ACTA MATHEMATICA VIETNAMICA
Volume 30, Number 1, 2005, pp. 59-68

A CERTAIN SUBCLASS OF P-VALENTLY ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS OF COMPLEX ORDER

AJAB AKBARALLY AND MASLINA DARUS

Abstract. The object of this paper is to show some properties of functions belonging to a subclass $M_p(A, B, b, n)$ (where b is a complex number with $\mathrm{Re}(b) > 0$ and A and B are two arbitrary constants with $-1 \leqslant B < A \leqslant 1$). Coefficient estimates and some distortion theorems for this class of functions are found. Radii of close-to-convexity, starlikeness and convexity are derived. An application to fractional calculus is given.

1. Introduction

Let S denote the family of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$.

We denote $T(p)$ by the subclass of S consisting of functions of the form

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in N). \tag{1.2}$$

We denote by $M_p(A, B, b, n)$ the class of functions $f \in T(p)$ that satisfy the condition

$$1 + \frac{1}{b}\Big(\frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)} - p\Big) \prec \frac{1 + Az}{1 + Bz} \tag{1.3}$$

where $\prec$ denotes subordination, $b \neq 0$ is any complex number with $\mathrm{Re}\, b > 0$, A and B are arbitrary fixed numbers, $-1 \leqslant B < A \leqslant 1$. $D^{n+p}f(z)$ is the extension of the familiar operator $D^n f(z)$ of Ruscheweyh Derivatives [3], $n \in N_0 = N \cup \{0\}$. This operator was considered by Sekine, Owa and Obradovic [4] where

$$D^{n+p}f(z) = z^p - \sum_{k=1}^{\infty} C_{p,k}(n) a_{p+k} z^{p+k}$$

Received June 25, 2004.
2000 *Mathematics Subject Classification.* Primary 30C45.
Key words and phrases. Coefficient estimates, distortion theorem, close-to-convexity, starlikeness, convexity, fractional calculus.

with

$$C_{p,k}(n) = \frac{(n+p+k)\cdots(1+k)}{(n+p)!}\cdot$$

The subclass $M_p(A,B,b,n)$ is obtained from the subclass $S^b_\alpha(A,B,n)$ defined by Aouf and Amri [1].

2. Coefficient Estimates

Theorem 2.1. *A necessary and sufficient condition for a function $f \in T(p)$ to be in the class $M_p(A,B,b,n)$ is*

$$\frac{\sum_{k=1}^{\infty}[k+|b(A-B)-Bk|]C_{p,k}(n)|a_{p+k}|}{|b|(A-B)} \leqslant 1 \tag{2.1}$$

Proof. ($\Rightarrow$) By definition of subordination we can write (1.3) as

$$1+\frac{1}{b}\Big(\frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)}-p\Big) = \frac{1+Aw(z)}{1+Bw(z)} \qquad (w(z)\in U),$$

$$\frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)}-p = \Big(b(A-B)-B\Big(\frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)}-p\Big)\Big)w(z), \tag{2.2}$$

$$\frac{pz^p-\sum_{k=1}^{\infty}(p+k)C_{p,k}(n)a_{p+k}z^{p+k}}{z^p-\sum_{k=1}^{\infty}C_{p,k}(n)a_{p+k}z^{p+k}}-p$$

$$=\Big(b(A-B)-B\Big(\frac{pz^p-\sum_{k=1}^{\infty}(p+k)C_{p,k}(n)a_{p+k}z^{p+k}}{z^p-\sum_{k=1}^{\infty}C_{p,k}(n)a_{p+k}z^{p+k}}-p\Big)\Big)w(z),$$

i.e.

$$\frac{\sum_{k=1}^{\infty}(-k)C_{p,k}(n)a_{p+k}z^k}{1-\sum_{k=1}^{\infty}C_{p,k}(n)a_{p+k}z^k} = \Big(b(A-B)-B\Big(\frac{\sum_{k=1}^{\infty}(-k)C_{p,k}(n)a_{p+k}z^k}{1-\sum_{k=1}^{\infty}C_{p,k}(n)a_{p+k}z^k}\Big)\Big)w(z).$$

Since $|w(z)|<1$,

$$\Big|\sum_{k=1}^{\infty}(-k)C_{p,k}(n)a_{p+k}z^k\Big| \leqslant \Big|b(A-B)-\sum_{k=1}^{\infty}[b(A-B)-Bk]C_{p,k}(n)a_{p+k}z^k\Big|.$$

Letting $z\to 1^-$ through real values we have

$$\sum_{k=1}^{\infty}[k+|b(A-B)-Bk|]C_{p,k}(n)|a_{p+k}| \leqslant |b|(A-B),$$

i.e.

$$\frac{\sum_{k=1}^{\infty}[k+|b(A-B)-Bk|]C_{p,k}(n)|a_{p+k}|}{|b|(A-B)} \leqslant 1.$$

($\Leftarrow$) Let (2.1) be true. From (2.2) we see that since $|w(z)|<1$,

$$\left|\frac{z(D^{n+p}f(z))'-pD^{n+p}f(z)}{b(A-B)D^{n+p}-B(z(D^{n+p}f(z))'-pD^{n+p}f(z))}\right|$$

$$= \left|\frac{\sum_{k=1}^{\infty}(-k)C_{p,k}(n)a_{p+k}z^k}{b(A-B)-\sum_{k=1}^{\infty}[b(A-B)-Bk]C_{p,k}(n)a_{p+k}z^k}\right| < 1. \tag{2.3}$$

We need to prove that (2.3) is true.

By applying the hypothesis (2.1) and letting $|z|=1$ we find that

$$\left|\frac{\sum_{k=1}^{\infty}(-k)C_{p,k}(n)a_{p+k}z^k}{b(A-B)-\sum_{k=1}^{\infty}[b(A-B)-Bk]C_{p,k}(n)a_{p+k}z^k}\right|$$

$$\leqslant \frac{\sum_{k=1}^{\infty}kC_{p,k}(n)|a_{p+k}|}{|b|(A-B)-\sum_{k=1}^{\infty}|b(A-B)-Bk|C_{p,k}(n)|a_{p+k}|}$$

$$\leqslant \frac{|b|(A-B)-\sum_{k=1}^{\infty}|b(A-B)-Bk|C_{p,k}(n)|a_{p+k}|}{|b|(A-B)-\sum_{k=1}^{\infty}|b(A-B)-Bk|C_{p+k}(n)|a_{p+k}|}$$

$$\leqslant 1.$$

Hence we find that (2.3) is true. Therefore $f \in M_p(A,B,b,n)$. □

3. Distortion Theorems

Theorem 3.1. *If $f \in M_p(A,B,b,n)$ then*

$$r^p - r^{p+1}\frac{|b|(A-B)}{[1+|b(A-B)-B|]C_{p,1}(n)} \leqslant |f(z)|$$

$$\leqslant r^p + r^{p+1}\frac{|b|(A-B)}{[1+|b(A-B)-B|]C_{p,1}(n)} \quad (|z|=r)$$

with equality for

$$f(z) = z^p - z^{p+1}\frac{|b|(A-B)}{[1+|b(A-B)-B|]C_{p,1}(n)}.$$

Proof. From (2.1) we obtain

$$\sum_{k=1}^{\infty}[k+|b(A-B)-Bk|]C_{p,k}(n)|a_{p+k}| \leqslant |b|(A-B)$$

$$\sum_{k=1}^{\infty}|a_{p+k}| \leqslant \frac{|b|(A-B)}{[1+|b(A-B)-B|]C_{p,1}(n)}. \tag{3.1}$$

From (1.2) and (3.1) it follows that

$$\begin{aligned}|f(z)| &\geqslant |z|^p - \sum_{k=1}^{\infty}|a_{p+k}||z|^{p+k} \\ &\geqslant r^p - r^{p+1}\sum_{k=1}^{\infty}|a_{p+k}| \\ &\geqslant r^p - r^{p+1}\frac{|b|(A-B)}{[1+|b(A-B)-B|]C_{p,1}(n)}.\end{aligned}$$

Similarly,

$$\begin{aligned}|f(z)| &\leqslant |z|^p + \sum_{k=1}^{\infty}|a_{p+k}||z|^{p+k} \\ &\leqslant r^p + r^{p+1}\sum_{k=1}^{\infty}|a_{p+k}| \\ &\leqslant r^p + r^{p+1}\frac{|b|(A-B)}{[1+|b(A-B)-B|]C_{p,1}(n)}.\end{aligned}$$

□

Theorem 3.2. *If $f \in M_p(A,B,b,n)$ then*

$$\begin{aligned}&pr^{p-1} - (p+1)r^p\frac{(p+1)|b|(A-B)}{[1+|b(A-B)-B|]C_{p,1}(n)} \leqslant |f'(z)| \\ &\leqslant pr^{p-1} + (p+1)r^p\frac{(p+1)|b|(A-B)}{[1+|b(A-B)-B|]C_{p,1}(n)} \quad (|z|=r)\end{aligned}$$

with equality for

$$f(z) = z^p - z^{p+1}\frac{|b|(A-B)}{[1+|b(A-B)-B|]C_{p,1}(n)}.$$

Proof. By (3.1) we have

$$\sum_{k=1}^{\infty}(p+k)|a_{p+k}| \leqslant \frac{(p+1)|b|(A-B)}{[1+|b(A-B)-B|]C_{p,1}(n)}. \tag{3.2}$$

From (1.2) and (3.2) it follows that

$$\begin{aligned}|f'(z)| &\geqslant p|z|^{p-1} - \sum_{k=1}^{\infty}(p+k)|a_{p+k}||z|^{p+k-1}\\ &\geqslant pr^{p-1} - r^p\sum_{k=1}^{\infty}(p+k)|a_{p+k}|\\ &\geqslant pr^{p-1} - r^p\frac{(p+1)|b|(A-B)}{[1+|b(A-B)-B|]C_{p,1}(n)}\,.\end{aligned}$$

Similarly,

$$\begin{aligned}|f'(z)| &\leqslant p|z|^{p-1} + \sum_{k=1}^{\infty}(p+k)|a_{p+k}||z|^{p+k-1}\\ &\leqslant pr^{p-1} + r^p\sum_{k=1}^{\infty}(p+k)|a_{p+k}|\\ &\leqslant pr^{p-1} - r^p\frac{(p+1)|b|(A-B)}{[1+|b(A-B)-B|]C_{p,1}(n)}\,.\end{aligned}$$

□

4. Close-To-Convexity, Starlikeness and Convexity

A function $f \in T(p)$ is said to be close-to-convex of order δ $(0 \leqslant \delta < 1)$ if

$$\mathrm{Re}\{f'(z)\} > \delta, \tag{4.1}$$

for all $z \in U$, see [1]. A function $f \in T(p)$ is said to be starlike of order δ if

$$\mathrm{Re}\Big\{\frac{zf'(z)}{f(z)}\Big\} > \delta. \tag{4.2}$$

A function $f \in T(p)$ is said to be convex of order δ if and only if $zf'(z)$ is starlike of order δ, that is, if

$$\mathrm{Re}\Big\{1 + \frac{zf''(z)}{f'(z)}\Big\} > \delta. \tag{4.3}$$

Theorem 4.1. *If $f \in M_p(A,B,b,n)$, then f is close-to-convex of order δ in $|z| < r_1(p,A,B,b,n,\delta)$ where*

$$r_1(p,A,B,b,n,\delta) = \inf_k\Big[\frac{(p-\delta)[k+|b(A-B)-Bk|]C_{p,k}(n)}{(p+k)[|b|(A-B)]}\Big]^{1/k}.$$

Proof. It is sufficient to show that

$$\Big|\frac{f'(z)}{z^{p-1}} - p\Big| \leqslant \sum_{k=1}^{\infty}(p+k)|a_{p+k}||z|^k \leqslant p-\delta. \tag{4.4}$$

By (2.1) we have

$$\sum_{k=1}^{\infty}[k+|b(A-B)-Bk|]C_{p,k}(n)|a_{p+k}| \leqslant |b|(A-B). \tag{4.5}$$

Observe that (4.4) is true if

$$\frac{(p+k)|z|^k}{p-\delta} \leqslant \frac{[k+|b(A-B)-Bk|]C_{p,k}(n)}{|b|(A-B)}. \tag{4.6}$$

Solving (4.6) for $|z|$ we obtain

$$|z| \leqslant \Big[\frac{(p-\delta)[k+|b(A-B)-Bk|]C_{p,k}(n)}{(p+k)[|b|(A-B)]}\Big]^{1/k}, \quad (p \in N).$$

□

Theorem 4.2. *If $f \in M_p(A,B,b,n)$ then f is starlike of order δ in $|z| < r_2(p,A,B,b,n,\delta)$ where*

$$r_2(p,A,B,b,n,\delta) = \inf_k \Big[\frac{(p-\delta)[k+|b(A-B)-Bk|]C_{p,k}(n)}{(p+k-\delta)[|b|(A-B)]}\Big]^{1/k}.$$

Proof. We must show that

$$\Big|\frac{zf'(z)}{f(z)} - p\Big| \leqslant \frac{\sum_{k=1}^{\infty} k|a_{p+k}||z|^k}{1-\sum_{k=1}^{\infty}|a_{p+k}||z|^k} \leqslant p-\delta. \tag{4.7}$$

We see from (4.5) that (4.7) is true if

$$\frac{(p+k-\delta)|z|^k}{p-\delta} \leqslant \frac{[k+|b(A-B)-Bk|]C_{p,k}(n)}{|b|(A-B)}. \tag{4.8}$$

Solving (4.8) for $|z|$ we obtain

$$|z| \leqslant \Big[\frac{(p-\delta)[k+|b(A-B)-Bk|]C_{p,k}(n)}{(p+k-\delta)[|b|(A-B)]}\Big]^{1/k}, \quad p \in N.$$

□

Theorem 4.3. *If $f \in M_p(A,B,b,n)$ then f is convex of order δ in*

$$|z| < r_3(p,A,B,b,n,\delta)$$

where

$$r_3(p,A,B,b,n,\delta) = \inf_k \Big[\frac{p(p-\delta)|k+|b(A-B)-Bk|]C_{p,k}(n)}{(p+k)(p+k-\delta)[|b|(A-B)]}\Big]^{1/k}.$$

Proof. We must show that

$$\Big|1+\frac{zf''(z)}{f'(z)} - p\Big| \leqslant \frac{\sum_{k=1}^{\infty} k(p+k)|a_{p+k}||z|^k}{p-\sum_{k=1}^{\infty}(p+k)|a_{p+k}||z|^k} \leqslant p-\delta. \tag{4.9}$$

From (4.5) we see that (4.9) is true if

$$\frac{(p+k)(p+k-\delta)|z|^k}{p(p-\delta)} \leqslant \frac{[k+|b(A-B)-Bk|]C_{p,k}(n)}{|b|(A-B)}. \tag{4.10}$$

Solving (4.10) for $|z|$ we obtain

$$|z| \leqslant \Big[\frac{p(p-\delta)[k+|b(A-B)-Bk|]C_{p,k}(n)}{(p+k)(p+k-\delta)[|b|(A-B)]}\Big]^{1/k}, \quad (p \in N).$$

□

5. An application in the fractional calculus

We recall here the following definitions of the fractional calculus given by Owa [3].

Definition 5.1. The fractional integral of order δ is defined, for a function $f(z)$, by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(t)}{(z-t)^{1-\delta}} dt$$

where $\delta > 0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\delta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-t) > 0$.

Definition 5.2. The fractional derivative of order δ is defined, for a function $f(z)$, by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\delta}} dt$$

where $0 \leqslant \delta < 1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{-\delta}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-t) > 0$.

Definition 5.3. Under the condition of Definition 5.2, the fractional derivative of order $n+\delta$ is defined by

$$D^{n+\delta} f(z) = \frac{d^n}{dz^n} D_z^{\delta} f(z)$$

where $0 \leqslant \delta < 1$ and $n = 0, 1, 2, \ldots$.

Now we shall prove the following theorems using the definition above (cf. [2]).

Theorem 5.4. *If $f \in T(p)$ is in the class $M_p(A, B, b, n)$ then*

(5.1)

$$|D_z^{-\delta} f(z)| \leqslant \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} |z|^{p+\delta} \Big[1 + \frac{(p+1)[|b|(A-B)]}{(p+\delta+1)[1+|b(A-B)-B|]C_{p,1}(n)} |z|\Big]$$

and

(5.2)
$$|D_z^{-\delta} f(z)| \geqslant \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)}|z|^{p+\delta}\Big[1 - \frac{(p+1)[|b|(A-B)]}{(p+\delta+1)[1+|b(A-B)-B|]C_{p,1}(n)}|z|\Big].$$

Proof. From Definition 5.1 we see that

$$D_z^{-\delta} f(z) = \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} z^{p+\delta} - \sum_{k=1}^{\infty} \frac{\Gamma(p+k+1)}{\Gamma(p+k+\delta+1)} a_{p+k} z^{p+k+\delta} \tag{5.3}$$
$$(\delta > 0; \quad k \geqslant 1; \quad p \in N).$$

For convenience let

$$\phi(k) = \frac{\Gamma(p+k+1)}{\Gamma(p+k+\delta+1)} \cdot$$

Clearly the function $\phi(k)$ is a decreasing function of k and

$$0 < \phi(k) \leqslant \phi(1) = \frac{\Gamma(p+2)}{\Gamma(p+\delta+2)} \cdot$$

By (2.1) we have that

$$\sum_{k=1}^{\infty} |a_{p+k}| \leqslant \frac{|b|(A-B)}{[1+|b(A-B)-B|]C_{p,1}(n)} \cdot \tag{5.4}$$

From (5.3) and (5.4) it follows that

$$|D_z^{-\delta} f(z)| \leqslant |z|^{p+\delta}\Big\{\frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} + \phi(1)|z| \sum_{k=1}^{\infty} |a_{p+k}|\Big\}$$
$$\leqslant \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)}|z|^{p+\delta}\Big\{1 + \frac{(p+1)[|b|(A-B)]}{(p+\delta+1)[1+|b(A-B)-B|]C_{p,1}(n)}|z|\Big\}$$

which is equivalent to (5.1) and

$$|D_z^{-\delta} f(z)| \geqslant |z|^{p+\delta}\Big\{\frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} - \phi(1)|z| \sum_{k=1}^{\infty} |a_{p+k}|\Big\}$$
$$\geqslant \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)}|z|^{p+\delta} \times$$
$$\times \Big\{1 - \frac{(p+1)[|b|(A-B)]}{(p+\delta+1)[1+|b|(A-B)-B]C_{p,1}(n)}|z|\Big\}$$

which is equivalent to (5.2). □

Theorem 5.5. *If $f \in T(p)$ is in the class $M_p(A, B, b, n)$ then*

(5.5)
$$|D_z^{\delta} f(z)| \leqslant \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)}|z|^{p-\delta}\Big[1 + \frac{(p+1)[|b|(A-B)]}{(p-\delta+1)[1+|b(A-B)-B|]C_{p,1}(n)}|z|\Big]$$

and

(5.6)
$$|D_z^\delta f(z)| \geqslant \frac{\Gamma(p+1)}{\Gamma(p-\delta+1)}|z|^{p-\delta}\Big[1-\frac{(p+1)[|b|(A-B)]}{(p-\delta+1)[1+|b(A-B)-B|]C_{p,1}(n)}|z|\Big].$$

Proof. By Definition 5.2 we have

$$D_z^\delta f(z) = \frac{\Gamma(p+1)}{\Gamma(p-\delta+1}z^{p-\delta} - \sum_{k=1}^{\infty}\frac{\Gamma(p+k+1)}{\Gamma(p+k-\delta+1)}a_{p+k}z^{p+k-\delta}$$
$$(0 \leqslant \delta < 1; \quad k \geqslant 1; \quad p \in N)$$

(5.7)
$$D_z^\delta f(z) = \frac{\Gamma(p+1)}{\Gamma(p-\delta+1)}z^{p-\delta} - \sum_{k=1}^{\infty}\frac{(p+k)\Gamma(p+k)}{\Gamma(p+k-\delta+1)}a_{p+k}z^{p+k-\delta}.$$

Let $\Psi(k) = \dfrac{\Gamma(p+k)}{\Gamma(p+k-\delta+1)}$. Since $\Psi(k)$ is a decreasing function of k we have

$$0 < \Psi(k) \leqslant \Psi(1) = \frac{\Gamma(p+1)}{\Gamma(p-\delta+2}\,.$$

By (5.4) we have

(5.8)
$$\sum_{k=1}^{\infty}(p+k)|a_{p+k}| \leqslant \frac{(p+1)|b|(A-B)}{[1+|b(A-B)-B|]C_{p,1}(n)}\,.$$

From (5.7) and (5.8) it follows that

$$\begin{aligned}|D_z^\delta f(z)| &\leqslant |z|^{p-\delta}\Big\{\frac{\Gamma(p+1)}{\Gamma(p-\delta+1)} + \Psi(1)|z|\sum_{k=1}^{\infty}(p+k)|a_{p+k}|\Big\}\\ &\leqslant \frac{\Gamma(p+1)}{\Gamma(p-\delta+1)}|z|^{p-\delta} \times\\ &\quad\times\Big\{1+\frac{(p+1)[|b|(A-B)]}{(p-\delta+1)[1+|b(A-B)-B|]C_{p,1}(n)}|z|\Big\}\end{aligned}$$

which is equivalent to (5.5) and

$$\begin{aligned}|D_z^\delta f(z)| &\geqslant |z|^{p-\delta}\Big\{\frac{\Gamma(p+1)}{\Gamma(p-\delta+1)} - \Psi(1)|z|\sum_{k=1}^{\infty}(p+k)|a_{p+k}|\Big\}\\ &\geqslant \frac{\Gamma(p+1)}{\Gamma(p-\delta+1)}|z|^{p-\delta} \times\\ &\quad\times\Big\{1-\frac{(p+1)[|b|(A-B)]}{(p-\delta+1)[1+|b(A-B)-B|]C_{p,1}(n)}|z|\Big\}\end{aligned}$$

which is equivalent to (5.6). □

References

[1] M. K. Aouf and B. A. Al-Amri, *On certain subclass of analytic functions with complex order*, Demonstratio Mathematica **4** (2003), 827-838.
[2] S. Owa, *On the distortion theorems*, I. Kyungpook Math. J. **18** (1978), 53-59.
[3] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Maths. Soc. **49** (1975), 109-115.
[4] T. Sekine, S. Owa and M. Obradovic, *A certain class of p-valent functions with negative coefficients*, Current Topics in Analytic Function Theory, H. M. Srivastava and S. Owa (Eds.), World Scientific Publishing Co. Pte Ltd, 1992.

School Of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600, Selangor Darul Ehsan, Malaysia
E-mail address: maslina@pkrisc.cc.ukm.my