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MATCHING THEOREMS, FIXED POINT THEOREMS AND MINIMAX INEQUALITIES IN TOPOLOGICAL ORDERED SPACES

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Abstract. We obtain a generalization of the KKM theorem in topological ordered spaces. Applications concerning matching theorems, fixed point theorems, minimax inequalities are given.

1. Introduction

In [12], [13], by applying his infinite dimensional generalization [11] of the classical Knaster-Kuratowski-Mazurkiewicz (KKM) theorem [21], Fan obtained a minimax inequality and a matching theorem which have numerous applications to various and diverse branches of mathematics. Since then many generalizations of the results have been obtained in a topological vector space or hyperconvex metric space setting; see for example [1], [2], [3], [5], [9], [10], [13], [14], [15], [23], [25], [27] and [30]. In [14], Horvath obtained minimax inequalities by replacing convexity with contractibility in a topological space but only in a compact setting. These results are themselves generalizations of the well known theorems of Ky Fan. In 1996, by applying the classical KKM theorem and a well known result of Brown [7] in topological semilattices, Horvath and Llinares Ciscar [17] obtained a version of KKM theorem and Browder-Fan theorem in topological ordered spaces. In 2001, using Horvath and Llinares Ciscar's results and the methods from [17], [29] and [30], Luo [23] proved a new form of the KKM theorem and minimax inequality of Ky Fan in a noncompact setting for mappings taking values in a topological ordered space.

In this paper, first we shall employ the results and methods used in [2], [6], [9], [17] to obtain some generalizations of Horvath and Llinares Ciscar's results [17]. Next by applying these results, some matching theorems and fixed point theorems are obtained in a noncompact setting. Finally, some coincidence theorems and several general Ky Fan type minimax inequalities are presented. In particular, a Sion-Neumann type minimax theorem is given.

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2. Preliminaries

We need the following definitions from [17].

Definition 2.1. A partially ordered set $(X, \leqslant)$ is a sup-semilattice if any two elements x, y have a least upper bound $\sup\{x, y\}$. It is bounded if there is an element $1 \in X$ such that $x \leqslant 1$ for any $x \in X$. $(X, \leqslant)$ is a topological semilattice if X is a topological space and the mapping $X \times X \to X$, $(x, y) \mapsto \sup\{x, y\}$, is continuous.

Besides the above definition of a sup-semilattice, we can consider inf-semilattices. When no confusion can arise we will simply use the word semilattice. It is evident that each nonempty finite set A of X will have a least upper bound, denoted by $\sup A$.

In a partially ordered set $(X, \leqslant)$, two arbitrary elements x_1 and x_2 do not have to be comparable. In the case where $x_1 \leqslant x_2$, the set

$$[x_1, x_2] := \{y \in X : x_1 \leqslant y \leqslant x_2\}$$

is called an order interval.

Now assume that $(X, \leqslant)$ is a semilattice and $A \subseteq X$ is a nonempty finite subset. Then the set $\Delta(A) := \bigcup\limits_{a \in A} [a, \sup A]$ is well defined and one can see that it has the following properties:

1. $A \subseteq \Delta(A)$;
2. If $A \subseteq A'$ then $\Delta(A) \subseteq \Delta(A')$.

Definition 2.2. We say that a subset $E \subseteq X$ is Δ-convex if, for any nonempty finite subset $A \subseteq E$, $\Delta(A) \subseteq E$.

One can see without difficulty that E is Δ-convex if and only if the following conditions are satisfied:

(i) If $x_1, x_2 \in E$, then $\sup\{x_1, x_2\} \in E$;

(ii) If $x_1, x_2 \in E$ and $x_1 \leqslant x_2$, then $[x_1, x_2] \subseteq E$;

or if the following single condition holds:

(iii) If $x_1, x_2 \in E$, then $[x_1, \sup\{x_1, x_2\}] \subseteq E$.

Example 2.1. In $\mathbb{R}^2$, let

$$X := \{(x, y) : 0 \leqslant x \leqslant 1,\ y = 1\} \cup \{(x, y) : x = 1,\ 0 \leqslant y \leqslant 1\}.$$

The partial ordering of X is the usual partial ordering of $\mathbb{R}^2$. Then X is a topological semilattice.

This example have obvious generalizations to $\mathbb{R}^n$. Other examples can be seen in [17]. The following example is introduced in [23].

Example 2.2. In $\mathbb{R}^2$, let

$$X := \{(x, 1) : 0 \leqslant x < 1,\ y = 1\} \cup \{(x, y) : 0 \leqslant y \leqslant 1,\ x \geqslant 1,\ y \geqslant x - 1\}.$$

The partial ordering of $\mathbb{R}^2$ is defined by

$$(a,b) \leqslant (c,d) \Longleftrightarrow [c-a \geqslant 0,\ d-b \geqslant 0 \text{ and } d-b \leqslant c-a]\,.$$

Then X is Δ-convex.

For nonempty sets X and Y, we denote by 2^Y the family of all subsets of Y and $\langle X \rangle$ the family of all nonempty finite subsets of X. If $F : X \to 2^Y$, then $F^{-1},\ F^* : Y \to 2^X$ and $F^c : X \to 2^Y$ are defined by

$$F^{-1}(y) := \{x \in X : y \in F(x)\},\ F^*(y) := \{x \in X : y \notin F(x)\} \text{ and}$$
$$F^c(x) := \{y \in Y : y \notin F(x)\}.$$

We denote by Δ_n the standard n dimensional simplex with vertices $e_0, ..., e_n$. If J is a nonempty subset of $\{0, ..., n\}$, Δ_J denotes the convex hull of the vertices $\{e_j : j \in J\}$.

Let us recall some basic results that will be used in what follows.

Theorem 2.1 (KKM)**.** *Let $R_i \subseteq \Delta_n,\ i = 0, ..., n$. Assume that either all R_i are closed or all are open, and that for each nonempty subset J of $\{0, ..., n\}$,*

$$\Delta_J \subseteq \bigcup_{j \in J} R_j.$$

Then

$$\bigcap_{i=0}^{n} R_i \neq \emptyset.$$

In the orginal result, all the sets were supposed to be closed. The fact that they also can be open was shown by Kim [18], Shih [24] and Lassonde [22].

Definition 2.3. Let X be a topological semilattice and $F : D \to 2^X$ be a correspondence, where $D \subseteq X$. F is called a KKM map if for any nonempty finite subset $A \subseteq D$, we have

$$\bigcup_{x \in A} [x, \sup A] \subseteq \bigcup_{x \in A} F(x).$$

The proof of the following useful result is contained in the proof of Theorem 1 of [17], thus it is omittted.

Lemma 2.1. *Let X be a topological semilattice with path-connected intervals and $\{x_0, ..., x_n\}$ any nonempty finite subset of X. Then there exists a continuous map $f : \Delta_n \to X$ such that $f(\Delta_J) \subseteq \Delta(\{x_j : j \in J\})$ for any nonempty finite subset J of $\{0, ..., n\}$.*

The following theorem can be found in [16, 17].

Theorem 2.2. *Let X be a topological semilattice with path-connected intervals and $F : X \to 2^X$ a correspondence with nonempty values such that*

(a) *either all the values are closed or all are open;*

(b) *F is a KKM map.*

Then the family $\{F(x) : x \in X\}$ has the finite intersection property.

Throughout this paper, correspondence means multivalued mapping (multi-function).

3. Matching theorems

The following result is a variant of Theorem 1 of [17].

Lemma 3.1. *Let X be a topological semilattice with path-connected intervals and $\{R_i : i = 0, ..., n\}$ be a family of subsets of X. Suppose*

(a) *there exist points $x_0, ..., x_n$ of X such that, for any nonempty subset J of $\{0, ..., n\}$,*

$$\Delta(\{x_j : j \in J\}) \subseteq \bigcup_{j\in J} R_j;$$

(b) *the sets $\Delta(\{x_0, ..., x_n\}) \cap R_i,\ i = 0, ..., n,$ are either all open in $\Delta(\{x_0, ..., x_n\})$ or all closed in $\Delta(\{x_0, ..., x_n\})$.*
Then

$$\bigcap_{i=0}^{n} R_i \neq \emptyset.$$

Proof. By Lemma 2.1, there exists a continuous function $f : \Delta_n \to X$ such that $f(\Delta_J) \subseteq \Delta(\{x_j : j \in J\})$ for any nonempty finite subset J of $\{0, ..., n\}$. For each $i = 0, ..., n$, let $S_i := f^{-1}(\Delta(\{x_0, ..., x_n\}) \cap R_i)$. Then the sets $S_i,\ i \in \{0, ..., n\}$ are either all closed in Δ_n or all open in Δ_n. For each nonempty subset J of $\{0, ..., n\}$, we have

$$\begin{aligned}\bigcup_{j\in J} S_j &= f^{-1}\Big(\Delta(\{x_0, ..., x_n\}) \cap (\bigcup_{j\in J} R_j)\Big)\\ &\supset f^{-1}\Big(\Delta(\{x_0, ..., x_n\}) \cap \Delta(\{x_j : j \in J\})\Big)\\ &= f^{-1}\Big(\Delta(\{x_j : j \in J\})\Big) \supset \Delta_J.\end{aligned}$$

Therefore,

$$conv\{e_j : j \in J\} \subseteq \bigcup_{j\in J} S_j.$$

Using the "closed" variant, respectively the "open" variant of the KKM map principle we obtain $\bigcap_{i=0}^{n} S_i \neq \emptyset$. Taking any $\mu \in \bigcap_{i=0}^{n} S_i$, we have

$$f(\mu) \in \bigcap_{i=0}^{n} \Big(\Delta(\{x_0, ..., x_n\}) \cap R_i\Big),$$

so $\bigcap_{i=0}^{n} R_i \neq \emptyset$. □

From Lemma 3.1 we have the following KKM map principle which generalizes Theorem 2.2.

Theorem 3.1. *Let X be a topological semilattice with path-connected intervals and $F : X \to 2^X$ be a correspondence with nonempty values such that*

(a) *F is KKM, i.e., for any nonempty finite subset $A \subseteq X$,*

$$\bigcup_{x\in A} [x, \sup A] \subseteq \bigcup_{x\in A} F(x);$$

(b) *the sets $F(x) \cap \Delta(A)$, $x \in X$, are either all closed in $\Delta(A)$ or all open in $\Delta(A)$ for each $A \in \langle X\rangle$.*

Then the family $\{F(x) : x \in X\}$ has the finite intersection property.

Remark 3.1. If the sets $F(x) \cap \Delta(A)$, $x \in X$, are all closed in $\Delta(A)$ for each $A \in \langle X\rangle$, $F(x_0)$ is compact for some $x_0 \in X$ and for each $x \in X$, $F(x_0) \cap F(x)$ is closed in $F(x_0)$, then $\bigcap_{x\in X} F(x) \neq \emptyset$.

As an application of Lemma 3.1, we have the following matching theorem.

Theorem 3.2. *Let X be a topological semilattice with path-connected intervals and $A_1, ..., A_n$ be n subsets of X. Assume that the sets A_i, $i = 1, ..., n$, are either all closed or all open such that $\bigcup_{i=1}^{n} A_i = X$. Then for any n points $x_1, ..., x_n$ (not necessarily distinct) of X, there exists a nonempty subset J_0 of $\{1, ..., n\}$ such that*

$$\Delta\big(\{x_j : j \in J_0\}\big) \cap \big(\bigcap_{j\in J_0} A_j\big) \neq \emptyset.$$

Proof. Suppose the conclusion of the theorem was false. Then we would have

$$\Delta\big(\{x_j : j \in J\}\big) \cap \big(\bigcap_{j\in J} A_j\big) = \emptyset$$

for each nonempty subset J of $\{1, ..., n\}$. For each $j = 1, ..., n$, let $G_j := X \setminus A_j$. Then the sets G_j, $j \in \{1, ..., n\}$, are either all open in X or all closed in X. It follows that $\Delta\big(\{x_j : j \in J\}\big) \subseteq \bigcup_{j\in J} G_j$ for each nonempty subset J of $\{1, ..., n\}$. By Lemma 3.1 we have $\bigcap_{j=1}^{n} G_j \neq \emptyset$, which contradicts the assumption $\bigcup_{i=1}^{n} A_i = X$. This completes the proof. □

Next, by using Theorem 3.2 we obtain the Berge [4] and Klee [20] type intersection theorem.

Theorem 3.3. *Let X be a topological semilattice with path-connected intervals and $Y_1, ..., Y_n$ be Δ-convex subsets of X such that every $n - 1$ members of them have a common point and let $\{Z_i : 1 \leqslant i \leqslant n\}$ be a covering of X having all its* *members closed or all open. Then there exists a nonempty subset I of $\{1, ..., n\}$ such that*

$$\cap\{Y_j : j \in \bar{I}\ \} \cap (\cap\{Z_i : i \in I\}) \neq \emptyset,$$

where $\bar{I}$ denotes the complement of I in $\{1, ..., n\}$.

Proof. For each $i \in \{1, ..., n\}$, select a single point x_i in the intersection $\bigcap_{j\neq i} Y_j$ and let $D := \{x_1, ..., x_n\}$ be the set of all selected points. By Theorem 3.2 there exists a nonempty subset I of $\{1, ..., n\}$ such that

$$\Delta\big(\{x_i : i \in I\}\big) \cap \big(\cap \{Z_i : i \in I\}\big) \neq \emptyset.$$

Since $x_i \in \cap\{Y_j : j \in \bar{I}\}$ for each $i \in I$ and the set $\cap\{Y_j : j \in \bar{I}\}$ is Δ-convex it follows that $\Delta(\{x_i : i \in I\}) \subset \cap\{Y_j : j \in \bar{I}\}$ and thereby

$$\cap\{Y_j : j \in \bar{I}\} \cap (\cap\{Z_i : i \in I\}) \neq \emptyset.$$

This completes the proof. □

As corollaries of Theorem 3.1, we have the following.

Theorem 3.4. *Let X be a topological semilattice with path-connected intervals and $F : X \to 2^X$ be a correspondence with nonempty values such that*

(a) $\bigcup_{x\in X} F(x) = X$;

(b) *for some $x_0 \in X$, $F^c(x_0)$ is compact and for each $x \in X$, $F^c(x_0) \cap F^c(x)$ is closed in $F^c(x_0)$;*

(c) *for each $x \in X$ and for each $A \in \langle X\rangle$, $\Delta(A) \cap F^c(x)$ is closed in $\Delta(A)$.*

Then there exists $A \in \langle X\rangle$ such that $\Delta(A) \cap (\bigcap_{x\in A} F(x)) \neq \emptyset$.

Proof. Suppose the assertion of the theorem was false, then for each $A \in \langle X\rangle$ we would have

$$\Delta(A) \cap (\bigcap_{x\in A} F(x)) = \emptyset,$$

so

$$\Delta(A) \subseteq X \setminus \bigcap_{x\in A} F(x) = \bigcup_{x\in A} F^c(x).$$

Define $G : X \to 2^X$ by $G(x) := F^c(x)$ for each $x \in X$, then G is a KKM map. By (c), for each $x \in X$ and for each $A \in \langle X\rangle$, $\Delta(A) \cap G(x)$ is closed in $\Delta(A)$. Thus by Theorem 3.1 the family $\{G(x) : x \in X\}$ has the finite intersection property. By (b), $G(x_0)$ is compact and for each $x \in X$, $G(x_0) \cap G(x)$ is closed in $G(x_0)$. Thus by Remark 3.1 we have $\bigcap_{x\in X} G(x) \neq \emptyset$ which contradicts (a). Hence the assertion must hold. □

Theorem 3.5. *Let X be a topological semilattice with path-connected intervals and $F,\ G : X \to 2^X$ be correspondences with nonempty values such that*

(a) *for each $x \in X$, $F(x) \subseteq G(x)$ and $x \in F(x)$;*

(b) *for each $x \in X$, $F^*(x)$ is Δ-convex ;*

(c) *for some $x_0 \in X$, $G(x_0)$ is compact and for each $x \in X$, $G(x_0) \cap G(x)$ is closed in $G(x_0)$;*

(d) *for each $x \in X$ and for each $A \in \langle X \rangle$, $\Delta(A) \cap G(x)$ is closed in $\Delta(A)$.*

Then $\bigcap_{x \in X} G(x) \neq \emptyset$.

Proof. By Theorem 3.1, we only need to show that G is a KKM map. If G were not KKM, then there would exist $A \in \langle X \rangle$ such that $\Delta(A)$ is not contained in $\bigcup_{x \in A} G(x)$; let $y \in \Delta(A)$ be such that $y \notin \bigcup_{x \in A} G(x)$. It follows that $A \subseteq G^*(y) \subseteq F^*(y)$ by (a) so that $\Delta(A) \subseteq F^*(y)$ by (b). As $y \in \Delta(A)$, we must have $y \in F^*(y)$ so that $y \notin F(y)$ which contradicts (a). This completes the proof. □

The next statement is an immediate consequence of Theorem 3.1.

Corollary 3.1. *Let X be a topological semilattice with path-connected intervals and $F : X \to 2^X$ a correspondence with nonempty values such that*

(a) *F is a KKM map;*

(b) *for some $x_0 \in X$, $F(x_0)$ is compact and for each $x \in X$, $F(x)$ is closed in X.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

4. Fixed point theorems

We first shall apply Lemma 3.1 to obtain the following fixed point theorem.

Theorem 4.1. *Let X be a topological semilattice with path-connected intervals, $x_0, ..., x_n \in X$ and $S,\ T : X \to 2^X$ be correspondences with nonempty values such that*

(a) *for each $i = 0, ..., n$, $S(x_i) \subseteq T(x_i)$;*

(b) *for each $i = 0, ..., n$, $\Delta(\{x_0, ..., x_n\}) \cap S(x_i)$ is closed in $\Delta(\{x_0, ..., x_n\})$;*

(c) *for each nonempty subset A of $\{x_0, ..., x_n\}$ with $A \subseteq T^{-1}(y)$ for some $y \in X$, $\Delta(A) \subseteq T^{-1}(y)$;*

(d) *$\bigcup_{i=0}^{n} S(x_i) = X$.*

Then there exists $x^* \in X$ such that $x^* \in T(x^*)$.

Proof. For each $x \in X$, let $F(x) := T^c(x)$ and $G(x) := S^c(x)$. We suppose that $\Delta(A) \subseteq \bigcup_{x \in A} G(x)$ for each nonempty subset A of $\{x_0, ..., x_n\}$. By (b), for each $i = 0, ..., n$, $\Delta(\{x_0, ..., x_n\}) \cap G(x_i)$ is open in $\Delta\big(\{x_0, ..., x_n\}\big)$. By Lemma 3.1,

$\bigcap\limits_{i=0}^{n} G(x_i) \neq \emptyset$, which contradicts (d). Thus, there exists a nonempty subset A of $\{x_0, ..., x_n\}$ such that $\Delta(A)$ is not contained in $\bigcup\limits_{x \in A} G(x)$. Take any $x^* \in \Delta(A)$ with $x^* \notin \bigcup\limits_{x \in A} G(x)$. It follows that for each $x \in A$, $x^* \in S(x) \subseteq T(x)$ by (a) so that $x \in T^{-1}(x^*)$. Therefore $A \subseteq T^{-1}(x^*)$ and hence $\Delta(A) \subseteq T^{-1}(x^*)$ by (c). As $x^* \in \Delta(A)$, we have $x^* \in T^{-1}(x^*)$, which implies $x^* \in T(x^*)$. The proof is complete. □

Theorem 4.2. *Let X be a topological semilattice with path-connected intervals and $S,\ T : X \to 2^X$ be correspondences with nonempty values such that*

(a) *for each $x \in X,\ S(x) \subseteq T(x)$;*

(b) $\bigcup\limits_{x \in X} S(x) = X$;

(c) *for some $x_0 \in X,\ S^c(x_0)$ is compact and for each $x \in X,\ S^c(x_0) \cap S^c(x)$ is closed in $S^c(x_0)$;*

(d) *for each $x \in X$ and for each $A \in \langle X \rangle,\ \Delta(A) \cap S^c(x)$ is closed in $\Delta(A)$;*

(e) *for each $x \in X,\ T^{-1}(x)$ is Δ-convex.*

Then there exists $x^* \in X$ such that $x^* \in T(x^*)$.

Proof. By Theorem 3.4, there exists $A \in \langle X \rangle$ such that

$$\Delta(A) \cap \left(\bigcap_{x \in A} S(x) \right) \neq \emptyset.$$

Take any $x^* \in \Delta(A) \cap \big(\bigcap\limits_{x \in A} S(x) \big)$. Then $x^* \in \Delta(A)$ and $A \subseteq S^{-1}(x^*) \subseteq T^{-1}(x^*)$ by (a). By (e), $\Delta(A) \subseteq T^{-1}(x^*)$. Therefore $x^* \in T^{-1}(x^*)$, and so $x^* \in T(x^*)$. The proof is complete. □

The following is an immediate consequence of Theorem 4.2.

Corollary 4.1. *Let X be a topological semilattice with path-connected intervals and $S,\ T : X \to 2^X$ be correspondences with nonempty values such that*

(a) *for each $x \in X,\ S(x) \subseteq T(x)$;*

(b) $\bigcup\limits_{x \in X} S(x) = X$;

(c) *for some $x_0 \in X,\ S^c(x_0)$ is compact and for each $x \in X,\ S(x)$ is open in X;*

(d) *for each $x \in X,\ T^{-1}(x)$ is Δ-convex.*

Then there exists $x^* \in X$ such that $x^* \in T(x^*)$.

Based on Theorem 3.2 we shall extend the Browder-Fan type fixed point theorems in [16, 17].

Theorem 4.3. *Let X be a topological semilattice with path-connected intervals and $T : X \to 2^X$ be a correspondence such that*

(a) *$T(x)$ is Δ-convex for each $x \in X$;*
(b) *there exists a finite set $D \in \langle X \rangle$ satisfying*
 (b1) *$T(x) \cap D \neq \emptyset$ for each $x \in X$;*
 (b2) *for all $y \in D$, the fibers $T^{-1}(y)$ are either all closed or all open.*

Then there exists $x^* \in X$ such that $x^* \in T(x^*)$.

Proof. From (b1), $X = \bigcup\{T^{-1}(y) : y \in D\}$. By Theorem 3.2, there are a nonempty set $A \subset D$ and a point x^* such that

$$x^* \in \Delta(A) \cap \big(\cap \{T^{-1}(y) : y \in A\}\big).$$

From $x^* \in \cap\{T^{-1}(y) : y \in A\}$ it follows that $A \subset T(x^*)$ and by (a) we have

$$x^* \in \Delta(A) \subset T(x^*).$$

The proof is complete. □

The following corollary of Theorem 4.3 was proved in [17].

Corollary 4.2. *Let X be a compact topological semilattice with path-connected intervals and $T : X \to 2^X$ be a correspondence such that*

(a) *for each $x \in X$, $T(x)$ is nonempty and Δ-convex ;*
(b) *for each $y \in X$, the fiber $T^{-1}(y)$ is open.*

Then there exists $x^* \in X$ such that $x^* \in T(x^*)$.

Proof. Observe that $X = \cup\{T^{-1}(y) : y \in X\}$ so, by compactness, there exist $y_1, ..., y_n \in X$ such that $X = \bigcup_{i=1}^{n} T^{-1}(y_i)$. It suffices to set $D = \{y_1, ..., y_n\}$. □

From Theorem 4.3, we prove the following Fan type section theorem.

Theorem 4.4. *Let X be a topological semilattice with path-connected intervals and let E be a subset of $X \times X$, having the following properties:*

(a) *$(x, x) \in E$ for all $x \in X$;*
(b) *for each $x \in X$ the set $\{y \in X : (x, y) \notin E\}$ is Δ-convex ;*
(c) *the sets $\{x \in X : (x, y) \in E\}$ are either (c_1) all closed or (c_2) all open, for $y \in X$.*

Then for every nonempty finite set $D \subset X$ there exists an element $x_D \in X$ such that $\{x_D\} \times D \subset E$.

Proof. Suppose that the assertion of the theorem is false. Then there exists a nonempty finite set $D \subset X$ such that

$$\{x\} \times D \not\subset E \text{ for every } x \in X.$$

Define a map $T : X \to 2^X$ by $T(x) := \{y \in X : (x, y) \notin E\}$. Then for each $x \in X$, $T(x)$ is Δ-convex, $T(x) \cap D \neq \emptyset$, and we have the fibers

$$T^{-1}(y) = X \setminus \{x \in X : (x, y) \in E\}$$

which are either all open (in case (c_1)), or all closed (in case (c_2)).

By Theorem 4.3, there exists $x^* \in X$ such that $x^* \in T(x^*)$, hence $(x^*, x^*) \notin E$, which contradicts (a). The proof is complete. □

Corollary 4.3. *Let X be a compact topological semilattice with path-connected intervals and let E be a subset of $X \times X$ satisfying the conditions (a), (b) and (c_1) in Theorem 4.4. Then there exists $x^* \in X$ such that $\{x^*\} \times X \subset E$.*

5. Minimax inequalities

In this section, first we use Theorem 3.2 to obtain the following coincidence theorem.

Theorem 5.1. *Let X be a topological semilattice with path-connected intervals, $S : X \to 2^X$ a KKM map, and $T : X \to 2^X$ a map. Suppose that there exists a nonempty finite set $D \subset X$ such that*

(a) *$T(x) \cap D \neq \emptyset$ for all $x \in X$;*

(b) *for all $y \in D$, the fibers $T^{-1}(y)$ are either all closed or all open.*

Then there exists an element $x_0 \in X$ such that $T(x_0) \cap S^{-1}(x_0) \neq \emptyset$.

Proof. From (a), $X = \cup\{T^{-1}(y) : y \in D\}$. By Theorem 3.2 for the closed (open respectively) covering $\{T^{-1}(y) : y \in D\}$ there exists a nonempty set $A \subset D$ and an element $x_0 \in \Delta(A) \cap (\cap\{T^{-1}(y) : y \in A\})$.

Since S is a KKM map, $x_0 \in \Delta(A) \subset \cup\{S(y) : y \in A\}$ hence for at least one $y_0 \in A$, we have $x_0 \in S(y_0)$, so $y_0 \in S^{-1}(x_0)$. On the other hand, $x_0 \in T^{-1}(y_0)$ implies $y_0 \in T(x_0)$. Therefore $T(x_0) \cap S^{-1}(x_0) \neq \emptyset$ and the proof is complete. □

Corollary 5.1. *Let X be a compact topological semilattice with path-connected intervals, $S : X \to 2^X$ a KKM map, and $T : X \to 2^X$ a map. Suppose that for each $x \in X$, $T(x)$ is a nonempty subset of X and $T^{-1}(x)$ is open. Then there exists $x_0 \in X$ such that $T(x_0) \cap S^{-1}(x_0) \neq \emptyset$.*

From this corollary, we get the following Fan type minimax inequality.

Theorem 5.2. *Let X be a compact topological semilattice with path-connected intervals and f, g be real-valued functions defined on $X \times X$ such that*

(a) *$f(x, y) \leqslant g(x, y)$ for all $(x, y) \in X \times X$;*

(b) *for each $x \in X$, $f(x, .)$ is a lower semicontinuous function on X;*

(c) *for each $y \in X$, the set $\{x \in X : g(x, y) > 0\}$ is Δ-convex ;*

(d) *$g(x, x) \leqslant 0$ for all $x \in X$.*

Then there exists $y_0 \in X$ such that $f(x, y_0) \leqslant 0$ for all $x \in X$.

Proof. Suppose that the conclusion of the theorem is false, i.e, for each $y \in X$ the set $\{x \in X : f(x,y) > 0\}$ is nonempty. Define the maps $S,\ T : X \to 2^X$ by

$$S(x) := \{y \in X : g(x,y) \leqslant 0\} \text{ and } T(y) := \{x \in X : f(x,y) > 0\},\ x,\ y \in X.$$

We shall show that S is a KKM map. Suppose that there exists a nonempty finite set $D \subset X$ and a point $y \in \Delta(D) \setminus \cup\{S(x) : x \in D\}$. Then $g(x,y) > 0$ for each $x \in D$ and by (c), $\Delta(D) \subset \{x \in X : g(x,y) > 0\}$. Hence we get $g(y,y) > 0$, a contradiction to (d). By hypothesis, for each $x \in X$, $T^{-1}(x) = \{y \in X : f(x,y) > 0\}$ is open. Thus, by Corollary 5.1, there exist $x_0,\ y_0$ such that $x_0 \in T(y_0)$ and $y_0 \in S(x_0)$. These relations and (a) lead to the following contradiction

$$0 < f(x_0,y_0) \leqslant g(x_0,y_0) \leqslant 0.$$

Hence the conclusion of the theorem must hold. □

Definition 5.1. Let X be a topological semilattice and f be a real-valued function on X. Then f is said to be Δ-quasiconcave (resp. Δ-quasiconvex) on X if the set $\{x \in X : f(x) > r\}$ (resp. $\{x \in X : f(x) < r\}$) is Δ-convex for every $r \in \mathbb{R}$.

Remark 5.1. Note that the condition (c) in the above theorem is implied by the following condition:
(c') for each $y \in X$, $g(.,y)$ is a Δ-quasiconcave function on X.

If we put aside the condition (d) and replace the condition (c) by (c') then the conclusion of Theorem 5.2 can be given by the following minimax inequality:

$$\inf_{y\in X} \sup_{x\in X} f(x,y) \leqslant \sup_{x\in X} g(x,x).$$

The next result is a variant of Theorem 5.2 and it admits a similar proof by using Theorem 5.1.

Theorem 5.3. *Let X be a compact topological semilattice with path-connected intervals and $f,\ g$ be real-valued functions defined on $X \times X$ such that*

(a) *$f(x,y) \leqslant g(x,y)$ for all $(x,y) \in X \times X$;*

(b) *for each $x \in X$, $f(x,.)$ is an upper semicontinuous function on X;*

(c) *for each $y \in X$, the set $\{x \in X : g(x,y) \geqslant 0\}$ is Δ-convex ;*

(d) *$g(x,x) < 0$ for all $x \in X$.*

Then for every nonempty finite set $D \subset X$ there exists a point $y_D \in X$ such that $f(x,y_D) < 0$ for all $x \in D$.

Finally, we prove a version of the Sion-Neumann type minimax theorem in topological ordered spaces. First, we shall prove a new coincidence theorem and then using this result, von Neumann-Sion's minimax equality is obtained.

We shall need the following selection theorem of [17].

Theorem 5.4. *Let X be a compact topological space, Y a topological semilattice and $F : X \to 2^Y$ a correspondence such that*

(a) *for each $x \in X$, $F(x)$ is nonempty and Δ-convex ;*

(b) *for each* $y \in Y$, $F^{-1}(y)$ *is open in* X.

Then there is a continuous function $f : X \to Y$ *such that for any* $x \in X$ *we have* $f(x) \in F(x)$.

We have the following coincidence theorem.

Theorem 5.5. *Let* X *be a compact topological semilattice with path-connected intervals,* Y *a topological semilattice and* $A : X \to 2^Y$, $B : Y \to 2^X$ *correspondences such that*

(a) *for each* $x \in X$, $A(x)$ *is nonempty and* Δ*-convex,* $B^{-1}(x)$ *is open in* Y;

(b) *for each* $y \in Y$, $A^{-1}(y)$ *is open in* X, $B(y)$ *is nonempty and* Δ*-convex.*

Then there exists an element x_0 *such that* $A(x_0) \cap B^{-1}(x_0) \neq \emptyset$.

Proof. From Theorem 5.4, A has a continuous selection $f : X \to Y$ such that $f(x) \in F(x)$ for any $x \in X$.

We define a new mapping $R : X \to 2^X$ by setting

$$R(x) := B(f(x)).$$

One can see that R has the following properties:

(i) for each $x \in X$, $R(x)$ is nonempty and Δ-convex by hypothesis of B;

(ii) for each $x \in X$, $R^{-1}(x) = f^{-1}(B^{-1}(x))$ is open by the continuity of f and hypothesis of B.

Hence by Corollary 4.2, there exists $x^* \in X$ such that $x^* \in R(x^*)$. Then $x^* \in B(f(x^*))$, i.e., $f(x^*) \in B^{-1}(x^*)$. Moreover, $f(x^*) \in A(x^*)$, so $y^* \in A(x^*) \cap B^{-1}(x^*)$ with $y^* := f(x^*)$ and the proof is complete. □

Now we are in a position to prove a Sion-Neumann type minimax theorem.

Theorem 5.6. *Let* X *be a compact topological semilattice with path-connected intervals,* Y *a topological semilattice. Let* f, $g : X \times Y \to \mathbb{R}$ *be functions such that*

(a) $f(x, y) \leqslant g(x, y)$ *for all* $(x, y) \in X \times Y$;

(b) *for all* $x \in X$, $g(x, .)$ *is* Δ*-quasiconvex on* Y *and* $f(x, .)$ *is lower semicontinuous on* Y;

(c) *for all* $y \in Y$, $g(., y)$ *is upper semicontinuous on* X *and* $f(., y)$ *is* Δ*-quasiconcave on* X.

Then the following inequality holds:

$$\inf_{y\in Y} \sup_{x\in X} f(x, y) \leqslant \sup_{x\in X} \inf_{y\in Y} g(x, y).$$

Proof. Arguing by contradiction, suppose that there exists a real number r such that

$$\inf_Y \sup_X f > r > \sup_X \inf_Y g.$$

This implies that multifunctions $A : X \to 2^Y$ and $B : Y \to 2^X$, defined by

$$A(x) := \{y \in Y : g(x, y) < r\} \quad \text{and} \quad B(y) := \{x \in X : f(x, y) > r\},$$

have nonempty values. Moreover, the values of A and B are Δ-convex, because g and f are Δ-quasiconvex on Y and Δ-quasiconcave on X, respectively. Since $A^{-1}(y) = \{x \in X : g(x, y) < r\}$ and $B^{-1}(x) := \{y \in Y : f(x, y) > r\}$, we find that for each $y \in Y$, $A^{-1}(y)$ is open because g is upper semicontinuous on X. Similarly, $B^{-1}(x)$ is open for each $x \in X$. Then by Theorem 5.5 there exists some (x_0, y_0) with $y_0 \in A(x_0) \cap B^{-1}(x_0)$, which gives $g(x_0, y_0) < r < f(x_0, y_0)$. This contradicts (a). Hence the conclusion of the theorem must hold. □

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