|+|s_k-s_0|+\|x_k-x_0\|\big) + h(u(t_k,s_k,x_k)).$$

Thus,

$$h(u(t_0,s_0,x_0)) \le h(l) \quad \text{as } k\to+\infty,$$

$$u(t_0,s_0,x_0) \le l = \lim_{k\to+\infty} u(t_k,s_k,x_k). \tag{2.12}$$

Combining (2.11) and (2.12) we see that u is continuous in U_T.

Step 2. We claim that

$$\lim_{\substack{(t,s,y)\to(0^+,0^+,x)\\ (t,s,y)\in U_T}} u(t,s,y) = u_0(x)\,,\ \text{for each fixed } x\in\mathbb{R}^n.$$

Indeed, on the one hand, the definition (2.4) clearly shows that

$$h(u(t,s,y)) \le h(u_0(y)) + (tH_1+sH_2)^*(0),\ \forall (t,s,y)\in U_T.$$

This yields

$$\limsup_{\substack{(t,s,y)\to(0^+,0^+,x)\\ (t,s,y)\in U_T}} h\big(u(t,s,y)\big) \le h(u_0(x)).$$

Thus,

$$\limsup_{\substack{(t,s,y)\to(0^+,0^+,x)\\ (t,s,y)\in U_T}} u(t,s,y) \le u_0(x). \tag{2.13}$$

On the other hand,

$$\begin{aligned} h(u(t,s,x')) &= h(u_0(y)) + (tH_1+sH_2)^*(x'-y)\\ &\ge h(u_0(y)) - (tH_1+sH_2)(0) \end{aligned} \tag{2.14}$$

for all $(t,s,x')\in(0,T]^2\times B(0;r)$; $\forall y\in L(t,s,x')$.

We now show that

$$\lim_{\substack{y\in L(t,s,x')\\ (t,s,x')\to(0^+,0^+,x)}} y = x. \tag{2.15}$$

Assume the contrary. By passing to a subsequence, we may assume that there are $(t_k,s_k,x_k)\in(0,T]^2\times B(0,r)$, $y_k\in L(t_k,s_k,x_k)$, $k\in\mathbb{N}$, $|x_k-y_k|\ge\varepsilon>0$.

Since $||L(t,s,x)||\le N$, $\forall (t,s,x)\in(0,T]^2\times B(0,r)$, it holds for k quite large and $y_k\in L(t_k,s_k,x_k)$ that

$$\begin{aligned} +\infty > C &= \max_{|x|<r} h(u_0(x)) + T.(H_1+H_2)^*(0)\\ &\ge h(u_0(x_k)) + (t_kH_1+s_kH_2)^*(0) \ge h\big(u(t_k,s_k,x_k)\big), \end{aligned} \tag{2.16}$$

$$\begin{aligned} h\big(u(t_k,s_k,x_k)\big) &= h(u_0(y_k)) + (t_kH_1+s_kH_2)^*(x_k-y_k)\\ &= h(u_0(y_k)) + \frac{(t_kH_1+s_kH_2)^*(x_k-y_k)}{|x_k-y_k|}|x_k-y_k|\\ &\ge \min_{|y|\le N} h(u_0(y)) + \varepsilon\ \frac{(t_kH_1+s_kH_2)^*(x_k-y_k)}{|x_k-y_k|}\ \to +\infty \end{aligned}$$

as $k\to\infty$. This contradicts (2.16). Hence (2.15) is verified. From (2.14) and (2.15) we obtain

$$\liminf_{\substack{(t,s,x')\to(0^+,0^+,x)\\ (t,s,x')\in U_T}} u(t,s,x') \ge u_0(x). \tag{2.17}$$

Therefore, (2.3) immediately follows from (2.13) and (2.17).

Step 3. Now we will show that u is a subsolution.

Let us fix s. If $\varphi \in C^1(U_T)$, $u(t_0,s,x_0) = \varphi(t_0,s,x_0)$ and $u-\varphi$ has a local maximum at (t_0,s,x_0). For (t,s,x) in a neighbourhood of (t_0,s,x_0) $(0 \le t < t_0)$

$$\begin{aligned} h(\varphi(t_0,s,x_0)) &= h(u(t_0,s,x_0)) \\ &\le h(u(t,s,x)) + (t_0-t)H_1^*(\frac{x_0-x}{t_0-t}) \quad \text{(by Lemma 2.2)} \\ &\le h(\varphi(t,s,x)) + lH_1^*(\lambda), \end{aligned}$$

where $l = t_0 - t,\ x = x_0 - l\lambda,\ \lambda \in \mathbb{R}^n$. Hence

$$\frac{h(\varphi(t_0,s,x_0)) - h(\varphi(t_0-l,s,x_0-l\lambda))}{l} \le H_1^*(\lambda).$$

Letting $l \to 0$, we obtain at (t_0,s,x_0)

$$\begin{aligned} &h'(\varphi)\varphi_t + \langle h'(\varphi)D\varphi, \lambda\rangle \le H_1^*(\lambda) \\ &h'(\varphi)\varphi_t + \sup_{\lambda\in\mathbb{R}^n}\{\langle h'(\varphi)D\varphi,\lambda\rangle - H_1^*(\lambda)\} \le 0, \end{aligned}$$

and then

$$h'(\varphi)\varphi_t + H_1(h'(\varphi)D\varphi) \le 0. \tag{2.18}$$

Since H_1 is a convex and homogeneous function of degree $m > 1$,

$$H_1(h'(\varphi)D\varphi) = \big(h'(\varphi)\big)^m H_1(D\varphi) \ \text{ at } \ (t_0,s,x_0).$$

Moreover, $h'(\varphi) = f(\varphi)^{\frac{1}{m-1}}$ at (t_0,s,x_0). Then (2.18) becomes

$$\begin{aligned} f(\varphi)^{\frac{1}{m-1}}\varphi_t + \big(f(\varphi)^{\frac{1}{m-1}}\big)^m H_1(D\varphi) \le 0 \text{ at } (t_0,s,x_0) \\ \big(f(\varphi)^{\frac{1}{m-1}}\big)\big[\varphi_t + f(\varphi)H_1(D\varphi)\big] \le 0 \text{ at } (t_0,s,x_0). \end{aligned}$$

Using the assumptions on f, we get

$$\varphi_t(t_0,s,x_0) + f(\varphi(t_0,s,x_0))H_1(D\varphi(t_0,s,x_0)) \le 0,$$

thus

$$\varphi_t(t_0,s,x_0) + f(u(t_0,s,x_0))H_1(D\varphi(t_0,s,x_0)) \le 0.$$

Similarly, for fixed t, if $\varphi \in C^1(U_T)$ such that $u(t,s_0,x_0) = \varphi(t,s_0,x_0)$ and $u-\varphi$ has a local maximum at (t,s_0,x_0), we have

$$\varphi_s(t,s_0,x_0) + f(u(t,s_0,x_0))H_2(D\varphi(t,s_0,x_0)) \le 0.$$

Step 4. Finally, we will show that the function u is supersolution.

Let us fix s. Assume for the contrary that $u = u(t,s,x)$ is not a supersolution of (2.1). Then there exist $\varepsilon_0 > 0$, $(t_0,s,x_0) \in U_T$, a neighbourhood $V(t_0,s,x_0)$ of (t_0,s,x_0) and $\varphi \in C^1(U_T)$ such that $u - \varphi$ attains its minimum zero at (t_0,s,x_0) on $V(t_0,s,x_0)$ and $h'(\varphi(t_0,s,x_0)) \ne 0$,

$$\begin{aligned} u(t_0,s,x_0) &= \varphi(t_0,s,x_0), \\ u - \varphi \ge 0 \quad &\text{in} \quad V(t_0,s,x_0), \\ \varphi_t(t_0,s,x_0) + f(u(t_0,s,x_0))&H_1(D\varphi(t_0,s,x_0)) < -\varepsilon_0 < 0. \end{aligned}$$

Then

$$\varphi_t(t_0, s, x_0) + f(\varphi(t_0, s, x_0))H_1(D\varphi(t_0, s, x_0)) < -\varepsilon_0 < 0.$$

Thus,

$$h'(\varphi)\varphi_t + h'(\varphi)f(\varphi)H_1(D\varphi) < -\varepsilon_0 h'(\varphi) < 0 \text{ at } (t_0, s, x_0) \tag{2.19}$$

since $h'(\varphi(t_0, s, x_0)) > 0$.

From the Dynamic Programming Principle (Lemma 2.2), for $l > 0$ sufficiently small, there exists x^* such that $x^* \to x_0$ as $l \to 0$, and

$$h(u(t_0, s, x_0)) = h(u(t_0 - l, s, x^*)) + lH_1^*(\frac{x_0 - x^*}{l}).$$

Let $\lambda = \dfrac{x_0 - x^*}{l}$. Then $\lambda \in \mathbb{R}^n$,

$$\begin{aligned} h(\varphi(t_0, s, x_0)) &= h(u(t_0, s, x_0)) \\ &= h(u(t_0 - l, s, x^*)) + lH_1^*(\frac{x_0 - x^*}{l}) \\ &\geq h(\varphi(t_0 - l, s, x^*)) + lH_1^*(\frac{x_0 - x^*}{l}) \\ &= h(\varphi(t_0 - l, s, x_0 - l\lambda)) + lH_1^*(\lambda). \end{aligned}$$

Hence

$$\frac{h(\varphi(t_0, s, x_0)) - h(\varphi(t_0 - l, s, x_0 - l\lambda))}{l} \geq H_1^*(\lambda).$$

Computing as in Step 3 we obtain

$$h'(\varphi)\varphi_t + h'(\varphi)f(\varphi)H_1(D\varphi) \geq 0\,, \text{ at } (t_0, s, x_0)$$

which conflicts with (2.19).

This proves that, for fixed s, u is a supersolution of (2.1). Arguing similarly as above, we can also prove that for fixed t, u is a supersolution of (2.2). □

Remark 2.5. The results in Theorem 2.3 can be extended to an arbitrary number of times, that is, k equations in $(0, T]^k \times \mathbb{R}^n$ involving k different Hamiltonians $H_i = H_i(\gamma, p),\ (i = 1, ..., k)\,, k > 2$. In the case $k = 1$ we get the same result as in [1].

We explain our result by the following examples.

Example 1. Consider the Cauchy problem

$$\begin{aligned} u_t + 3u^2.(Du)^2 &= 0 \text{ in } (0, T]^2 \times \mathbb{R}, \\ u_s + 3u^2.(Du)^2 &= 0 \text{ in } (0, T]^2 \times \mathbb{R}, \\ u(0, 0, x) = u_0(x) &= \ln(x^2 + 1) \text{ on } \mathbb{R}. \end{aligned}$$

Clearly, the Hamiltonians $f(\gamma)H_1(p) = f(\gamma)H_2(p) = 3\gamma^2.(p)^2$ and the initial function $u_0(x) = \ln(x^2+1)$ satisfy all the assumptions of Theorem 2.3. Hence, we can find a multi-time viscosity solution of the above problem:

$$u(t,s,x) = \Big(\ln^3(y_0^2+1) + \frac{(x-y_0)^2}{4(t+s)}\Big)^{\frac{1}{3}},$$

where y_0 is a solution of the equation

$$\frac{6y\ln^2(y^2+1)}{y^2+1} + \frac{y}{2(t+s)} = \frac{x}{2(t+s)}.$$

Example 2. Consider the Cauchy problem

$$u_t + e^{3u}.(Du)^4 = 0 \text{ in } (0,T]^2 \times \mathbb{R},$$
$$u_s + e^{3u}.(Du)^4 = 0 \text{ in } (0,T]^2 \times \mathbb{R},$$
$$u(0,0,x) = u_0(x) = \ln(x^2+1) \text{ on } \mathbb{R}.$$

It is easy to check that all the conditions of Theorem 2.3 are fulfilled. A multi-time viscosity solution of problem is given by

$$u(t,s,x) = \ln\Big(y_0^2 + 1 + \frac{3}{4}\frac{\sqrt[3]{(x-y_0)^4}}{\sqrt[3]{4(t+s)}}\Big),$$

where y_0 is the unique solution of

$$8y^3 + \frac{y}{4(t+s)} = \frac{x}{4(t+s)}.$$

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