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SUFFICIENT CONDITIONS FOR STRONG STABILITY OF NONLINEAR TIME-VARYING CONTROL SYSTEMS WITH STATE DELAY

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Abstract. This paper deals with a strong stability problem of a class of nonlinear time-varying control systems with state delays. Under appropriate growth conditions on the nonlinear perturbation, new sufficient conditions for the strong stabilizability are established based on the global null-controllability of the nominal linear system. These conditions are presented in terms of the solution of a standard Riccati differential equation. A constructive procedure for finding feedback stabilizing controls is also given.

1. Introduction

Consider a nonlinear time-varying control system with state delays of the form

$$\begin{aligned} &\dot{x}(t) = f(t, x(t), x(t-h), u(t)), \quad t \geq 0, \\ &x(t) = \phi(t),\ \ t \in [-h, 0], \end{aligned} \tag{1}$$

where $h \geqslant 0$, $x(t) \in X$- the state, $u(t) \in U$- the control,

$$f(t, x, y, u) : [0, \infty) \times X \times X \times U \to X,$$

and $\phi(t) : [-h, 0] \to X$- a given function.

The topic of Lyapunov stability of control systems described by a system of differential equations is an interesting research area in the past decades. An integral part of the stability analysis of differential equations is the existence of inherent time delays. Time delays are frequently encountered in many physical and chemical processes as well as in the models of hereditary systems, Lotka-Volterra systems, control of the growth of global economy, control of epidemics, etc. Therefore, stability problems of time-delay control systems have been the subject of numerous investigations, see; e.g. [1, 6, 16, 24, 28, 31] and references therein. The standard stability problem is to find a control function $u(t) = h(x(t))$ in order to keep the zero solution of the closed-loop system

$$\dot{x}(t) = f(t, x(t), x(t-h), h(x(t)))$$

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exponentially stable in the Lyapunov sense [13, 31], i.e., the solution $x(t,\phi)$ of the closed-loop system satisfies the condition

$$\exists N > 0,\ \delta > 0 : \quad \|x(t,\phi)\| \le Ne^{-\delta t}\|\phi\|, \quad \forall t \ge 0,$$

where

$$\|\phi\| = \sup_{s\in[-h,0]} \|\phi(s)\|.$$

In this case one says that the system is stabilizable by the feedback control $u(t) = h(x(t))$ and this control is called a stabilizing feedback control of the system. The positive number $\delta > 0$ depending on the stabilizing control is commonly called a Lyapunov stability exponent. In the literature on control theory of dynamical systems the stabilizability is one of the important qualitative properties and the investigation of the stabilizability has attracted the attention of many researchers, see; e.g. [1, 7, 19, 20, 23, 26, 32] and references therein. In practice, various stabilizability concepts have been defined to improve the efficiency of the stability of control systems. One of the extended stability properties of control systems is the concept of the strong (or complete) stabilizability, originally introduced by Wonham [29], which plays an important role in many mechanical and control engineering problems [1, 30, 32]. This property relates to a strong exponential stability of the control system, namely, control system (1) is strongly stabilizable if for every given number $\delta > 0$, there exists a feedback control function $u(t) = h(x(t))$ such that the solution $x(t,\phi)$ of the closed-loop system satisfies the condition

$$\exists N > 0 : \quad \|x(t,\phi)\| \le Ne^{-\delta t}\|\phi\|, \quad \forall t \ge 0. \tag{2}$$

This means that for any given positive number $\delta > 0$, the system zero-input response of the closed-loop system decays faster than $e^{-\delta t}$. In other words, for any given in advance Lyapunov stability exponent $\delta > 0$, the system can be $\delta-$exponentially stabilizable. Such definition may arise because of controlling of the speed of the real models in many mechanical and physical control systems [1, 6, 30, 32]. First results on the strong stabilizability of linear time-invariant control systems in finite-dimensional spaces can be found in [29, 30], where by studying the spectrum of the system matrices or by solving a modified algebraic Riccati equation, it was proved that the global-null controllability [10] implies the strong stabilizability. Further extensions on the relationship between the strong stabilizability and controllability of infinite-dimensional time-invariant control systems are given in [15, 21, 27]. However, the strong stabilizability and control design problems for time-varying control systems have not been examined fully in the literature, which are more complicated and given results are lacking. The difficulties increase to the same extent as passing from undelayed to delayed time-varying control systems as well as from linear to nonlinear time-varying delay systems.

The aim of this paper is to study the strong stabilizability problem for the following time-varying control delay system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + A_1(t)x(t-h) + B(t)u(t)\\ &\quad + f(t, x(t), x(t-h), u(t)), \quad t \geq 0,\\ x(t) &= \phi(t), \quad t \in [-h, 0],\end{aligned} \tag{3}$$

where $A(t)$, $A_1(t) : X \to X$, $B(t) : U \to X$-are linear matrix/operator functions and the given nonlinear perturbation term $f(t, x, y, u) : [0, \infty) \times X \times X \times U \to X$ could result from errors in modelling the general linear system (1), adding parameters, or uncertainties and disturbances which exist in any realistic systems. A common approach is to treat the stability of the nominal linear control system. Then, when the nonlinearities satisfy some appropriate growth conditions, one can use the Lyapunov direct method to design a stabilizing feedback control. Based on the global null-controllability assumption of the nominal linear time-varying control system, sufficient conditions for the strong stabilizability are established by solving a standard Riccati differential equation. These conditions depending on the size of the delay do not involve any spectrum of the evolution operator/matrix, and hence are easy to be verified and constructed.

For a systematic exposition of the results, we start with the case of finite-dimensional control systems. Then, the results are directed to infinite-dimensional control systems by extending the relationship between the global null-controllability and the existence of the solution of a Riccati operator equation. A constructive algorithm to find feedback stabilizing controls via the controllability and the solution of certain Riccati equations is also given. The stability conditions obtained in this paper are even new in the context of linear time-varying control systems, and they can be considered as further extensions of [9, 15, 22, 27, 29] to nonlinear and time-delayed systems.

The organization of this paper is as follows. Section 2 gives sufficient conditions for the strong stabilizability of system (3) in finite-dimensional spaces. The result is extended to infinite-dimensional control systems in Hilbert spaces in Section 3. A step-by-step procedure for constructing feedback stabilizing controls as well as illustrating examples are also given. Finally, conclusions are made in Section 4.

Notation. Standard notations are adapted throughout this paper. R^+ denotes the set of all real non-negative numbers; R^n denotes n finite-dimensional Euclidean space, with the Euclidean norm $\|.\|$ and the scalar product of two vectors $x^T y$; ^{T} denotes the transpose of the vector/matrix; $R^{n\times m}$ denotes the set of all $(n \times m)$-matrices; A matrix A is symmetric if $A = A^T$;

A matrix A is called non-negative definite ($A \geq 0$) if $x^T Ax \geq 0$, for all $x \in R^n$; A is positive definite ($A > 0$) if $x^T Ax > 0 \;\; \forall x \neq 0$; $M(R^n_+)$ denotes the set of all symmetric non-negative definite matrix functions in $R^{n\times n}$ continuous in $t \in R^+$;

Let X, U denote infinite-dimensional real Hilbert spaces with inner product $\langle ., .\rangle$; $L(X)$ (respectively, $L(U, X)$) denotes the Banach space of all linear bounded operators mapping X into X (respectively, U into X); $L_2([0,t],X)$ denotes the set of all L_2-integrable and X-valued functions on $[0,t]$; $C([0,t],X)$ denotes the set of all $X-$valued continuous function on $[0,t]$; $D(A)$ and A^* denotes the domain and the adjoint of the operator A, respectively; clM denotes the closure of a set M; I denotes the identity operator;

An operator $Q \in L(X)$ is called non-negative definite ($Q \geq 0$) if $\langle Qx, x\rangle \geq 0$, for all $x \in X$; $Q \in L(X)$ is called self-adjoint if $Q = Q^*$; $LO([0,+\infty), X^+)$ denotes the set of all linear bounded set-adjoint non-negative definite operator-valued functions in X continuous in $t \in [0,+\infty)$.

Sometimes, the scalar product of two vectors x, y will be used by $\langle x, y\rangle$ instead of $x^T y$. Furthermore, for the sake of brevity, we will omit the arguments of matrix/operator functions, if it does not cause any confusion.

2. Finite-dimensional systems

In this section we consider the control delay system (3) in finite-dimensional spaces: $X = R^n$, $U = R^m$, $n \geq m$, $A(t) \in R^{n\times n}$, $A_1(t) \in R^{n\times n}$, $B(t) \in R^{n\times m}$, $\phi(s) \in C([-h,0],R^n)$. Throughout this section we consider the class of admissible controls $u(t) \in L_2([0,T],R^m)$ for every $T > 0$. Furthermore, to guarantee the existence of the solution of the control system, the following conditions will be made throughout this section:

A.1. $A(.)x$, $A_1(.)y$, $B(.)u$, $f(.,x,y,u)$ are continuous functions on R^+ for all $x \in R^n$, $y \in R^n$, $u \in R^m$.

A.2. There are non-negative continuous functions $a(t), a_1(t), b(t) : R^+ \to R^+$ such that

$$\|f(t,x,y,u)\| \leq a(t)\|x\| + a_1(t)\|y\| + b(t)\|u\|, \quad \forall (t,x,y,u) \in R^+ \times X \times X \times U.$$

Definition 2.1. Let $\delta > 0$ be a positive number. Control system (3) is said to be $\delta-$ stabilizable if there is a feedback control $u = h(x)$ such that the solution of the closed-loop system satisfies the condition (2).

Definition 2.2. Control system (3) is said to be strongly stabilizable if it is $\delta-$stabilizable for every $\delta > 0$.

In order to study the strong stabilizability problem, it is important to introduce the global null-controllability definition given by Kalman [10]. Consider the nominal linear time-varying control system $[A(t), B(t)]$ of system (3):

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in R^+. \tag{4}$$

Definition 2.3. Linear control system (4) is said to be globally null-controllable (GNC) in finite time if for every state $x \in R^n$, there exist a finite time $T > 0$ and an admissible control $u(t) \in L_2([0,T], R^m)$ such that

$$U(T,0)x + \int_0^T U(T,s)B(s)u(s)ds = 0,$$

where $U(t,s)$ is the fundamental matrix of the linear system $\dot{x}(t) = A(t)x(t)$.

The following well-known controllability criteria will be used later.

Proposition 2.1. [1] *Linear time-varying control system* (4) *is GNC in finite time if and only if one of the following conditions holds:*

(i) $\exists T > 0$: *The matrix* $\int_0^T U(T,s)B(s)B^T(s)U^T(T,s)ds$ *is positive definite,*

(ii) $\exists t_0 > 0$: *rank* $[M_0(t_0), M_1(t_0), \ldots, M_{n-1}(t_0)] = n$, *where*

$$M_0(t) = B(t)$$
$$M_{k+1}(t) = -A(t)M_k(t) + \frac{d}{dt}M_k(t), \quad k = 0,1,2,...,n-1,$$

and $A(t)$, $B(t)$ *are assumed to be analytical functions on* $[0,\infty)$.

In the sequel, the solution to the stabilizability problem involves a Riccati differential equation (RDE) of the form

(5)
$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) - P(t)B(t)B^T(t)P(t) + Q(t) = 0, \quad P(0) = P_0,$$

where $P(t)$ is an unknown matrix function. Before proceeding to the main result, a sufficient condition for the existence of non-negative positive solution of the RDE (5) is provided in the following proposition.

Proposition 2.2. [10] *Assume that the linear control system* $[A(t), B(t)]$ *is GNC. Then for every no-negative positive definite bounded function* $Q(t) \geqslant 0$ *and for every initial matrix* $P_0 \geqslant 0$, *the RDE* (5) *has a solution* $P(t) \in M(R^n_+)$, *which is a bounded function on* $[0,\infty)$.

For every $\delta > 0$, we denote $\tilde{A}(t) = A(t) + \delta I$, and consider the following RDE

$$\dot{P}(t) + \tilde{A}^T(t)P(t) + P(t)\tilde{A}(t) - P(t)B(t)B^T(t)P(t) + I = 0. \tag{6}$$

Let us set

$$b = \sup_{t\in R^+} b(t), \quad B = \sup_{t\in R^+} \|B(t)\|,$$

$$p = \sup_{t\in R^+} \|P(t)\|, \quad a_1 = \sup_{t\in R^+} a_1(t), \quad A_1 = \sup_{t\in R^+} \|A_1(t)\|.$$

The following theorem gives a sufficient condition for $\delta-$stabilizability of the nonlinear control delay system (3).

Theorem 2.1. *Assume that the conditions A.1, A.2 hold and the linear control system* $[A(t), B(t)]$ *is GNC in finite time. Nonlinear control delay system* (3) *is* δ*-stabilizable if*

$$0 < b < \frac{1}{2Bp^2}, \tag{7}$$

$$a_1 + A_1 < \frac{\sqrt{1 - 2p^2 bB}}{2pe^{\delta h}}, \tag{8}$$

$$\sup_{t \in R^+} a(t) < \frac{1}{4p} - \frac{1}{2}pbB - p(a_1 + A_1)^2, \tag{9}$$

and the stabilizing feedback control is given by

$$u(t) = -\frac{1}{2}B^T(t)P(t)x(t), \tag{10}$$

where $P(t) \in M(R^n_+)$ *is the solution of the RDE* (6) *with any initial condition* $P_0 \geq 0$.

Proof. Let us set $y(t) = e^{\delta t}x(t)$. The nonlinear control system (3) is transformed into the following system

(11)
$$\dot{y}(t) = \tilde{A}(t)y(t) + \tilde{A}_1(t)y(t-h) + \tilde{B}(t)u(t) + \tilde{f}(t, y(t), y(t-h), u(t)), \quad t \in R^+,$$
$$y(t) = \tilde{\phi}(t) = e^{\delta t}\phi(t), \quad \forall t \in [-h, 0],$$

where

$$\tilde{A}_1(t) = e^{\delta h}A_1(t), \quad \tilde{B}(t) = e^{\delta t}B(t),$$

$$\tilde{f}(t, y(t), y(t-h), u(t)) = e^{\delta t}f(t, e^{-\delta t}y(t), e^{-\delta(t-h)}y(t-h), u(t)).$$

By the assumption that the linear control system $[A(t), B(t)]$ is GNC, it is to verify that the linear control system $[\tilde{A}(t), B(t)]$ is also GNC. Indeed, we first note that the fundamental matrix $U_{\tilde{A}}(t, s)$ of linear differential equation $\dot{x}(t) = \tilde{A}(t)x(t)$ is given by

$$U_{\tilde{A}}(t, s) = e^{\delta(t-s)}U_A(t, s),$$

where $U_A(t, s)$ is the fundamental matrix of the former system $\dot{x}(t) = A(t)x(t)$. Hence, from the GNC of the system $[A(t), B(t)]$ it follows, by Definition 2.3, that for every $x \in R^m$, there exist a finite time $T > 0$ and admissible control $u(t) \in L_2([0, T), R^m)$ such that

$$U_A(T, 0)x + \int_0^T U_A(T, s)B(s)u(s)ds = 0.$$

Multiplying both sides of the above equation with $e^{\delta T} > 0$ we obtain that

$$\begin{aligned} &e^{\delta T}U_A(T,0)x + \int\limits_0^T e^{\delta(T-s)}U_A(T,s)B(s)e^{\delta s}u(s)ds \\ &= U_{\tilde{A}}(T,0)x + \int\limits_0^T U_{\tilde{A}}(T,s)B(s)\tilde{u}(s)ds = 0, \end{aligned}$$

which implies that the system $[\tilde{A}(t), B(t)]$ is GNC in the time $T > 0$ with the admissible control $\tilde{u}(t) = e^{\delta t}u(t)$. Therefore, by Proposition 2.2., the RDE (6) with any initial condition $P_0 \geqslant 0$ has a bounded solution $P(t) \in M(R^n_+)$. Let

$$p = \sup_{t\in R^+} \|P(t)\| < +\infty.$$

Note that RDE (6) cannot have the zero solution. Then we have $p > 0$. Let us consider a Lyapunov-like function for the closed-loop system of the system (11):

$$V(t, y_t) = y^T(t)P(t)y(t) + \frac{1}{2}\int\limits_{t-h}^{t} \|y(s)\|^2 ds. \tag{12}$$

Observe that

$$\begin{aligned} \|\tilde{f}(t, y(t), y(t-h), u(t))\| &= e^{\delta t}\|f(t, e^{-\delta t}y(t), e^{-\delta(t-h)}y(t-h), u(t))\| \\ &\leqslant e^{\delta t}[a(t)e^{-\delta t}\|y(t)\| + a_1(t)e^{-\delta(t-h)}\|y(t-h)\| \\ &\quad + b(t)\|u(t)\|] \\ &= a(t)\|y(t)\| + a_1(t)e^{\delta h)}\|y(t-h)\| + b(t)e^{\delta t}\|u(t)\|. \end{aligned}$$

Taking the feedback control (10):

$$u(t) = -\frac{1}{2}B^T(t)P(t)x(t) = -\frac{1}{2}e^{-2\delta t}\tilde{B}^T(t)P(t)y(t),$$

the derivative of the Lyapunov function $V(t, y_t)$ along the solution $y(t)$ is defined as

$$\begin{aligned} \dot{V}(t, y_t) &= \langle \dot{P}(t)y(t), y(t)\rangle + 2\langle P(t)\dot{y}(t), y(t)\rangle + \frac{1}{2}[\|y(t)\|^2 - \|y(t-h)\|^2] \\ &= \langle [-\tilde{A}^T P - P\tilde{A} + e^{-2\delta t}P\tilde{B}\tilde{B}^T P - I]y, y\rangle \\ &\quad + 2\langle P[\tilde{A}y + \tilde{A}_1 y(t-h) + \tilde{B}u + \tilde{f}(.)], y\rangle + \frac{1}{2}[\|y(t)\|^2 - \|y(t-h)\|^2] \end{aligned}$$

$$
\begin{aligned}
&= -\langle Py, \tilde{A}y\rangle - \langle P\tilde{A}y, y\rangle + e^{-2\delta t}\langle P\tilde{B}\tilde{B}^T Py, y\rangle - \langle Iy, y\rangle + 2\langle P\tilde{A}y, y\rangle \\
&\quad + 2\langle P\tilde{B}u, y\rangle + 2\langle P\tilde{A}_1 y(t-h), y\rangle + 2\langle P\tilde{f}(.), y\rangle + \frac{1}{2}[\|y(t)\|^2 - \|y(t-h)\|^2] \\
&= -\langle Py, \tilde{A}y\rangle - \langle P\tilde{A}y, y\rangle + e^{-2\delta t}\langle P\tilde{B}\tilde{B}^T Py, y\rangle - \langle Iy, y\rangle + 2\langle P\tilde{A}y, y\rangle \\
&\quad + 2\langle P\tilde{B}[-\frac{1}{2}e^{-2\delta t}\tilde{B}^T Py], y\rangle + 2\langle P\tilde{A}_1 y(t-h), y\rangle + 2\langle P\tilde{f}(.), y\rangle \\
&\quad + \frac{1}{2}\|y(t)\|^2 - \frac{1}{2}\|y(t-h)\|^2 \\
&= -\langle Py, \tilde{A}y\rangle - \langle P\tilde{A}y, y\rangle + e^{-2\delta t}\langle P\tilde{B}\tilde{B}^T Py, y\rangle - \langle Iy, y\rangle + 2\langle P\tilde{A}y, y\rangle \\
&\quad - e^{-2\delta t}\langle P\tilde{B}\tilde{B}^T Py, y\rangle + 2\langle P\tilde{A}_1 y(t-h), y\rangle + 2\langle P\tilde{f}(.), y\rangle \\
&\quad + \frac{1}{2}[\|y(t)\|^2 - \|y(t-h)\|^2].
\end{aligned}
$$

Since $P(t)$ is symmetric:

$$\langle Py, \tilde{A}y\rangle = \langle y, P\tilde{A}y\rangle,$$

we have

$$
\begin{aligned}
\dot{V}(t, y_t) &= -\langle Iy, y\rangle + 2\langle P\tilde{A}_1 y(t-h), y\rangle + 2\langle P\tilde{f}(.), y\rangle \\
&\quad + \frac{1}{2}\|y(t)\|^2 - \frac{1}{2}\|y(t-h)\|^2 \\
&= -\|y(t)\|^2 + 2\langle P\tilde{A}_1 y(t-h), y\rangle + 2\langle P(t)\tilde{f}(t, y(t), y(t-h)), u(t))\rangle \\
&\quad + \frac{1}{2}[\|y(t)\|^2 - \|y(t-h)\|^2].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\|\tilde{A}_1(t)\| &\leqslant A_1 e^{\delta h}, \\
2\langle P\tilde{A}_1 y(t-h), y\rangle &\leqslant 2pA_1 e^{\delta h}\|y(t-h)\|\|y(t)\|, \\
2\langle P(t)\tilde{f}(t, y(t), y(t-h), u(t)), y(t)\rangle &\leqslant 2p\|\tilde{f}(t, y(t), y(t-h), u(t))\|\|y(t)\|.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\dot{V}(t, y_t) &\leqslant -\frac{1}{2}\|y(t)\|^2 + 2pA_1 e^{\delta h}\|y(t-h)\|\|y(t)\| \\
&\quad + 2p\|\tilde{f}(t, y(t), y(t-h), u(t))\|\|y(t)\| \\
&\leqslant -\frac{1}{2}\|y(t-h)\|^2 - \frac{1}{2}\|y(t)\|^2 \\
&\quad + 2pA_1 e^{\delta h}\|y(t-h)\|\|y(t)\| - \frac{1}{2}\|y(t-h)\|^2 \\
&\quad + 2p[a(t)\|y(t)\| + a_1(t)e^{\delta h}\|y(t-h)\| + e^{\delta t}b(t)\|u(t)\|]\|y(t)\|
\end{aligned}
$$

$$\begin{aligned}
&\leqslant -\frac{1}{2}\|y(t)\|^2 + 2pA_1e^{\delta h}\|y(t-h)\|\|y(t)\| - \frac{1}{2}\|y(t-h)\|^2 \\
&\quad + 2pa(t)\|y(t)\|^2 + 2pa_1e^{\delta h}\|y(t-h)\|\|y(t)\| + 2pbe^{\delta t}\|u(t)\|\|y(t)\|, \\
&\leqslant \Big[-\frac{1}{2} + 2pa(t) + p^2Bb\Big]\|y(t)\|^2 + 2p(a_1+A_1)e^{\delta h}\|y(t-h)\|\|y(t)\| \\
&\quad - \frac{1}{2}\|y(t-h)\|^2.
\end{aligned}$$

By completing the square, we obtain that

$$\dot V(t,y_t) \leqslant \Big[-\frac{1}{2} + 2pa(t) + p^2Bb + 2p^2(A_1+a_1)^2e^{2\delta h}\Big]\|y(t)\|^2.$$

Taking the conditions (7), (8), (9) into account, there is a number $\gamma > 0$ such that

$$\dot V(t,y_t) \leqslant -\gamma\|y(t)\|^2, \quad \forall t \in R^+. \tag{13}$$

Integrating both side of (12) from 0 to t we have

$$\begin{aligned}
\langle P(t)y(t),y(t)\rangle + \frac{1}{2}\int_{t-h}^{t}\|y(s)\|^2ds - \langle P(0)y(0),y(0)\rangle - \frac{1}{2}\int_{-h}^{0}\|y(s)\|^2ds \\
\leqslant -\gamma\int_0^t\|y(s)\|^2ds,
\end{aligned}$$

which implies

$$\begin{aligned}
\int_0^t\|y(s)\|^2ds &\leqslant -\frac{1}{\gamma}\langle P(t)y(t),y(t)\rangle - \frac{1}{2\gamma}\int_{t-h}^{t}\|y(s)\|^2ds \\
&\quad + \frac{1}{\gamma}\langle P(0)y(0),y(0)\rangle + \frac{1}{2\gamma}\int_{-h}^{0}\|y(s)\|^2ds \\
&\leqslant \frac{1}{\gamma}\langle P(0)y(0),y(0)\rangle + \frac{1}{2\gamma}\int_{-h}^{0}\|y(s)\|^2ds.
\end{aligned}$$

Taking $P(0) = I$, and since $y(s) = e^{\delta s}x(s) = e^{\delta s}\phi(s)$ for all $t \in [-h,0]$ we have

$$\|y(s)\| \leq e^{\delta s}\|\phi(s)\| \leqslant e^{\delta s}\sup_{s\in[-h,0]}\|\phi(s)\| := M.$$

Therefore,

$$\int_0^t\|y(s)\|^2ds \leqslant \beta + \frac{1}{2\gamma}hM^2,$$

where $\beta = \dfrac{1}{\gamma}\langle y(0), y(0)\rangle$. Letting $t \to \infty$ gives

$$\int_0^\infty \|y(s)\|^2 ds < +\infty,$$

which implies that the solution $y(t, \tilde{\phi})$ of the system (11) belongs to $L_2([0, +\infty), R^n)$ and hence there is a number $N > 0$ such that $\|y(t)\| \leqslant N\|\tilde{\phi}\|$, or

$$\|x(t, \phi)\| \leqslant Ne^{-\delta t}\|\phi\|.$$

The proof of the theorem is completed. □

Note that if $A_1(t) = 0$, $f(t, x, y, u) = 0$, i.e., $a(t) = a_1(t) = b(t) = 0$, the conditions (7)-(9) automatically hold and then Theorem 2.1 can be applied to the linear control system $[A(t), B(t)]$ in finite-dimensional spaces as follows.

Corollary 2.1. *The finite-dimensional linear control system* $[A(t), B(t)]$ *is strongly stabilizable if it is GNC in finite time.*

Remark 2.1. Corollary 2.1 extends a result of [29] to time-varying case and it improves a result of [9], where the controllability assumption was assumed to be more strict: the uniform global controllability.

From the proof of Theorem 2.1, the following procedure of finding stabilizing feedback control can be applied:

Step 1. Verify the GNC of linear control system $[A(t), B(t)]$ by Proposition 2.1.

Step 2. For given $\delta > 0$, find the solution $P(t) \in M(R^n_+)$ of RDE (6).

Step 3. Compute the numbers p, b, B, A_1, a_1 and check the conditions (7)-(9).

Step 4. The stabilizing feedback control $u(t)$ is given by (10).

Example 2.1. Consider the nonlinear control delay system (3) in R^2, where $h = \dfrac{1}{8}$, $\delta = 2$ and

$$A(t) = \begin{pmatrix} \dfrac{1}{20}e^{-4t}\sin^2 t - 5e^{4t} & 0 \\ 0 & \dfrac{1}{20}e^{-4t}\cos^2 t - 5e^{4t} \end{pmatrix},$$

$$A_1(t) = \begin{pmatrix} e^{-\frac{1}{2}t}\sin t & 0 \\ 0 & e^{-\frac{1}{2}t}\cos t \end{pmatrix}, \quad B(t) = \begin{pmatrix} \sin t & 0 \\ 0 & \cos t \end{pmatrix},$$

$$f(t, x(t), x(t-h), u(t)) = x(t)\sin^2 t + e^{-\frac{1}{2}t}x(t-h) + e^{-\frac{9}{2}t}u(t), \quad \forall t \geqslant 0.$$

We have

$$a(t) = \sin^2 t, \quad a_1(t) = e^{-\frac{1}{2}t}, \quad b(t) = e^{-\frac{9}{2}t}.$$

We can easily verify the GNC of the linear control system $[A(t), B(t)]$ by Proposition 2.1 (ii), where

$$M_0(t) = B(t) = \begin{pmatrix} \sin t & 0 \\ 0 & \cos t \end{pmatrix},$$

$$M_1(t) = \begin{pmatrix} -\big[\frac{1}{20}e^{-4t}\sin^2 t - 5e^{4t}\big]\sin t + \cos t & 0 \\ 0 & -\big[\frac{1}{20}e^{-4t}\cos^2 t - 5e^{4t}\big]\cos t - \sin t \end{pmatrix},$$

and hence, rank $[M_0(t_2), M_1(t_2)] = 2$, with $t_0 = \frac{\pi}{2}$. On the other hand, for $\delta = 2$, and for the defined matrices $\tilde{A}(t)$, $B(t)$, upon some computations we can find that the solution $P(t)$ of the RDE (6) is given by

$$P(t) = \begin{pmatrix} \frac{1}{10}e^{-4t} & 0 \\ 0 & \frac{1}{10}e^{-4t} \end{pmatrix}.$$

Thus, computing the numbers b, B, p, a_1, A_1, we verify the conditions (7)-(9). The system is then 2-stabilizable with the feedback control

$$u(t) = -\frac{1}{2}\begin{pmatrix} \frac{1}{10}e^{-4t}\sin t & 0 \\ 0 & \frac{1}{10}e^{-4t}\cos t \end{pmatrix}.$$

3. Infinite-dimensional systems

We now consider the system (3) in infinite-dimensional spaces: $x \in X$, $u \in U$; X, U-are real Hilbert spaces, for every $t \in R^+$, $A(t) : X \to X$ is a linear operator, $A_1(t) \in L(X)$, $B(t) \in L(U, X)$, $f(t, x, y, u) : R^+ \times X \times X \times U \to X$. Throughout this section we consider the class of admissible controls $u(t) \in L_2([0, T], U)$ for every $T > 0$. As in [2, 7], for guarantying the existence of the solution of infinite-dimensional control system (3), throughout this section we assume that

B.1. The operator functions $A(.)x$, $A_1(.)x \in L(X)$, $B(.)u \in L(U, X)$, $f(., x, y, u)$ are continuous on $[0, \infty)$ for every $x \in X$, $y \in X$, $u \in U$.

B.2. The linear operator function $A(t) : X \to X$, $\mathrm{cl}(D(A(t))) = X$, generates an evolution semigroup operator $U(t, s)$ [18].

B.3. The nonlinear function $f(t, x, y, u)$ satisfies the condition: there exist non-negative continuous functions $a(t), a_1(t), b(t) : R^+ \to R^+$ such that

$$\|f(t, x, y, u)\| \le a(t)\|x\| + a_1(t)\|y\| + b(t)\|u\|, \quad \forall (t, x, y, u) \in R^+ \times X \times X \times U.$$

In this case, the mild solution of the nonlinear system (1) in Hilbert space is given by

$$x(t,\phi) = U(t,0)\phi(t) + \int_0^t U(t,\tau)\big[A_1 x(\tau-h) + B(\tau)u(\tau) + f(\tau, x(\tau), x(\tau-h), u(\tau))\big]d\tau.$$

Before proceeding further, we state the following well-known infinite-dimensional controllability criterion, which will be used later.

Proposition 3.1. [5, 32] *Infinite-dimensional linear control system* $[A(t), B(t)]$ *is GNC iff*

$$\exists T > 0, c > 0: \quad \int_0^T \|B^*(s)U^*(T,s)x^*\|^2 ds \geqslant c\|U^*(T,0)x^*\|^2, \quad \forall x^* \in X^*.$$

Associated with the infinite-dimensional linear control system $[A(t), B(t)]$, we consider a Riccati operator equation (ROE) described formally by the form

$$\dot{P}(t) + A^*(t)P(t) + P(t)A(t) - P(t)B(t)B^*(t)P(t) + Q(t) = 0. \tag{14}$$

Since $A(t), t \in R^+$ is an unbounded operator, the solution of ROE will be defined as follows.

Definition 3.1. The solution of ROE (13) is a linear operator function $P(t) \in L(X)$ satisfying the following two conditions:

(i) The scalar function $\langle P(\cdot)x, y\rangle$ is continuously differentiabe on $[0, \infty)$ for every $x, y \in D(A(.))$.

(ii) For all $x, y \in D(A(t)), t \in R^+$:

$$\frac{d}{dt}\langle P(t)x, y\rangle + \langle P(t)x, A(t)y\rangle + \langle P(t)A(t)x, y\rangle - \langle P(t)B(t)B^*(t)P(t)x, y\rangle + \langle Q(t)x, y\rangle = 0.$$

The existence problem of the solution of ROE (13) in infinite-dimensional case was studied (see; e.g. [4, 8, 17] and references therein). We first state the following sufficient condition guaranteed the existence of a bounded solution $P(t)$ of ROE (13), which is given in [2, 25] as follows.

Proposition 3.2. *Let* $Q(t) \in LO([0,\infty), X^+)$ *be a bounded operator function. If the linear control system* $[A(t), B(t)]$ *is* $Q(t)$*-stabilizable in the sense that for every initial state* x_0*, there is an admissible control* $u(t) \in L_2([0,+\infty), U)$ *such that the cost function*

$$J(u) = \int_0^\infty [\|u(t)\|^2 + \langle Q(t)x(t,x_0), x(t,x_0)\rangle]dt, \tag{15}$$

exists and is finite, then the ROE (13) *with any initial condition* $P_0 \geqslant 0$ *has the solution* $P(t) \in LO([0,\infty), X^+)$*, which is also a bounded function:*

$$\sup_{t\in R^+} \|P(t)\| < +\infty.$$

The following proposition will play a key role in the derivation of the existence of the solution of ROE (13) from the global null-controllability of the system $[A(t), B(t)]$.

Proposition 3.3. *If linear control system* $[A(t), B(t)]$ *is GNC in finite time, then for any bounded operator function* $Q(t) \in LO([0,\infty), X^+)$*, the ROE (13) with* $P_0 \geqslant 0$ *has a bounded solution* $P(t) \in LO([0,\infty), X^+)$*.*

Proof. Assume that the system $[A(t), B(t)]$ is globally null-controllable in some time $T > 0$. Let us take any operator $Q(t) \in LO([0,\infty), X^+)$ and consider the cost function (14). Due to the global null-controllability, for every initial state $x_0 \in X$ there is a control $u(t) \in L_2([0,T], U)$ such that the solution $x(t, x_0)$ of the system, according to the control $u(t)$, satisfies

$$x(0) = x_0, \quad x(T, x_0) = 0.$$

Let us denote by $u_x(t)$ an admissible control according to the solution $x(t, x_0)$ of the system. Define

$$\tilde{u}(t) = \begin{cases} u_x(t), & t \in [0,T], \\ 0 & t > T. \end{cases}$$

If $\tilde{x}(.)$ is the solution corresponding to $\tilde{u}(.)$, then $\tilde{x}(t) = 0$ for all $t > T$. Therefore, for every initial state x_0, there is a control $\tilde{u}(t) \in L_2([0,\infty), U)$ such that

$$\begin{aligned} J(\tilde{u}) &= \int_0^\infty \Big[\langle Q(s)\tilde{x}(s,x_0), \tilde{x}(s,x_0)\rangle + \|\tilde{u}(s)\|^2\Big] ds \\ &= \int_0^T \Big[\langle Q(s)x(s,x_0), x(s,x_0)\rangle + \|u(s)\|^2\Big] ds < +\infty \end{aligned}$$

which means that the system $[A(t), B(t)]$ is $Q(t)$-stabilizable and hence by Proposition 3.2, the ROE (13) has a bounded solution $P(t) \in LO([0,\infty), X^+)$. The proof of the proposition is completed. $\square$

Let $\delta > 0$ be a given number. Putting $\tilde{A}(t) = A(t) + \delta I$, we consider a ROE of the form

$$\dot{P}(t) + \tilde{A}^*(t)P(t) + P(t)\tilde{A}(t) - P(t)\tilde{B}(t)\tilde{B}^*(t)P(t) + I = 0. \tag{16}$$

Let us set

$$p = \sup_{t\in R^+} \|P(t)\|, \quad b = \sup_{t\in R^+} b(t), \quad B = \sup_{t\in R^+} \|B^*(t)\|,$$

$$a_1 = \sup_{t\in R^+} a_1(t), \quad A_1 = \sup_{t\in R^+} \|A_1(t)\|.$$

Now, we give the following sufficient conditions for the strong stabilizability of the nonlinear control delay system (3) in Hilbert spaces.

Theorem 3.1. *Assume the conditions B.1 - B.3. Assume that linear control system $[A(t), B(t)]$ is GNC in finite time. The infinite-dimensional nonlinear control delay system* (3) *is $\delta-$stabilizable if the following conditions hold:*

$$0 < b < \frac{1}{2p^2B}, \quad a_1 + A_1 < \frac{\sqrt{1-2p^2bB}}{2pe^{\delta h}}, \tag{17}$$

$$\sup_{t\in R^+} a(t) < \frac{1}{4p} - \frac{1}{2}pbB - p(a_1+A_1)^2e^{2\delta h}. \tag{18}$$

The stabilizing feedback control is given by

$$u(t) = -\frac{1}{2}B^*(t)P(t)x(t), \tag{19}$$

where $P(t)$ is the solution of the ROE (15) *with any initial condition $P_0 \geqslant 0$.*

Proof. Let us set $y(t) = e^{\delta t}x(t)$. As in Theorem 2.1, the nonlinear control delay system (1) is transformed into the nonlinear control system (11), where

$$\tilde{A}_1(t) = e^{\delta h}A_1(t),$$
$$\tilde{B}(t) = e^{\delta t}B(t),$$
$$\tilde{f}(t, y(t), y(t-h), u(t)) = e^{\delta t}f(t, e^{-\delta t}y(t), e^{-\delta(t-h)}y(t-h), u(t)).$$

By the assumption of the global null-controllability of the system $[A(t), B(t)]$, we shall prove that the system $[\tilde{A}(t), B(t)]$ is also globally null-controllable. Indeed, we note that (see, e.g. [7, 18]) the evolution semigroup operator $U_{\tilde{A}}(t,s)$ generated by $\tilde{A}(t)$ is $U_{\tilde{A}}(t,s) = e^{\delta(t-s)}U_A(t,s)$, where $U_A(t,s)$ is the evolution semigroup operator generated by $A(t)$. Hence, by Proposition 3.1, we have

$$\exists T > 0, c > 0: \quad \int_0^T \|B^*(s)U_A^*(T,s)x^*\|^2 ds \geqslant c\|U_A^*(T,0)x^*\|^2, \quad \forall x^* \in X^*.$$

Multiplying both sides of the above inequality with $e^{\delta T}$ and noticing

$$\tilde{B}(s) = e^{\delta s}B(s), \quad U_{\tilde{A}}(T,s) = e^{\delta(T-s)}U_A(T,s), \quad U_{\tilde{A}}(T,0) = e^{\delta T}U_A(T,0),$$

we obtain that

$$\int_0^T \|\tilde{B}^*(s)U_{\tilde{A}}^*(T,s)x^*\|^2 ds \geqslant c\|U_{\tilde{A}}^*(T,0)x^*\|^2, \quad \forall x^* \in X^*,$$

which implies the global null-controllability of the system $[\tilde{A}(t), B(t]$. Thus, we can now apply Proposition 3.3 for the existence of the solution of ROE (13), with $Q(t) = I$. Let $P(t) \in LO([0,\infty), X^+)$ be the solution of ROE (15) with

$P_0 = I$. Taking the feedback control (18), we consider the following Lyapunov-like function for the closed loop system of the system (11):

$$V(t, y_t) = \langle P(t)y(t), y(t)\rangle + \frac{1}{2}\int_{t-h}^{t} \|y(s)\|^2 ds,$$

and by the same arguments used in the proof of Theorem 2.1, using the conditions (16), (17) we can prove the boundedness of the solution $y(t)$ of the transformed system (11), and then the exponential stability condition

$$\|x(t,\phi)\| \leqslant Ne^{-\delta t}\|\phi\|.$$

The proof of the theorem is completed. □

Remark 3.1. It is worth noticing that Theorem 3.1 improves a result of [22], where the growth condition on the nonlinear perturbation $f(.)$ without state delays was strictly assumed that:

$$\|f(t,x,y,u)\| \leq a(t)\|x\| + b(t), \quad \forall (t,x,y,u) \in R^+ \times X \times X \times U.$$

Note that if $f(t,x,y,u) = 0$, i.e. $a = b = a_1 = 0$, we have the following obvious consequence.

Corollary 3.1. *Assume that the infinite-dimensional linear control system* $[A(t), B(t)]$ *is GNC in finite time. The linear control delay system*

$$\dot{x}(t) = A(t)x(t) + A_1(t)x(t-h) + B(t)u(t),$$

is $\delta-$ *stabilizable if*

$$0 < A_1 < \frac{1}{2pe^{\delta h}}\,.$$

In the case if $A_1(.) = 0$, $f(t,x,y,u) = 0$, the conditions (16), (17) automatically hold and then we have the following subsequence for the strong stabilizability of linear control system, which extends the result of [15, 27] to the time-varying case.

Corollary 3.2. *The infinite-dimensional linear control system*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$

is strongly stabilizable if the system is GNC in finite time.

As in finite-dimensional case, the following procedure recalls all steps of finding stabilizing feedback control:

Step 1. Verify the GNC of infinite dimensional linear control system $[A(t), B(t)]$ by Proposition 3.1.

Step 2. For given $\delta > 0$, find the solution $P(t) \in LO([0,+\infty), X^+)$ of ROE (15).

Step 3. Compute the numbers p, b, B, A_1, a_1 and check the conditions (16)-(17).

Step 4. The stabilizing feedback control $u(t)$ is given by (18).

Example 3.1. Consider system (3) in the Hilbert spaces l_2, where

$$A(t) : (x_1, x_2, ...) \in l_2 \longrightarrow (\frac{1}{8}e^{-4t} - 2e^{4t})(x_1, x_2, ...) \in l_2,$$

$$A_1(t) : (x_1, x_2, ...) \in l_2 \longrightarrow e^{-\frac{1}{2t}}(x_1, x_2, ...) \in l_2,$$

$$B(t) : (u_1, u_2, ...) \in l_2 \longrightarrow e^{-2t}(u_1, u_2, ...) \in l_2,$$

$$f(t, x, y, u) = \frac{1}{3}x \sin^2 t + \frac{1}{3}e^{-\frac{1}{2}t}y + \frac{4}{5}e^{-\frac{9}{2}t}u, \quad \forall t \geqslant 0.$$

We have

$$a(t) = \frac{1}{3}\sin^2 t, \quad a_1(t) = \frac{1}{3}e^{-\frac{1}{2}t}, \quad b(t) = \frac{4}{5}e^{-\frac{9}{2}t}.$$

To verify the GNC of the system $[A(t), B(t)]$ we first find the evolution operator $U(t, s)$. Upon some computations we find that

$$U(t,\tau) = \begin{pmatrix} u_{11}(t,\tau) & 0\cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & u_{22}(t,\tau) & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & u_{nn}(t,\tau) & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where

$$u_{11}(t,\tau) = e^{-\frac{1}{32}(e^{-4t}-e^{-4\tau})-\frac{1}{2}(e^{4t}-e^{4\tau})},$$
$$u_{22}(t,\tau) = e^{-\frac{1}{32}(e^{-4t}-e^{-4\tau})-\frac{1}{2}(e^{4t}-e^{4\tau})},$$
$$u_{nn}(t,\tau) = e^{-\frac{1}{32}(e^{-4t}-e^{-4\tau})-\frac{1}{2}(e^{4t}-e^{4\tau})}.$$

Therefore, defining

$$\|U^*(T,0)x^*\|^2 = \sum_{n=1}^{\infty} e^{-\frac{1}{16}e^{-4T}} e^{\frac{1}{16}} e^{-e^{4T}} e^1 x_n^2,$$

$$\|B^*(\tau)U^*(T,\tau)x^*\|^2 = \sum_{n=1}^{\infty} [e^{-\frac{1}{16}.e^{-4T}}.e^{\frac{1}{16}}.e^{-e^{4T}}.e^1.x_n^2].[e^{-4\tau}.e^{-\frac{1}{16}}.e^{-1}.e^{\frac{1}{16}e^{-4\tau}}.e^{e^{4\tau}}]$$
$$\geqslant \sum_{n=1}^{\infty}[e^{-\frac{1}{16}.e^{-4T}}.e^{\frac{1}{16}}.e^{-e^{4T}}.e^1.x_n^2].[e^{-4\tau}.e^{-\frac{17}{16}}],$$

and applying Proposition 3.1, where $c = 0.08$, $T = 1$, we can verify the GNC of the system $[A(t), B(t)]$. On the other hand, we have

$$\tilde{A}(t)x = (\frac{1}{8}e^{-4t} - 2e^{4t} + 2)x,$$

the ROE

$$\dot{P}(t) + \tilde{A}^*(t)P(t) + P(t)\tilde{A}(t) - P(t)\tilde{B}(t)\tilde{B}^*(t)P(t) + I = 0,$$

has the solution

$$P(t) = \begin{pmatrix} \frac{1}{4}e^{-4t} & 0 \\ 0 & \frac{1}{4}e^{-4t} \end{pmatrix}.$$

and all the conditions (16), (17) are satisfied with

$$b = 4/5, \quad p = 1/4, \quad a_1 = 1/3, \quad A_1 = 1.$$

By Theorem 3.1, the system is 2-stabilizable.

Remark 3.2. Note that the sufficient conditions for the strong stability of infinite-dimensional system (3) involve solving ROE (15), which is, in general, still a difficult problem. However, some effective approaches to this problem can be found in [3, 8, 14, 17].

4. Conclusions

In this paper, we have studied the strong stability problem for a class of nonlinear time-varying control systems with state delays. Based on the controllability of the nominal linear control system, sufficient conditions depending on the size of the delay for the strong stabilizability have been established by solving a standard Riccati matrix/operator equation. The feature of this work is that the strong stability conditions do not involve any spectrum of the evolution operator/matrix, and hence are easy to verify and construct. A constructive procedure for finding the stabilizing feedback control and illustrative examples of the results are given. It is worth mentioning that the results presented in this paper do not involve multiple delays as well as the constraints on both the state and control of the system. These issues will be the subject of the future investigations.

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