ACTA MATHEMATICA VIETNAMICA 261
Volume 30, Number 3, 2005, pp. 261-274

ADDITIVE REVERSES OF THE GENERALIZED TRIANGLE INEQUALITY IN NORMED SPACES

S. S. DRAGOMIR

Abstract. Some additive reverses of the generalized triangle inequality in normed linear spaces are given. Applications for complex numbers are provided as well.

1. Introduction

In [2], Diaz and Metcalf established the following reverse of the generalized triangle inequality in real or complex normed linear spaces.

If $F : X \to \mathbb{K}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is a linear functional of unit norm defined on the normed linear space X endowed with the norm $\|\cdot\|$ and the vectors $x_1, \ldots, x_n$ satisfy the condition

$$0 \leq r \|x_i\| \leq \operatorname{Re} F(x_i), \qquad i \in \{1, \ldots, n\}; \tag{1.1}$$

then

$$r \sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\|, \tag{1.2}$$

where equality holds if and only if both

$$F\left(\sum_{i=1}^{n} x_i\right) = r \sum_{i=1}^{n} \|x_i\| \tag{1.3}$$

and

$$F\left(\sum_{i=1}^{n} x_i\right) = \left\| \sum_{i=1}^{n} x_i \right\| \tag{1.4}$$

hold.

If $X = H$, $(H; \langle \cdot, \cdot \rangle)$ is an inner product space and $F(x) = \langle x, e \rangle$, $\|e\| = 1$, then the condition (1.1) may be replaced with the simpler assumption

$$0 \leq r \|x_i\| \leq \operatorname{Re} \langle x_i, e \rangle, \qquad i = 1, \ldots, n, \tag{1.5}$$

Received December 9, 2004; in revised form March 4, 2005.
2000 *Mathematics Subject Classification.* 46B05, 46C05, 26D15, 26D10.
Key words and phrases. Triangle inequality, reverse inequality, normed linear spaces, inner product spaces, complex numbers.

which implies the reverse of the generalized triangle inequality (1.2). In this case the equality holds in (1.2) if and only if [2]

$$\sum_{i=1}^{n} x_i = r\left(\sum_{i=1}^{n} \|x_i\|\right) e. \tag{1.6}$$

Let $F_1, \ldots, F_m$ be linear functionals on X, each of unit norm. Let [2]

$$c = \sup_{x \neq 0} \left[\frac{\sum_{k=1}^{m} |F_k(x)|^2}{\|x\|^2}\right];$$

it then follows that $1 \leq c \leq m$. Suppose the vectors $x_1, \ldots, x_n$ whenever $x_i \neq 0$, satisfy

$$0 \leq r_k \|x_i\| \leq \operatorname{Re} F_k(x_i), \qquad i = 1, \ldots, n,\ k = 1, \ldots, m. \tag{1.7}$$

Then [2]

$$\left(\frac{\sum_{k=1}^{m} r_k^2}{c}\right)^{1/2} \sum_{i=1}^{n} \|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|, \tag{1.8}$$

where equality holds if and only if both

$$F_k\left(\sum_{i=1}^{n} x_i\right) = r_k \sum_{i=1}^{n} \|x_i\|, \qquad k = 1, \ldots, m \tag{1.9}$$

and

$$\sum_{k=1}^{m} \left[F_k\left(\sum_{i=1}^{n} x_i\right)\right]^2 = c\left\|\sum_{i=1}^{n} x_i\right\|^2. \tag{1.10}$$

If $X = H$, an inner product space, then, for $F_k(x) = \langle x, e_k \rangle$, where $\{e_k\}_{k=\overline{1,n}}$ is an orthonormal family in H, i.e., $\langle e_i, e_j \rangle = \delta_{ij}$, $i, j \in \{1, \ldots, n\}$, δ_{ij} denotes the Kronecker delta, the condition (1.7) may be replaced by

$$0 \leq r_k \|x_i\| \leq \operatorname{Re} \langle x_i, e_k \rangle, \qquad i = 1, \ldots, n,\ k = 1, \ldots, m; \tag{1.11}$$

implying the following reverse of the generalized triangle inequality

$$\left(\sum_{k=1}^{m} r_k^2\right)^{\frac{1}{2}} \sum_{i=1}^{n} \|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|, \tag{1.12}$$

where the equality holds if and only if

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\|\right) \sum_{k=1}^{m} r_k e_k. \tag{1.13}$$

The main aim of this paper is to provide some new reverse results of the generalized triangle inequality in its additive form, namely, upper bounds for the nonnegative quantity

$$\sum_{i=1}^{n} \|x_i\| - \left\|\sum_{i=1}^{n} x_i\right\|$$

under various assumptions for the vectors x_i, $i \in \{1, \dots, n\}$ in a real or complex normed space $(X, \|\cdot\|)$. Applications for complex numbers are provided as well.

2. Semi-inner products and Diaz-Metcalf inequality

In 1961, G. Lumer [7] introduced the following concept.

Definition 1. Let X be a linear space over the real or complex number field $\mathbb{K}$. The mapping $[\cdot,\cdot] : X \times X \to \mathbb{K}$ is called a *semi-inner product* on X, if the following properties are satisfied (see also [3, p. 17]):

(i) $[x + y, z] = [x, z] + [y, z]$ for all $x, y, z \in X$;
(ii) $[\lambda x, y] = \lambda [x, y]$ for all $x, y \in X$ and $\lambda \in \mathbb{K}$;
(iii) $[x, x] \geq 0$ for all $x \in X$ and $[x, x] = 0$ implies $x = 0$;
(iv) $|[x, y]|^2 \leq [x, x] [y, y]$ for all $x, y \in X$;
(v) $[x, \lambda y] = \bar{\lambda} [x, y]$ for all $x, y \in X$ and $\lambda \in \mathbb{K}$.

It is well known that the mapping $X \ni x \longmapsto [x, x]^{\frac{1}{2}} \in \mathbb{R}$ is a norm on X and for any $y \in X$, the functional $x \xmapsto{\varphi_y} [x, y] \in \mathbb{K}$ is a continuous linear functional on X endowed with the norm $\|\cdot\|$ generated by $[\cdot,\cdot]$. Moreover, one has $\|\varphi_y\| = \|y\|$ (see for instance [3, p. 17]).

Let $(X, \|\cdot\|)$ be a real or complex normed space. If $J : X \to 2^{X^*}$ is the *normalized duality mapping* defined on X, i.e., (see for instance [3, p. 1])

$$J(x) = \{\varphi \in X^* | \varphi(x) = \|\varphi\| \|x\|, \ \|\varphi\| = \|x\|\}, \quad x \in X,$$

then we may state the following representation result (see for instance [3, p. 18]):

Each semi-inner product $[\cdot,\cdot] : X \times X \to \mathbb{K}$ that generates the norm $\|\cdot\|$ of the normed linear space $(X, \|\cdot\|)$ over the real or complex number field $\mathbb{K}$, is of the form

$$[x, y] = \left\langle \tilde{J}(y), x \right\rangle \quad \text{for any} \quad x, y \in X,$$

where $\tilde{J}$ is a selection of the normalized duality mapping and $\langle \varphi, x \rangle := \varphi(x)$ for $\varphi \in X^*$ and $x \in X$.

Utilizing the concept of semi-inner products, we can state the following particular case of the Diaz-Metcalf inequality.

Corollary 1. *Let $(X, \|\cdot\|)$ be a normed linear space, $[\cdot,\cdot] : X \times X \to \mathbb{K}$ a semi-inner product generating the norm $\|\cdot\|$ and $e \in X$, $\|e\| = 1$. If $x_i \in X$, $i \in$*

$\{1, \ldots, n\}$ *and* $r \geq 0$ *such that*

$$r \|x_i\| \leq \operatorname{Re}\ [x_i, e] \quad \text{for each} \ \ i \in \{1, \ldots, n\}, \tag{2.1}$$

then we have the inequality

$$r \sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\|. \tag{2.2}$$

The case of equality holds in (2.2) *if and only if both*

$$\left[\sum_{i=1}^{n} x_i, e \right] = r \sum_{i=1}^{n} \|x_i\| \tag{2.3}$$

and

$$\left[\sum_{i=1}^{n} x_i, e \right] = \left\| \sum_{i=1}^{n} x_i \right\|. \tag{2.4}$$

The proof is obvious from the Diaz-Metcalf theorem [2, Theorem 3] applied for the continuous linear functional $F_e(x) = [x, e]$, $x \in X$.

Before we provide a simpler necessary and sufficient condition of equality in (2.2), we need to recall the concept of strictly convex normed spaces and a classical characterization of these spaces.

Definition 2. A normed linear space $(X, \|\cdot\|)$ is said to be strictly convex if for every x, y from X with $x \neq y$ and $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1 - \lambda) y\| < 1$ for all $\lambda \in (0, 1)$.

The following characterization of strictly convex spaces is useful in what follows (see [1], [6], or [3, p. 21]).

Theorem 1. *Let* $(X, \|\cdot\|)$ *be a normed linear space over* $\mathbb{K}$ *and* $[\cdot, \cdot]$ *a semi-inner product generating its norm. The following statements are equivalent:*

(i) $(X, \|\cdot\|)$ *is strictly convex;*

(ii) *For every* $x, y \in X$, $x, y \neq 0$ *with* $[x, y] = \|x\| \|y\|$, *there exists a* $\lambda > 0$ *such that* $x = \lambda y$.

The following result may be stated.

Corollary 2. *Let* $(X, \|\cdot\|)$ *be a strictly convex normed linear space,* $[\cdot, \cdot]$ *a semi-inner product generating the norm* $\|\cdot\|$ *and* e, x_i $(i \in \{1, \ldots, n\})$ *as in Corollary* 1. *Then the case of equality holds in* (2.2) *if and only if*

$$\sum_{i=1}^{n} x_i = r \left(\sum_{i=1}^{n} \|x_i\| \right) e. \tag{2.5}$$

Proof. If (2.5) holds true, then, obviously

$$\left\| \sum_{i=1}^{n} x_i \right\| = r \left(\sum_{i=1}^{n} \|x_i\| \right) \|e\| = r \sum_{i=1}^{n} \|x_i\|,$$

which is the equality case in (2.2).

Conversely, if the equality holds in (2.2), then by Corollary 1, we have that (2.3) and (2.4) hold true. Utilizing Theorem 1, we conclude that there exists a $\mu > 0$ such that

$$\sum_{i=1}^{n} x_i = \mu e. \tag{2.6}$$

Inserting this in (2.3) we get

$$\mu \|e\|^2 = r \sum_{i=1}^{n} \|x_i\|$$

giving

$$\mu = r \sum_{i=1}^{n} \|x_i\| . \tag{2.7}$$

Finally, by (2.6) and (2.7) we deduce (2.5) and the corollary is proved. □

3. An additive reverse for the triangle inequality

In the following we provide an alternative of the Diaz-Metcalf reverse of the generalized triangle inequality.

Theorem 2. *Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$ and $F : X \to \mathbb{K}$ a linear functional with the property that $|F(x)| \le \|x\|$ for any $x \in X$. If $x_i \in X$, $k_i \ge 0$, $i \in \{1, \dots, n\}$ are such that*

$$(0 \le) \|x_i\| - \operatorname{Re} F(x_i) \le k_i \quad \text{for each} \quad i \in \{1, \dots, n\}, \tag{3.1}$$

then we have the inequality

$$(0 \le) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \le \sum_{i=1}^{n} k_i. \tag{3.2}$$

The equality holds in (3.2) *if and only if both*

$$F\left(\sum_{i=1}^{n} x_i\right) = \left\| \sum_{i=1}^{n} x_i \right\| \quad \text{and} \quad F\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} \|x_i\| - \sum_{i=1}^{n} k_i. \tag{3.3}$$

Proof. If we sum in (3.1) over i from 1 to n, then we get

$$\sum_{i=1}^{n} \|x_i\| \le \operatorname{Re}\left[F\left(\sum_{i=1}^{n} x_i\right)\right] + \sum_{i=1}^{n} k_i. \tag{3.4}$$

Taking into account that $|F(x)| \le \|x\|$ for each $x \in X$, then we may state that

$$\operatorname{Re}\left[F\left(\sum_{i=1}^{n} x_i\right)\right] \le \left|\operatorname{Re} F\left(\sum_{i=1}^{n} x_i\right)\right| \le \left|F\left(\sum_{i=1}^{n} x_i\right)\right| \le \left\|\sum_{i=1}^{n} x_i\right\|. \tag{3.5}$$

Now, making use of (3.4) and (3.5), we deduce (3.2).

Obviously, if (3.3) is valid, then the case of equality in (3.2) holds true.

Conversely, if the equality holds in (3.2), then it must hold in all the inequalities used to prove (3.2), therefore we have

$$\sum_{i=1}^{n}\|x_i\| = \operatorname{Re}\left[F\left(\sum_{i=1}^{n}x_i\right)\right]+\sum_{i=1}^{n}k_i$$

and

$$\operatorname{Re}\left[F\left(\sum_{i=1}^{n}x_i\right)\right]=\left|F\left(\sum_{i=1}^{n}x_i\right)\right|=\left\|\sum_{i=1}^{n}x_i\right\|,$$

which imply (3.3). □

The following corollary may be stated.

Corollary 3. *Let $(X,\|\cdot\|)$ be a normed linear space, $[\cdot,\cdot]:X\times X\to\mathbb{K}$ a semi-inner product generating the norm $\|\cdot\|$ and $e\in X$, $\|e\|=1$. If $x_i\in X$, $k_i\geq 0$, $i\in\{1,\dots,n\}$ are such that*

$$(0\leq)\,\|x_i\|-\operatorname{Re}[x_i,e]\leq k_i \quad \textit{for each}\quad i\in\{1,\dots,n\}, \tag{3.6}$$

then we have the inequality

$$(0\leq)\sum_{i=1}^{n}\|x_i\|-\left\|\sum_{i=1}^{n}x_i\right\|\leq\sum_{i=1}^{n}k_i. \tag{3.7}$$

The equality holds in (3.7) *if and only if both*

$$\left[\sum_{i=1}^{n}x_i,e\right]=\left\|\sum_{i=1}^{n}x_i\right\| \quad \textit{and} \quad \left[\sum_{i=1}^{n}x_i,e\right]=\sum_{i=1}^{n}\|x_i\|-\sum_{i=1}^{n}k_i. \tag{3.8}$$

Moreover, if $(X,\|\cdot\|)$ is strictly convex, then the case of equality holds in (3.7) *if and only if*

$$\sum_{i=1}^{n}\|x_i\|\geq\sum_{i=1}^{n}k_i \tag{3.9}$$

and

$$\sum_{i=1}^{n}x_i=\left(\sum_{i=1}^{n}\|x_i\|-\sum_{i=1}^{n}k_i\right)\cdot e. \tag{3.10}$$

Proof. The first part of the corollary is obvious by Theorem 2 applied for the continuous linear functional of unit norm F_e, $F_e(x)=[x,e]$, $x\in X$. The second part may be shown on utilizing a similar argument to the one from the proof of Corollary 2. We omit the details. □

Remark 1. If $X=H$, $(H;\langle\cdot,\cdot\rangle)$ is an inner product space, then from Corollary 3 we deduce the additive reverse inequality obtained in Theorem 7 of [4]. For further similar results in inner product spaces, see [4] and [5].

4. Reverse inequalities for m functionals

The following result generalising Theorem 2 may be stated.

Theorem 3. *Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. If F_k, $k \in \{1, \dots, m\}$ are bounded linear functionals defined on X and $x_i \in X$, $M_{ik} \geq 0$ for $i \in \{1, \dots, n\}$, $k \in \{1, \dots, m\}$ such that*

$$\|x_i\| - \operatorname{Re} F_k(x_i) \leq M_{ik} \quad \text{for each} \quad i \in \{1, \dots, n\}, \ k \in \{1, \dots, m\}, \tag{4.1}$$

then we have the inequality

$$\sum_{i=1}^{n} \|x_i\| \leq \left\| \frac{1}{m} \sum_{k=1}^{m} F_k \right\| \left\| \sum_{i=1}^{n} x_i \right\| + \frac{1}{m} \sum_{k=1}^{m} \sum_{i=1}^{n} M_{ik}. \tag{4.2}$$

The case of equality holds in (4.2) *if both*

$$\frac{1}{m} \sum_{k=1}^{m} F_k \left(\sum_{i=1}^{n} x_i \right) = \left\| \frac{1}{m} \sum_{k=1}^{m} F_k \right\| \left\| \sum_{i=1}^{n} x_i \right\| \tag{4.3}$$

and

$$\frac{1}{m} \sum_{k=1}^{m} F_k \left(\sum_{i=1}^{n} x_i \right) = \sum_{i=1}^{n} \|x_i\| - \frac{1}{m} \sum_{k=1}^{m} \sum_{i=1}^{n} M_{ik}. \tag{4.4}$$

Proof. If we sum (4.1) over i from 1 to n, then we deduce

$$\sum_{i=1}^{n} \|x_i\| - \operatorname{Re} F_k \left(\sum_{i=1}^{n} x_i \right) \leq \sum_{i=1}^{n} M_{ik}$$

for each $k \in \{1, \dots, m\}$.

Summing these inequalities over k from 1 to m, we deduce

$$\sum_{i=1}^{n} \|x_i\| \leq \frac{1}{m} \sum_{k=1}^{m} \operatorname{Re} F_k \left(\sum_{i=1}^{n} x_i \right) + \frac{1}{m} \sum_{k=1}^{m} \sum_{i=1}^{n} M_{ik}. \tag{4.5}$$

Utilizing the continuity property of the functionals F_k and the properties of the modulus, we have

$$\begin{aligned} \sum_{k=1}^{m} \operatorname{Re} F_k \left(\sum_{i=1}^{n} x_i \right) &\leq \left| \sum_{k=1}^{m} \operatorname{Re} F_k \left(\sum_{i=1}^{n} x_i \right) \right| \\ &\leq \left| \sum_{k=1}^{m} F_k \left(\sum_{i=1}^{n} x_i \right) \right| \leq \left\| \sum_{k=1}^{m} F_k \right\| \left\| \sum_{i=1}^{n} x_i \right\|. \end{aligned} \tag{4.6}$$

Now, by (4.5) and (4.6), we deduce (4.2).

Obviously, if (4.3) and (4.4) hold true, then the case of equality is valid in (4.2).

Conversely, if the case of equality holds in (4.2), then it must hold in all the inequalities used to prove (4.2). Therefore we have

$$\sum_{i=1}^{n}\|x_i\| = \frac{1}{m}\sum_{k=1}^{m} \operatorname{Re} F_k\left(\sum_{i=1}^{n} x_i\right) + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n} M_{ik},$$

$$\sum_{k=1}^{m} \operatorname{Re} F_k\left(\sum_{i=1}^{n} x_i\right) = \left\|\sum_{k=1}^{m} F_k\right\|\left\|\sum_{i=1}^{n} x_i\right\|$$

and

$$\sum_{k=1}^{m} \operatorname{Im} F_k\left(\sum_{i=1}^{n} x_i\right) = 0.$$

These imply that (4.3) and (4.4) hold true, and the theorem is completely proved. □

Remark 2. If F_k, $k \in \{1, \dots, m\}$ are of unit norm, then, from (4.2), we deduce the inequality

$$\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n} x_i\right\| + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n} M_{ik}, \tag{4.7}$$

which is obviously coarser than (4.2), but perhaps more useful for applications.

The case of inner product spaces, in which we may provide a simpler condition of equality, is of interest in applications.

Theorem 4. *Let* $(H, \|\cdot\|)$ *be an inner product space over the real or complex number field* $\mathbb{K}$, e_k, $x_i \in H\backslash\{0\}$, $k \in \{1, \dots, m\}$, $i \in \{1, \dots, n\}$. *If* $M_{ik} \ge 0$ *for* $i \in \{1, \dots, n\}$, $k \in \{1, \dots, m\}$ *such that*

$$\|x_i\| - \operatorname{Re}\langle x_i, e_k\rangle \le M_{ik} \quad \text{for each} \quad i \in \{1, \dots, n\}, \ k \in \{1, \dots, m\}, \tag{4.8}$$

then we have the inequality

$$\sum_{i=1}^{n}\|x_i\| \le \left\|\frac{1}{m}\sum_{k=1}^{m} e_k\right\|\left\|\sum_{i=1}^{n} x_i\right\| + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n} M_{ik}. \tag{4.9}$$

The case of equality holds in (4.9) *if and only if*

$$\sum_{i=1}^{n}\|x_i\| \ge \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n} M_{ik} \tag{4.10}$$

and

$$\sum_{i=1}^{n} x_i = \frac{m\left(\sum_{i=1}^{n}\|x_i\| - \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n} M_{ik}\right)}{\left\|\sum_{k=1}^{m} e_k\right\|^2}\sum_{k=1}^{m} e_k. \tag{4.11}$$

Proof. As in the proof of Theorem 3, we have

$$\sum_{i=1}^{n}\|x_i\| \leq \operatorname{Re}\left\langle \frac{1}{m}\sum_{k=1}^{m}e_k, \sum_{i=1}^{n}x_i\right\rangle + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n}M_{ik}, \tag{4.12}$$

and $\sum_{k=1}^{m} e_k \neq 0$.

On utilizing the Schwarz inequality in the inner product space $(H;\langle\cdot,\cdot\rangle)$ for $\sum_{i=1}^{n} x_i$, $\sum_{k=1}^{m} e_k$, we have

$$\begin{aligned}\left\|\sum_{i=1}^{n}x_i\right\|\left\|\sum_{k=1}^{m}e_k\right\| &\geq \left|\left\langle \sum_{i=1}^{n}x_i, \sum_{k=1}^{m}e_k\right\rangle\right| \geq \left|\operatorname{Re}\left\langle \sum_{i=1}^{n}x_i, \sum_{k=1}^{m}e_k\right\rangle\right| \\ &\geq \operatorname{Re}\left\langle \sum_{i=1}^{n}x_i, \sum_{k=1}^{m}e_k\right\rangle.\end{aligned}\tag{4.13}$$

By (4.12) and (4.13) we deduce (4.9).

Taking the norm in (4.11) and using (4.10), we have

$$\left\|\sum_{i=1}^{n}x_i\right\| = \frac{m\left(\sum_{i=1}^{n}\|x_i\| - \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n}M_{ik}\right)}{\left\|\sum_{k=1}^{m}e_k\right\|},$$

showing that the equality holds in (4.9).

Conversely, if the case of equality holds in (4.9), then it must hold in all the inequalities used to prove (4.9). Therefore we have

$$\|x_i\| = \operatorname{Re}\langle x_i, e_k\rangle + M_{ik} \quad \text{for each} \quad i \in \{1,\dots,n\},\ k \in \{1,\dots,m\}, \tag{4.14}$$

$$\left\|\sum_{i=1}^{n}x_i\right\|\left\|\sum_{k=1}^{m}e_k\right\| = \left|\left\langle \sum_{i=1}^{n}x_i, \sum_{k=1}^{m}e_k\right\rangle\right| \tag{4.15}$$

and

$$\operatorname{Im}\left\langle \sum_{i=1}^{n}x_i, \sum_{k=1}^{m}e_k\right\rangle = 0. \tag{4.16}$$

From (4.14), on summing over i and k, we get

$$\operatorname{Re}\left\langle \sum_{i=1}^{n}x_i, \sum_{k=1}^{m}e_k\right\rangle = m\sum_{i=1}^{n}\|x_i\| - \sum_{k=1}^{m}\sum_{i=1}^{n}M_{ik}. \tag{4.17}$$

On the other hand, by the use of the following identity in inner product spaces,

$$\left\|u - \frac{\langle u,v\rangle v}{\|v\|^2}\right\|^2 = \frac{\|u\|^2\|v\|^2 - |\langle u,v\rangle|^2}{\|v\|^2}, \quad v \neq 0;$$

the relation (4.15) holds if and only if

$$\sum_{i=1}^{n} x_i = \frac{\left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle}{\left\| \sum_{k=1}^{m} e_k \right\|^2} \sum_{k=1}^{m} e_k,$$

giving, from (4.16) and (4.17), that

$$\sum_{i=1}^{n} x_i = \frac{m \sum_{i=1}^{n} \|x_i\| - \sum_{k=1}^{m} \sum_{i=1}^{n} M_{ik}}{\left\| \sum_{k=1}^{m} e_k \right\|^2} \sum_{k=1}^{m} e_k.$$

If the equality case holds in (4.9), then obviously (4.10) is valid, and the theorem is proved. □

Remark 3. If in the above theorem the vectors $\{e_k\}_{k=\overline{1,m}}$ are assumed to be orthogonal, then (4.9) becomes:

$$\sum_{i=1}^{n} \|x_i\| \le \frac{1}{m} \left(\sum_{k=1}^{m} \|e_k\|^2 \right)^{\frac{1}{2}} \left\| \sum_{i=1}^{n} x_i \right\| + \frac{1}{m} \sum_{k=1}^{m} \sum_{i=1}^{n} M_{ik}. \tag{4.18}$$

Moreover, if $\{e_k\}_{k=\overline{1,m}}$ is an orthonormal family, then (4.18) becomes

$$\sum_{i=1}^{n} \|x_i\| \le \frac{\sqrt{m}}{m} \left\| \sum_{i=1}^{n} x_i \right\| + \frac{1}{m} \sum_{k=1}^{m} \sum_{i=1}^{n} M_{ik}, \tag{4.19}$$

which has been obtained in [5].

Before we provide some natural consequences of Theorem 4, we need some preliminary results concerning reverses of Schwarz's inequality in inner product spaces (see for instance [4, p. 27]).

Lemma 1. *Let $(H, \|\cdot\|)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $x, a \in H$, $r > 0$. If $\|x - a\| \le r$, then we have the inequality*

$$\|x\| \|a\| - \operatorname{Re} \langle x, a \rangle \le \frac{1}{2} r^2. \tag{4.20}$$

The case of equality holds in (4.20) *if and only if*

$$\|x - a\| = r \quad \text{and} \quad \|x\| = \|a\|. \tag{4.21}$$

Proof. The condition $\|x - a\| \le r$ is clearly equivalent to

$$\|x\|^2 + \|a\|^2 \le 2 \operatorname{Re} \langle x, a \rangle + r^2. \tag{4.22}$$

Since

$$2 \|x\| \|a\| \le \|x\|^2 + \|a\|^2, \tag{4.23}$$

with equality if and only if $\|x\| = \|a\|$, hence by (4.22) and (4.23) we deduce (4.20).

The case of equality is obvious. □

Utilizing the above lemma we may state the following corollary of Theorem 4.

Corollary 4. *Let* $(H;\langle\cdot,\cdot\rangle)$, e_k, x_i *be as in Theorem* 4. *If* $r_{ik}>0$, $i\in\{1,\dots,n\}$, $k\in\{1,\dots,m\}$ *are such that*

$$\|x_i-e_k\|\le r_{ik} \quad \text{for each } \ i\in\{1,\dots,n\} \ \text{ and } k\in\{1,\dots,m\}, \tag{4.24}$$

then we have the inequality

$$\sum_{i=1}^{n}\|x_i\|\le\left\|\frac{1}{m}\sum_{k=1}^{m}e_k\right\|\left\|\sum_{i=1}^{n}x_i\right\|+\frac{1}{2m}\sum_{k=1}^{m}\sum_{i=1}^{n}r_{ik}^2. \tag{4.25}$$

The equality holds in (4.25) *if and only if*

$$\sum_{i=1}^{n}\|x_i\|\ge\frac{1}{2m}\sum_{k=1}^{m}\sum_{i=1}^{n}r_{ik}^2$$

and

$$\sum_{i=1}^{n}x_i=\frac{m\left(\sum\limits_{i=1}^{n}\|x_i\|-\frac{1}{2m}\sum\limits_{k=1}^{m}\sum\limits_{i=1}^{n}r_{ik}^2\right)}{\left\|\sum\limits_{k=1}^{m}e_k\right\|^2}\sum_{k=1}^{m}e_k.$$

The following lemma may provide another sufficient condition for (4.8) to hold (see also [4, p. 28]).

Lemma 2. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$ *and* $x,y\in H$, $M\ge m>0$. *If either*

$$\operatorname{Re}\langle My-x,x-my\rangle\ge 0 \tag{4.26}$$

or, equivalently,

$$\left\|x-\frac{m+M}{2}y\right\|\le\frac{1}{2}(M-m)\|y\|, \tag{4.27}$$

holds, then

$$\|x\|\,\|y\|-\operatorname{Re}\langle x,y\rangle\le\frac{1}{4}\cdot\frac{(M-m)^2}{m+M}\|y\|^2. \tag{4.28}$$

The case of equality holds in (4.28) *if and only if the equality case is realized in* (4.26) *and*

$$\|x\|=\frac{M+m}{2}\|y\|.$$

The proof is obvious by Lemma 1 for $a=\dfrac{M+m}{2}y$ and $r=\dfrac{1}{2}(M-m)\|y\|$.

Finally, the following corollary of Theorem 4 may be stated.

Corollary 5. *Assume that* $(H,\langle\cdot,\cdot\rangle)$, e_k, x_i *are as in Theorem 4. If* $M_{ik}\ge m_{ik}>0$ *satisfy the condition*

$$\operatorname{Re}\langle M_{ik}e_k-x_i,x_i-m_{ik}e_k\rangle\ge 0 \quad \text{for each } \ i\in\{1,\dots,n\} \ \text{ and } k\in\{1,\dots,m\},$$

then

$$\sum_{i=1}^{n}\|x_i\| \le \left\|\frac{1}{m}\sum_{k=1}^{m} e_k\right\| \left\|\sum_{i=1}^{n} x_i\right\| + \frac{1}{4m}\sum_{k=1}^{m}\sum_{i=1}^{n}\frac{(M_{ik}-m_{ik})^2}{M_{ik}+m_{ik}}\|e_k\|^2 .$$

5. Applications for complex numbers

Let $\mathbb{C}$ be the field of complex numbers. If $z = \operatorname{Re} z + i \operatorname{Im} z$, then by $|\cdot|_p : \mathbb{C} \to [0,\infty)$, $p \in [1,\infty]$ we define the $p-$*modulus* of z as

$$|z|_p := \begin{cases} \max\{|\operatorname{Re} z|, |\operatorname{Im} z|\} & \text{if } p = \infty, \\ \\ (|\operatorname{Re} z|^p + |\operatorname{Im} z|^p)^{\frac{1}{p}} & \text{if } p \in [1,\infty), \end{cases}$$

where $|a|$, $a \in \mathbb{R}$ is the usual modulus of the real number a. Obviously, for $p = 2$, we recapture the usual modulus of a complex number.

It is well known that $\left(\mathbb{C}, |\cdot|_p\right)$, $p \in [1,\infty]$ is a Banach space over the real number field $\mathbb{R}$.

Consider the Banach space $(\mathbb{C}, |\cdot|_1)$ and $F : \mathbb{C} \to \mathbb{C}$, $F(z) = az$ with $a \in \mathbb{C}$, $a \neq 0$. Obviously, F is linear on $\mathbb{C}$. For $z \neq 0$, we have

$$\frac{|F(z)|}{|z|_1} = \frac{|a||z|}{|z|_1} = \frac{|a|\sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2}}{|\operatorname{Re} z| + |\operatorname{Im} z|} \le |a| .$$

Since, for $z_0 = 1$, we have $|F(z_0)| = |a|$ and $|z_0|_1 = 1$, hence

$$\|F\|_1 := \sup_{z \neq 0}\frac{|F(z)|}{|z|_1} = |a| ,$$

showing that F is a bounded linear functional on $(\mathbb{C}, |\cdot|_1)$ and $\|F\|_1 = |a|$.

We can apply Theorem 3 to state the following reverse of the generalized triangle inequality for complex numbers.

Proposition 1. *Let $a_k, x_j \in \mathbb{C}$, $k \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$. If there exist the constants $M_{jk} \ge 0$, $k \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$ such that*

$$|\operatorname{Re} x_j| + |\operatorname{Im} x_j| \le \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j + M_{jk} \tag{5.1}$$

for each $j \in \{1,\dots,n\}$ and $k \in \{1,\dots,m\}$, then

$$\sum_{j=1}^{n}[|\operatorname{Re} x_j| + |\operatorname{Im} x_j|] \le \frac{1}{m}\left|\sum_{k=1}^{m} a_k\right| \left[\left|\sum_{j=1}^{n}\operatorname{Re} x_j\right| + \left|\sum_{j=1}^{n}\operatorname{Im} x_j\right|\right] + \frac{1}{m}\sum_{k=1}^{m}\sum_{j=1}^{n} M_{jk}. \tag{5.2}$$

The proof follows by Theorem 3 applied for the Banach space $(\mathbb{C}, |\cdot|_1)$ and $F_k(z) = a_k z$, $k \in \{1,\dots,m\}$ on taking into account that:

$$\left\|\sum_{k=1}^{m} F_k\right\|_1 = \left|\sum_{k=1}^{m} a_k\right| .$$

Now, consider the Banach space $(\mathbb{C}, |\cdot|_\infty)$. If $F(z) = dz$, then for $z \neq 0$ we have

$$\frac{|F(z)|}{|z|_\infty} = \frac{|d|\,|z|}{|z|_\infty} = \frac{|d| \sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2}}{\max\{|\operatorname{Re} z|, |\operatorname{Im} z|\}} \leq \sqrt{2}\,|d|.$$

Since, for $z_0 = 1 + i$, we have $|F(z_0)| = \sqrt{2}\,|d|$, $|z_0|_\infty = 1$, hence

$$\|F\|_\infty := \sup_{z \neq 0} \frac{|F(z)|}{|z|_\infty} = \sqrt{2}\,|d|,$$

showing that F is a bounded linear functional on $(\mathbb{C}, |\cdot|_\infty)$ and $\|F\|_\infty = \sqrt{2}\,|d|$.

If we apply Theorem 3, then we can state the following reverse of the generalized triangle inequality for complex numbers.

Proposition 2. *Let $a_k, x_j \in \mathbb{C}$, $k \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. If there exist the constants $M_{jk} \geq 0$, $k \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$ such that*

$$\max\{|\operatorname{Re} x_j|, |\operatorname{Im} x_j|\} \leq \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j + M_{jk}$$

for each $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$, then

$$\sum_{j=1}^{n} \max\{|\operatorname{Re} x_j|, |\operatorname{Im} x_j|\} \leq \frac{\sqrt{2}}{m} \left|\sum_{k=1}^{m} a_k\right| \max\left\{\left|\sum_{j=1}^{n} \operatorname{Re} x_j\right|, \left|\sum_{j=1}^{n} \operatorname{Im} x_j\right|\right\} + \frac{1}{m} \sum_{k=1}^{m} \sum_{j=1}^{n} M_{jk}. \tag{5.3}$$

Finally, consider the Banach space $\left(\mathbb{C}, |\cdot|_{2p}\right)$ with $p \geq 1$.

Let $F : \mathbb{C} \to \mathbb{C}$, $F(z) = cz$. By Hölder's inequality, we have

$$\frac{|F(z)|}{|z|_{2p}} = \frac{|c| \sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2}}{\left(|\operatorname{Re} z|^{2p} + |\operatorname{Im} z|^{2p}\right)^{\frac{1}{2p}}} \leq 2^{\frac{1}{2} - \frac{1}{2p}} |c|.$$

Since, for $z_0 = 1 + i$ we have $|F(z_0)| = 2^{\frac{1}{2}} |c|$, $|z_0| = 2^{\frac{1}{2p}}$ $(p \geq 1)$, hence

$$\|F\|_{2p} := \sup_{z \neq 0} \frac{|F(z)|}{|z|_{2p}} = 2^{\frac{1}{2} - \frac{1}{2p}} |c|,$$

showing that F is a bounded linear functional on $\left(\mathbb{C}, |\cdot|_{2p}\right)$, $p \geq 1$ and $\|F\|_{2p} = 2^{\frac{1}{2} - \frac{1}{2p}} |c|$.

If we apply Theorem 3, then we can state the following proposition.

Proposition 3. *Let a_k, x_j, M_{jk} be as in Proposition 2. If*

$$\left[|\operatorname{Re} x_j|^{2p} + |\operatorname{Im} x_j|^{2p}\right]^{\frac{1}{2p}} \leq \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j + M_{jk}$$

for each $j \in \{1, \ldots, n\}$ *and* $k \in \{1, \ldots, m\}$, *then*

$$\sum_{j=1}^{n} \left[|\mathrm{Re}\, x_j|^{2p} + |\mathrm{Im}\, x_j|^{2p} \right]^{\frac{1}{2p}} \tag{5.4}$$
$$\leq \frac{2^{\frac{1}{2} - \frac{1}{2p}}}{m} \left| \sum_{k=1}^{m} a_k \right| \left[\left| \sum_{j=1}^{n} \mathrm{Re}\, x_j \right|^{2p} + \left| \sum_{j=1}^{n} \mathrm{Im}\, x_j \right|^{2p} \right]^{\frac{1}{2p}} + \frac{1}{m} \sum_{k=1}^{m} \sum_{j=1}^{n} M_{jk},$$

where $p \geq 1$.

Remark 4. If in the above proposition we choose $p = 1$, then we have the following reverse of the generalized triangle inequality for complex numbers

$$\sum_{j=1}^{n} |x_j| \leq \left| \frac{1}{m} \sum_{k=1}^{m} a_k \right| \left| \sum_{j=1}^{n} x_j \right| + \frac{1}{m} \sum_{k=1}^{m} \sum_{j=1}^{n} M_{jk}$$

provided x_j, a_k, $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$ satisfy the assumption

$$|x_j| \leq \mathrm{Re}\, a_k \cdot \mathrm{Re}\, x_j - \mathrm{Im}\, a_k \cdot \mathrm{Im}\, x_j + M_{jk}$$

for each $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$. Here $|\cdot|$ is the usual modulus of a complex number and $M_{jk} > 0, j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$ are given.

References

[1] E. Berkson, *Some types of Banach spaces, Hermitian systems and Bade functionals,* Trans. Amer. Math. Soc. **106** (1965), 376-385.
[2] J. B. Diaz and F. T. Metcalf, *A complementary triangle inequality in Hilbert and Banach spaces,* Proc. Amer. Math. Soc. **17** (1966), 88-97.
[3] S. S. Dragomir, *Semi-Inner Products and Applications*, Nova Science Publishers Inc., New York, 2004.
[4] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*, Preprint [http://front.math.ucdavis.edu/math.FA/0309354].
[5] S. S. Dragomir, *Reverses of the triangle inequality in inner product spaces,* RGMIA Res. Rep. Coll. **7** (2004), Supplement, Article 7. [http://rgmia.vu.edu.au/v7(E).html].
[6] S. Gudder and D. Strawther, *Strictly convex normed linear spaces,* Proc. Amer. Math. Soc. **59**(2) (1976), 263-267.
[7] G. Lumer, *Semi-inner product spaces,* Trans. Amer. Math. Soc. **100** (1961), 29-43.
[8] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, Dordrecht-Boston-London, 1993.

School of Computer Science and Mathematics
Victoria University of Technology
PO Box 14428, MCMC 8001
Victoria, Australia

E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.vu.edu.au/dragomir/