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ON CASTELNUOVO-MUMFORD REGULARITY OF PRODUCTS OF MONOMIAL IDEALS

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Abstract. In this paper, we prove the formula

$$\operatorname{reg}(IJ) \leq \operatorname{reg}(I) + \operatorname{reg}(J),$$

where I is generated by a regular sequence consisting of two monomials and J is generated by an arbitrary regular sequence of monomials.

1. Introduction

Let $R = k[x_1, x_2, \ldots, x_u]$ be a polynomial ring over a field k and $I \subset R$ a homogeneous ideal. The Castelnuovo-Mumford regularity of I governs the degrees appearing in a minimal graded free resolution of I. Conca and Herzog ([C-H]) raised the following question: Whether it is true that

$$(*) \qquad \operatorname{reg}(I_1 I_2 \ldots I_d) \leq \operatorname{reg}(I_1) + \operatorname{reg}(I_2) + \ldots + \operatorname{reg}(I_d),$$

when each I_i is generated by a regular sequence? In the case I is an arbitrary monomial ideal, Hoa and Trung used the polarization method to find bounds for $\operatorname{reg}(I^d)$ (see [H-T]). Based on this method and a formula reduced by Hochster's Theorem, we prove (*) in the case $d = 2$, I_1 is generated by two monomials and I_2 is an arbitrary monomial ideal (see Main Theorem).

2. Preliminaries

First, we introduce some conventions. There are several characterizations for the Castelnuovo-Mumford regularity of a finitely generated graded R-module M (see [E-G]). We use here the following definition

$$\operatorname{reg}(M) := \min\{t \mid H^i_{\mathfrak{m}}(M)_{n-i} = 0 \text{ for all } n > t \text{ and } i \geq 0\},$$

where $H^i_{\mathfrak{m}}(M)_{n-i}$ denotes the $(n-i)$-th graded part of the i-th cohomology module of M with respect to the maximal graded ideal $\mathfrak{m}$ of R.

Let I be a monomial ideal in R. To estimate $\operatorname{reg}(I)$, we will need the technique of polarization to reduce I to a squarefree monomial ideal (see [S-V, Chap. 2]). Let $S := k[X]$, where $X = \{x_{ij} \mid 1 \leq i \leq u \text{ and } 1 \leq j \leq N_i\}$ and $N_i \gg 0$. Let h be a monomial $x_1^{\mu_1} x_2^{\mu_2} \ldots x_u^{\mu_u}$ of R. Then we define the *polarization of* h to be a

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squarefree monomial

$$p(h) := \prod_{i=1}^{u}\prod_{j=1}^{\mu_i} x_{ij}.$$

Let $m_1, m_2, \ldots, m_s$ form a minimal basis of I. Then the *polarization* of I is the squarefree monomial ideal of S defined by

$$p(I) := \big(p(m_1), p(m_2), \ldots, p(m_s)\big).$$

The following result gives a relation between the regularity of I and $p(I)$ (see [P, Chap. 3, Proposition 5]).

Proposition 2.1. $\mathrm{reg}(I) = \mathrm{reg}(p(I))$.

We may view $S/p(I)$ as the Stanley-Reisner ring $k[\Delta]$ of a simplicial complex Δ, where

$$\Delta := \Big\{\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \subseteq X \mid x_{i_1}x_{i_2}\ldots x_{i_r} \notin p(I)\Big\}.$$

Note that $k[\Delta]$ is a $\mathbb{Z}^q$-graded algebra and

$$k[\Delta]_s = \bigoplus_{a\in\mathbb{Z}^q,\, |a|=s} k[\Delta]_a$$

for all $s \in \mathbb{Z}$, where $|a| = a_1 + a_1 + \ldots + a_q$ and $q = \sharp X$. If F is a subset of X, then we define a subcomplex of Δ as follow:

$$lk_\Delta F := \{G \subseteq X \mid F \cap G = \emptyset, F \cup G \in \Delta\}.$$

The local cohomology modules of $k[\Delta]$ with respect to the maximal graded ideal $\mathfrak{m}$ of S can be computed by reduced homology groups of simplicial complexes with values in k from a formula due to Hochster [B-H, Theorem 5.3.8]. From this, we can deduce the following characterization for the Castelnuovo-Mumford regularity of $k[\Delta]$.

Proposition 2.2. *Let Δ be a simplicial complex on X and α a non-negative integer number. Then*

$$\mathrm{reg}(k[\Delta]) = \min\{\alpha \mid \widetilde{H}_t(lk_\Delta F) = 0 \text{ for all } t \geq \alpha, F \subseteq X\}.$$

Corollary 2.1. $\mathrm{reg}(k[\Delta]) \leq \dim(\Delta) + 1$.

3. MAIN THEOREM

The main result of the present paper is:

Main Theorem. *Let I, J be two ideals generated by regular sequences of monomials in R and $\mu(I) = 2$. Then*

$$\mathrm{reg}(IJ) \leq \mathrm{reg}(I) + \mathrm{reg}(J).$$

Lemma 3.1. *Let $I = (f_1, f_2, \ldots, f_n)$ and $J = (g_1, g_2, \ldots, g_m)$ be two ideals generated by regular sequences of monomials in R. Then*

$$\mathrm{reg}(p(p(I)p(J))) = \mathrm{reg}(IJ).$$

Proof. For each of $1 \le t \le u$ let a_t (resp. b_t) be the maximal degree of x_t in $f_1, f_2, \dots, f_n$ (resp. $g_1, g_2, \dots, g_m$). Let $m_t := \min(a_t, b_t)$ and $M_t := \max(a_t, b_t)$. We may assume that

$$p(IJ) \subseteq S = k[X_{1,1}, X_{1,2}, \dots, X_{1,a_1+b_1}; \dots ; X_{u,1}, X_{u,2}, \dots, X_{u,a_u+b_u}]$$

and

$$p(I), p(J) \subseteq k[X_{1,1}, X_{1,2}, \dots, X_{1,M_1}; \dots ; X_{u,1}, X_{u,2}, \dots, X_{u,M_u}].$$

Since $X_{t,r}$ appears in $f_i g_j$ with the degree at most one for each $r > m_t$, we may assume that

$$\begin{aligned} p(p(I)p(J)) \subseteq S' = k[&X_{1,1}, Y_{1,1}, \dots, X_{1,m_1}, Y_{1,m_1}, X_{1,m_1+1}, \dots, X_{1,M_1}; \dots ; \\ &X_{u,1}, Y_{u,1}, \dots, X_{u,m_u}, Y_{u,m_u}, X_{u,m_u+1}, \dots, X_{u,M_u}]. \end{aligned}$$

Note that $a_t + b_t = m_t + M_t$ for all t. Define an isomorphism $\varphi : S \longrightarrow S'$ as follows:

$$\varphi(X_{t,i}) = \begin{cases} X_{t,i} & \text{for } i = 1, \dots, M_t \\ Y_{t,i-M_t} & \text{for } i = M_t + 1, \dots, M_t + m_t \end{cases}$$

Let $i \neq i', j \neq j'$. By the condition of regular sequence, $\gcd(f_i, f_{i'}) = \gcd(g_j, g_{j'}) = 1$. Since $f_{i'} g_{j'} \mid f_i g_j$ if and only if $f_{i'} \mid g_j$ and $g_{j'} \mid f_i$, it follows that $f_i g_j$ is a minimal generator of IJ if and only if $p(f_i)p(g_j)$ is a minimal generator of $p(I)p(J)$. Moreover, if x_t appears both in f_i and g_j then we must have $\deg_{x_t}(f_i) = a_t$ and $\deg_{x_t}(g_j) = b_t$. From this statement if $X_{t,r} \mid p(f_i g_j)$ then $\varphi(X_{t,r}) \mid p(p(f_i)p(g_j))$. This implies that $\varphi(p(f_i g_j)) = p(p(f_i)p(g_j))$ for all i, j. Thus $\varphi(p(IJ)) = p(p(I)p(J))$. So the assertion is proved by virtue of Proposition 2.1. ☐

We put $\deg f_i = a_i$ and $\deg g_j = b_j$. Then

$$\begin{aligned} \operatorname{reg}(I) &= a_1 + a_2 - 1 \\ \operatorname{reg}(J) &= b_1 + b_2 + \cdots + b_m - m + 1. \end{aligned}$$

By Lemma 3.1, we only need to study product of squarefree monomial ideals. Moreover, it would not make any difference if we replace $p(IJ)$ by the ideal of $k[X]$ generated by $p(f_i g_j)$ for all i, j. We may assume that each of variable appears in $p(f_i g_j)$ for some i, j. Then

$$\sharp X = a_1 + a_2 + b_1 + b_2 + \cdots + b_m.$$

For $i = 1, 2$, let J_i be the ideal generated by the monomials $\dfrac{p(f_i g_j)}{p(f_i)}$, $j = 1, 2, \dots, m$. We may view J_i as a copy of J. In particular, $\operatorname{reg}(J_i) = \operatorname{reg}(J)$. Let Δ_I be the simplicial complex corresponding to the squarefree monomial ideal I.

Lemma 3.2. *Let $\Delta_1 = \Delta_{(p(f_1))}$, $\Delta_2 = \Delta_{(p(f_2))}$, $\Delta_3 = \Delta_{J_1}$ and $\Delta_4 = \Delta_{J_2}$. Then*

$$\Delta_{p(IJ)} = (\Delta_1 \cup \Delta_3) \cap (\Delta_2 \cup \Delta_4).$$

Proof. It is obvious that

$$p(IJ) = \big(p(f_1)J_1, p(f_2)J_2\big).$$

Let h be a squarefree monomial in S. Then $h \notin p(IJ)$ iff $h \notin p(f_1)J_1$ and $h \notin p(f_2)J_2$. From this the statement immediately follows. □

To compute the reduced homology of $\Delta_{p(IJ)}$, we need to use the *reduced Mayer-Vietoris sequence.*

Lemma 3.3 (see [St]). *Let δ be a simplicial complex and δ_1, δ_2 subcomplexes of δ. Then there is an exact sequence*

$$\cdots \to \widetilde{H}_q(\delta_1 \cap \delta_2) \to \widetilde{H}_q(\delta_1) \oplus \widetilde{H}_q(\delta_2) \to \widetilde{H}_q(\delta_1 \cup \delta_2) \to \widetilde{H}_{q-1}(\delta_1 \cap \delta_2) \to \cdots$$

We shall prove the vanishing of some reduced homology groups. Let

$$d := a_1 + a_2 + b_1 + b_2 + \cdots + b_m - m - 2.$$

Lemma 3.4. $\widetilde{H}_t(lk_{(\Delta_1 \cup \Delta_2)\cap \Delta_3} F) = 0$ *and* $\widetilde{H}_t(lk_{(\Delta_1 \cup \Delta_2)\cap \Delta_4} F) = 0$ *for* $t \geq d+1$, $F \subseteq X$.

Proof. We will only prove $\widetilde{H}_t(lk_{(\Delta_1 \cup \Delta_2)\cap \Delta_3} F) = 0$. Let

$$J' := \Big(p(f_1)p(f_2), \frac{p(f_1g_1)}{p(f_1)}, \dots, \frac{p(f_1g_m)}{p(f_1)}\Big).$$

By [H-T, Theorem 3.4], we have

$$\operatorname{reg}(J') \leq (a_1 + a_2) + b_1 + \ldots + b_m - m = d + 2.$$

Note that $\Delta_{J'} = (\Delta_1 \cup \Delta_2) \cap \Delta_3$. From this and Proposition 2.2, we obtain the result. □

For each of $i = 1, 2$ let X_i be the set of variables appearing in

$$M_{ij} := \frac{p(f_ig_j)}{\operatorname{lcm}(p(f_i), p(g_j))}$$

for $j = 1, 2, \dots, m$.

Lemma 3.5. *Suppose that* $X_1 \cup X_2 \neq \emptyset$. *Then*

$$\widetilde{H}_t(lk_{(\Delta_1 \cup \Delta_2)\cap \Delta_3 \cap \Delta_4} F) = 0$$

for $t \geq d$, $F \subseteq X$.

Proof. Let $\Delta := (\Delta_1 \cup \Delta_2) \cap \Delta_3 \cap \Delta_4$. Let h be the squarefree monomial corresponding to a face of Δ. Then $h \notin L := ((p(f_1)) \cap (p(f_2))) \cup J_1 \cup J_2 = ((p(f_1)) \cup J_1 \cup J_2) \cap ((p(f_2)) \cup J_1 \cup J_2)$. We may assume that $h \notin ((p(f_1)) \cup J_1 \cup J_2)$. Hence $p(f_1) \nmid h$ and $\dfrac{p(f_1g_j)}{p(f_1)} \nmid h$ for $j = 1, 2, \dots, m$. Since $p(f_1), \dfrac{p(f_1g_1)}{p(f_1)}, \dots, \dfrac{p(f_1g_m)}{p(f_1)}$ form a regular sequence, it follows that

$$\deg h \leq \sharp X - m - 1 = d + 1.$$

Note that the dimension of the face corresponding to h is $\deg h - 1$. So $\dim(\Delta) \le d$. Thus we only need to prove $\widetilde{H}_d(\Delta) = 0$ (i. e. $F = \emptyset$ and $t = d$) when $\dim(\Delta) = d$.

We will compute $\widetilde{H}_d(\Delta)$ by the *augmented oriented chain complex* $\widetilde{\mathsf{C}}_\bullet(\Delta)$ of Δ over k, where $\widetilde{\mathsf{C}}_\bullet(\Delta)$ is the chain complex of k-vector spaces with the differential map δ_i (see [B-H, Chap. 5])

$$0 \to \mathsf{C}_d(\Delta) \xrightarrow{\delta_d} \mathsf{C}_{d-1}(\Delta) \xrightarrow{\delta_{d-1}} \ldots \to \mathsf{C}_1(\Delta) \xrightarrow{\delta_1} \mathsf{C}_0(\Delta) \to 0.$$

Fix the squarefree monomial h corresponding to a maximal face of Δ. Then $\deg h = d + 1$ and $h \notin L$. Let

$$w_{ij} = \gcd(p(f_i), p(g_j)) \quad \text{and} \quad h_j = \frac{p(g_j)}{w_{1j}w_{2j}}.$$

We have

$$\frac{p(f_1g_j)}{p(f_1)} = h_jM_{1j}w_{2j} \quad \text{and} \quad \frac{p(f_2g_j)}{p(f_2)} = h_jM_{2j}w_{1j}.$$

Since $h \notin L$, $p(f_1)p(f_2) \nmid h$, $h_jM_{1j}w_{2j} \nmid h$ and $h_jM_{2j}w_{1j} \nmid h$ for all j. Let $T := \{j |\, h \in (h_j)\}$ and $\sharp T = t$. We consider several cases:

(i) $t = 0$. Since $\deg h = d + 1$ and $p(f_1)p(f_2), h_1, \ldots , h_m$ form a regular sequence, all variables but one in each of $p(f_1)p(f_2), h_1, \ldots , h_m$ are contained in h.

(ii) $t \ge 1$. Then $M_{1j}w_{2j} \nmid h$ and $M_{2j}w_{1j} \nmid h$ for all $j \in T$. Note that $M_{1j}w_{2j}$ and $M_{2j}w_{1j}$ do not have a common variable. So for each $j \in T$ at least two variables of $M_{1j}w_{2j}M_{2j}w_{1j}$ are not contained in h. Note that h_j for $j \notin T$ and $M_{1j}w_{2j}M_{2j}w_{1j}, j \in T$ form a regular sequence. Hence

$$d + 1 = \deg h \le \sharp X - (2t + (m - t)) = d + 2 - t.$$

So $t = 1$. Assume that $T = \{r\}$. Then $h_j \nmid h$ for each $j \ne r$, $M_{1r}w_{2r} \nmid h$ and $M_{2r}w_{1r} \nmid h$. Assume that $M_{1r} \nmid h$ and $M_{2r} \nmid h$. Since $p(f_1)p(f_2)$, $h_1, \ldots , h_{r-1}, M_{1r}, M_{2r}, h_{r+1}, \ldots , h_m$ form a regular sequence, we have

$$d + 1 = \deg h \le \sharp X - (3 + (m - 1)) = d,$$

a contradiction. If $M_{1r} \mid h$, then $w_{2r} \nmid h$. As above, all variables but one in each of $h_1, \ldots , h_{r-1}, w_{2r}, M_{2r}w_{1r}, h_{r+1}, \ldots , h_m$ are contained in h. If $M_{2r} \mid h$, then $w_{1r} \nmid h$. Similarly, all variables but one in each of $h_1, \ldots , h_{r-1}, w_{1r}, M_{1r}w_{2r}$, $h_{r+1}, \ldots , h_m$ are contained in h.

Let Y_i be the set of variables appear in $p(f_i)$ and

$$\Omega = \{F \in \Delta |\, \sharp F = d + 1\}.$$

According to the above cases, we divide Ω into three disjoint subsets:

- Ω_0 consists of subsets F such that: in each of the monomials $p(f_1)p(f_2), h_1$, $\ldots , h_m$ there is only one variable in $X\backslash F$ or in each of $h_1, \ldots , h_{r-1}, w_{1r}, w_{2r}$, $h_{r+1}, \ldots , h_m$ there is only one variable in $X\backslash F$ for some r. Hence $X_1 \cup X_2 \subset F$.

- Ω_1 consists of subsets F such that: in each of the monomials w_{2r}, M_{2r} and $h_1, \dots, h_{r-1}$, $h_{r+1}, \dots, h_m$ there is only one variable in $X \setminus F$ for some r. Hence $X_1 \subset F, X_2 \not\subset F, Y_1 \subset F$ and $Y_2 \not\subset F$ for $F \in \Omega_1$.
- Ω_2 consists of subsets F such that: in each of w_{1r}, M_{1r} and $h_1, \dots, h_{r-1}$, $h_{r+1}, \dots, h_m$ there is only one variable in $X \setminus F$ for some r. Hence $X_2 \subset F, X_1 \not\subset F, Y_2 \subset F$ and $Y_1 \not\subset F$ for $F \in \Omega_2$.

Let

$$Z := \sum_{\substack{F \in \Omega_1 \\ \alpha \in k}} \alpha F + \sum_{\substack{F \in \Omega_2 \\ \beta \in k}} \beta F + \sum_{\substack{F \in \Omega_0 \\ \gamma \in k}} \gamma F \in \operatorname{Ker}(\delta_d).$$

We distinguish two cases.

(1) $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$. If $F \in \Omega_1$ then there exists $u \in X_1 \subset F$ and $v \in (X_2 - F)$. Hence the face $(F-u)$ appears in $\delta_d(\alpha F)$. By definition, $v \notin (F-u)$ and $Y_1 \subset (F-u)$. Assume that there is a face $G \neq F$ in Ω such that $(F-u) \subset G$. If $G \in \Omega_1 \cup \Omega_0$, then $u \in G$. Hence $F = G$, a contradiction. If $G \in \Omega_2$, then $Y_1 \subset (F-u) \subset G$, contrary to our assumption. So the face $(F-u)$ is not contained in another face of Ω. It implies $\alpha = 0$. Next, if $F \in \Omega_2$ then we also have $\beta = 0$. Similarly, if $F \in \Omega_0$ then $X_1 \cup X_2 \subset F$. It also implies $\gamma = 0$. Hence $Z = 0$.

(2) We may assume that $X_1 \neq \emptyset$ and $X_2 = \emptyset$. In this case $\Omega_1 = \emptyset$. Now, we can proceed analogously to the proof of (1) to show that $Z = 0$. □

Lemma 3.6. *Suppose that $X_1 \cup X_2 \neq \emptyset$. Then*

$$\widetilde{H}_t(lk_{(\Delta_1 \cup \Delta_3) \cap (\Delta_2 \cup \Delta_4)} F) = 0$$

for $t \geq d+1$, $F \subseteq X$.

Proof. Applying Lemma 3.4, Lemma 3.5 and Lemma 3.3 to $lk_{(\Delta_1 \cup \Delta_2) \cap \Delta_3} F$ and $lk_{(\Delta_1 \cup \Delta_2) \cap \Delta_4} F$, we have

$$\widetilde{H}_t(lk_{(\Delta_1 \cup \Delta_2) \cap (\Delta_3 \cup \Delta_4)} F) = 0$$

for $t \geq d+1$. As in the proof of Lemma 3.5, we get $\dim(\Delta_3 \cup \Delta_4) \leq d+1$. So $\widetilde{H}_t(lk_{(\Delta_3 \cup \Delta_4)} F) = 0$ for $t \geq d+2$. As in the proof of Lemma 3.4, we have $\widetilde{H}_t(lk_{(\Delta_1 \cup \Delta_2)} F) = 0$ for $t \geq d+2$. Applying Lemma 3.3 to $lk_{(\Delta_1 \cup \Delta_2)} F$ and $lk_{(\Delta_3 \cup \Delta_4)} F$ implies

$$\widetilde{H}_t(lk_{(\Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4)} F) = 0$$

for all $t \geq d+2$. Note that

$$\operatorname{reg}(p(f_i)J_i) = \operatorname{reg}(J_i(-a_i)) = a_i + b_1 + b_2 + \ldots + b_m - m + 1.$$

Moreover,

$$\Delta_{(p(f_i)J_i)} = \Delta_{(p(f_i))} \cup \Delta_{J_i}.$$

On the other hand, $\operatorname{reg}(k[\Delta_1 \cup \Delta_3]) = a_1 + b_1 + b_2 + \ldots + b_m - m$. By Proposition 2.2, we get

$$\widetilde{H}_t(lk_{(\Delta_1 \cup \Delta_3)} F) = 0$$

for $t \geq d+1 \geq a_1 + b_1 + b_2 + \ldots + b_m - m$. Similarly, $\widetilde{H}_t(lk_{(\Delta_2 \cup \Delta_4)}F) = 0$ for $t \geq d+1$. Applying Lemma 3.3 again to $lk_{(\Delta_1 \cup \Delta_3)}F$ and $lk_{(\Delta_2 \cup \Delta_4)}F$, we obtain the statement. □

Now we are ready to prove Main Theorem.

Proof of Main Theorem. If $X_1 \cup X_2 \neq \emptyset$, then by Lemma 3.6 we have

$$\widetilde{H}_t(lk_{(\Delta_1 \cup \Delta_3) \cap (\Delta_2 \cup \Delta_4)}F) = 0$$

for $t \geq d+1$, $F \subseteq X$. By Lemma 3.2,

$$\widetilde{H}_t(lk_{\Delta_{p(IJ)}}F) = 0$$

for $t \geq d+1$, $F \subseteq X$. So the result follows from Proposition 2.2. If $X_1 \cup X_2 = \emptyset$, then $p(f_i g_j) = p(f_i)p(g_j)$. Hence

$$\Delta_{p(IJ)} = \Delta_I \cup \Delta_J.$$

It is clear that $\dim(\Delta_I \cap \Delta_J) \leq d-1$. Hence $\widetilde{H}_t(lk_{\Delta_I \cap \Delta_J}F) = 0$ for $t \geq d$, $F \subseteq X$. Using again the Mayer-Vietoris sequence of reduced homology groups associated with $lk_{\Delta_I}F$ and $lk_{\Delta_J}F$, we obtain $\widetilde{H}_t(lk_{\Delta_{p(IJ)}}F) = 0$ for all $t \geq d+1$ and $F \subseteq X$. Hence the conclusion follows from Proposition 2.2.

□

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