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ON THE REPRESENTATIVE THEOREMS FOR ONE-DIMENSIONAL ITERATIVE ARRAYS OF FINITE AUTOMATA

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Abstract. In this paper we show that there are representative theorems (or supply-demand theorems) for one-dimensional iterative arrays of finite automata and for one-dimensional iterative arrays of finite automata with a time-variant structure. Some applications are considered.

1. Introduction

In Computer Science, to study the capacity and the behaviour of processing systems we consider the languages representable by these systems. During the representation, the system needs to distinguish the non equivalent words by remembering each class of equivalent words into a state of the system. Therefore between the state growth speed of system (a supply) and the (non equivalent) word growth speed of the language representable by the system (a demand) there exists a nice supply-demand relation, which could be formulated as a very simple, but in no way trivial fact:

$$\text{“Representability} \Longrightarrow \text{Demand} \leqslant \text{Supply”}.$$

This relation is called the representative principle (or supply-demand principle) in Computer Science.

Our goal is to show that for all concrete processing systems in Computer Science such as the finite automata, the automata with a time-variant structure, the probabilistic automata, the Petri nets, the iterative arrays of finite automata, etc..., the above representative principle (or supply-demand principle) should become the representative theorems (or supply-demand theorems).

Following this approach we have shown in [12] that there are representative theorems for the finite automata and for the automata with a time-variant structure. As a corollary of these theorems we get again the well-known necessary

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conditions for the classes of languages representable by finite automata, by finite automata with a time-variant structure, by φ-automata with a time-variant structure, but now on an unified point of view.

For the class of languages representable by finite probabilistic automata (class of stochastic languages) we have the basic results of M. Rabin, A. Salomaa and P. D. Dieu (see [6-8]). By a glance, it seems that these results violated the supply-demand principle. With a constant number of states, the finite probabilistic automata could represent many rather complex languages, (see [5-6]). In [13] by analysing the representation of finite probabilistic automata, we have shown that there are also representative theorems for finite probabilistic automata but here the notion of state is understood in a more general sense. It is the notion of hyperstates.

For the class of languages representable by Petri nets, although we have had some examples of non-Petri net languages, we do not have any criterion for recognizing whether a given language is Petri net language or not. In [9-10] we proved that there are representative theorems for Petri net. Applying these theorems we get new necessary conditions for Petri net languages and give a series of simple languages but not being representable by any Petri net.

In this paper we enrich our line of research by exhibiting representative theorems for the one-dimensional iterative arrays of finite automata and for the one-dimensional iterative arrays of finite automata with a time-variant structure. Some applications are investigated.

The paper is organized as follows. In Section 2 we recall some definitions of one-dimensional iterative array of finite automata and the language representable by it. Section 3 deals with the notion of growth function for iterative array and gives the growth theorem for one-dimensional iterative array of finite automata. The first representative theorem for one-dimensional iterative array of finite automata is described in Section 4. Section 5 gives the second representative theorem for one-dimensional iterative array of finite automata. Finally, in Section 6 we introduce the notion of one-dimensional iterative array of finite automata with a time-variant structure and show that we also get representative theorems for the new model of iterative array.

2. Preliminaries

For a finite alphabet Σ, Σ^* (resp. Σ^r) denotes the set of all words (resp. of all words of length r) on the alphabet Σ. The empty word is denoted by Λ. For any word $w \in \Sigma^*$, $l(w)$ denotes the length of w. Every subset $L \subseteq \Sigma^*$ is called a language over the alphabet Σ. Let N be the set of all non-negative integers and $N^+ = N\backslash\{0\}$.

As has been well-known, the notion of iterative array of finite automata was first introduced by Von Neumann in model of self-reproduction (see [1]) and was investigated by many authors, e.g., by S. N. Cole, F. C. Hennie, A. J. Atrubin, P. C. Fischer, A. R. Smith, P. D. Dieu, D. L. Van, etc...

Loosely speaking, an one-dimensional iterative array of finite automata is an infinite sequence of finite automata $\mathit{\Gamma}_0, \mathit{\Gamma}_1, \cdots, \mathit{\Gamma}_n, \cdots$ connected in the following way (see the figures):

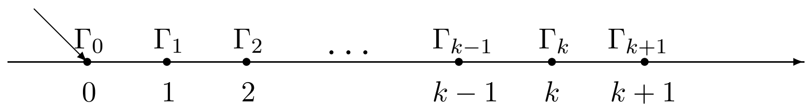

Fig 1a.

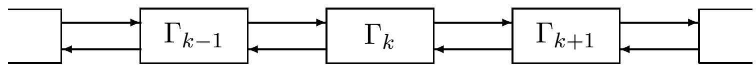

Fig 1b.

Fig. 1

(1) All automata $\mathit{\Gamma}_n$, $n \geqslant 0$ have the same structure. At the time $t = 0$, all $\mathit{\Gamma}_n$, $n \geqslant 1$ are at the quiescent state q and $\mathit{\Gamma}_0$ is at the initial state q_0.

(2) The state of $\mathit{\Gamma}_n$, $n \geqslant 1$, at the time t is defined by the states of $\mathit{\Gamma}_{n-1}$ and $\mathit{\Gamma}_{n+1}$ at the time $t - 1$.

(3) The state of $\mathit{\Gamma}_0$ at the time t is defined by the input symbol and the states of $\mathit{\Gamma}_0$ and $\mathit{\Gamma}_1$ at the time $t - 1$.

(4) The input of iterative array is connected to $\mathit{\Gamma}_0$.

Now we can define an one-dimensional iterative array of finite automata formally as follows.

An *one-dimensional iterative array of finite automata* (abbreviated 1IA) is given by a list

$$\mathit{\Gamma} = (\Sigma, Q, q_0, q, \lambda_0, \lambda, F),$$

where

Σ is a finite set of input symbols;

Q is the finite set of states of automata $\mathit{\Gamma}_n$, $n \geqslant 0$;

$q_0 \in Q$ is the initial state of $\mathit{\Gamma}_0$;

$q \in Q$ is the quiescent state of $\mathit{\Gamma}_n$, $n \geqslant 1$;

$\lambda_0 : \Sigma \times Q \times Q \longrightarrow Q$ is the state transition function of $\mathit{\Gamma}_0$;

$\lambda : Q \times Q \times Q \longrightarrow Q$ is the state transition function of $\mathit{\Gamma}_n$, $n \geqslant 1$, with the condition $\lambda(q_0, q, q) = \lambda(q, q, q) = q$;

$F \subseteq Q$ is the set of final states of $\mathit{\Gamma}_0$.

Let M be a mapping from $\Sigma \times Q^+ \longrightarrow Q^+$ defined by

$$M(a, p_0 p_1 \cdots p_n) = \lambda_0(a, p_0, p_1)\lambda(p_0, p_1, p_2) \cdots \lambda(p_{n-1}, p_n, q)\lambda(p_n, q, q)$$

with $n \geqslant 0$, $a \in \Sigma$, $p_i \in Q$, $0 \leqslant i \leqslant n$.

Here $M(\omega, u)$ is the state transition function of the iterative array $\mathit{\Gamma}$.

The mapping M can be extended to a mapping from $\Sigma^* \times Q^+ \longrightarrow Q^+$ by the recursive definition

$$\begin{cases} M(\Lambda, u) & = & u, \\ M(\omega a, u) & = & M(a, M(\omega, u)) \end{cases}$$

with $a \in \Sigma$, $\omega \in \Sigma^*$ and $u \in Q^+$.

After that we define N to be a mapping from $\Sigma^* \times Q^+ \longrightarrow Q$ such that for $\omega \in \Sigma^*$, $u \in Q^+$ and if $M(\omega, u) = p'_0 p'_1 \cdots p'_n$ then

$$N(\omega, u) = p'_0,$$

where $N(\omega, u)$ is the state transition function of Γ_0.

A word $\omega \in \Sigma^*$ is said to be representable by the iterative array Γ if $N(\omega, q_0) \in F$. The language representable by the iterative array Γ is the set

$$L(\Gamma) \ = \ \{ \omega \ \in \ \Sigma^* \,|\, N(\omega, q) \in F \}.$$

A language which is representable by an one-dimensional iterative array, is called an 1IA-language. The set of all 1IA-languages is denoted by $\mathcal{L}(1IA)$.

3. The growth theorem for 1IA

Let $\Gamma \ = \ (\Sigma \,, \, Q \,, \, q_0 \,, \, q \,, \, \lambda_0 \,, \, \lambda \,, \, F)$ be an 1IA. Each state of Γ_i, $i \geqslant 0$, describes a local state of Γ, each combination of non-quiescent states of Γ_i, $i \geqslant 0$, describes a global state of Γ and is called a configuration of Γ. The configuration $c_0 = (q_0, q, q, \cdots)$ is the initial configuration of Γ. After that we define the following sets:

$$\mathcal{C}_{\Gamma,r} \ = \ \{M(\omega, c_0) \,|\, \forall q_0 \in Q, \forall \omega \in \Sigma^r\},$$

$$\mathcal{C}_{\Gamma,\leq r} \ = \ \{M(\omega, c_0) \,|\, \forall q_0 \in Q, \forall \omega \in \Sigma^{\leq r}\}.$$

$\mathcal{C}_{\Gamma,r}$ (resp. $\mathcal{C}_{\Gamma,\leqslant r}$) is the set of all reachable configurations of Γ from any initial configuration c_0 and with any input $\omega \in \Sigma^r$ (resp. $\omega \in \Sigma^{\leqslant r}$).

Definition 1. The *growth functions of iterative array Γ* are

$$h_\Gamma(r) = |\mathcal{C}_{\Gamma,r}|,$$

$$g_\Gamma(r) = |\mathcal{C}_{\Gamma,\leqslant r}|;$$

where $|\mathcal{C}_{\Gamma,r}|$, is the cardinal of the set $\mathcal{C}_{\Gamma,r}$.

The following theorem gives us an upper bound of the growth functions for any 1IA.

Theorem 1 (The growth theorem for 1IA)**.** *Let Γ be an 1IA. We have*

$$h_\Gamma(r) = O(C^{P_1(r)}), \quad \forall r \in N^+,$$

$$g_\Gamma(r) = O(C^{P_1(r)}), \quad \forall r \in N^+.$$

where $C = \text{const}$ and $P_1(r)$ is a polynominal of degree 1.

Proof. Let $\Gamma = (\Sigma, Q, q_0, q, \lambda_0, \lambda, F)$ be an 1IA. From the definition of the growth functions, we have

$$h_\Gamma(r) = |\mathcal{C}_{\Gamma,r}| \leqslant |Q|^{r+1} = O(C^{P_1(r)}), \quad \forall r \in N^+,$$
$$g_\Gamma(r) = |\mathcal{C}_{\Gamma,\leq r}| \leqslant \frac{|Q|(|Q|^{r+1}-1)}{|Q|-1} = O(C^{P_1(r)}), \quad \forall r \in N^+.$$

where $C = |Q| = \text{const}$ and $P_1(r) = r + 1$.

Thus the growth functions of any 1IA is bounded by $C^{P_1(r)}$. This is an essential limitation of the 1IA. □

4. The first representative theorem for 1IA

Let $L \subseteq \Sigma^*$. We define the relations $L_r(modL)$ in Σ^r and $L_{\leq r}(modL)$ in $\Sigma^{\leq r}$ as follows:

$$uL_r v\,(modL) \Leftrightarrow \forall w \in \Sigma^* : wu \in L \leftrightarrow wv \in L\,, \quad \forall u, v \in \Sigma^r;$$
$$uL_{\leq r} v\,(modL) \Leftrightarrow \forall w \in \Sigma^* : wu \in L \leftrightarrow wv \in L\,, \quad \forall u, v \in \Sigma^{\leqslant r};$$

It is easy to show that the relations $L_r\,(modL)$ and $L_{\leqslant r}\,(modL)$ are reflexive, symmetric and transitive. They are equivalent relations. So we define:

$$H_L(r) = Rank L_r\,(modL);$$
$$G_L(r) = Rank L_{\leqslant r}\,(modL).$$

They are considered to be the representative complexities of the language L in Σ^r and in $\Sigma^{\leqslant r}$ respectively.

First, we give a simple but important property of $H_L(r)$, $G_L(r)$.

Let Σ be an alphabet, $|\Sigma| = m \geqslant 2$. We have

$$1 \leqslant H_L(r) \leqslant |\Sigma^r| = O(m^r) = O(C^{P_1(r)})\,, \quad \forall r \in N^+;$$
$$1 \leqslant G_L(r) \leqslant |\Sigma^{\leq r}| = O(m^r) = O(C^{P_1(r)})\,, \quad \forall r \in N^+.$$

where $C = \text{const}$ and $P_1(r)$ is a polynominal of degree 1.

There is a nice relation between the growth functions of an 1IA and the complexity functions of the language which is representable by it.

Theorem 2 (The first representative theorem for 1IA)**.** *Let Γ be an 1IA and $L = L(\Gamma)$. We have*

$$H_L(r) \leqslant h_\Gamma(r)\,, \quad \forall r \in N^+;$$
$$G_L(r) \leqslant g_\Gamma(r)\,, \quad \forall r \in N^+.$$

Proof. Let $\Gamma = (\Sigma, Q, q_0, q, \lambda_0, \lambda, F)$ be an 1IA and $L = L(\Gamma)$. We shall prove that, for example $G_L(r) \leqslant g_\Gamma(r)\,, \ \forall r \in N^+$. To do this, we assume the contrary, i.e., there exists an $r \in N^+$ such that $G_L(r) > g_\Gamma(r)$. Therefore, there are two words $u, v \in \Sigma^{\leqslant r}$ such that $u\overline{L}_{\leqslant r} v\,(modL)$, but $C_0(u) = C_0(v)$, where $C_0(\omega) = M(\omega, c_0)$, $\omega \in \Sigma^*$. It follows that

$$M(\omega, C_0(u)) \;=\; M(\omega, C_0(v))\,, \quad \forall \omega \in \Sigma^*;$$

$$M(\omega u, c_0) \ = \ M(\omega v, c_0) \ , \quad \forall \omega \in \Sigma^*.$$
$$N(\omega u, c_0) \ = \ N(\omega v, c_0) \ , \quad \forall \omega \in \Sigma^*;$$
$$\omega u \in L \longleftrightarrow \omega v \in L \ , \quad \forall \omega \in \Sigma^*;$$
$$u L_{\leqslant r} v \, (mod L).$$

It conflicts with the hypothesis $u\overline{L}_{\leqslant r} v \, (mod L)$. Therefore we get $G_L(r) \ \leqslant \ g_{\Gamma}(r)$, $\forall r \in N^+$.

By an analogous argument, we have $H_L(r) \ \leqslant \ h_{\Gamma}(r)$, $\forall r \in N^+$. □

Remark. We note that the upper bound for the growth functions $h_\Gamma(r)$, $g_\Gamma(r)$ of an 1IA and the upper bound for the comlexity functions $H_L(r)$, $G_L(r)$ of a language are the same $O(C^{P_1(r)})$. Therefore, from the Theorm 2 we can not get a necessary condition for the class $\mathcal{L}(1IA)$. The application of the first representative theorem is limited by it.

Now we consider a special case of $1IA$, whose growth functions are decreased in degree such that we could get some necessary condition.

Definition 2. An $1IA$ is K-*bounded*, $K \in N^+$, if there are only automata at $0, 1, \cdots, (K-1)$ and there are not automata at $K, K+1, \cdots$. An $1IA$ is *bounded* (abbreviated $1BIA$), if there exists an $K \in N^+$ such that the $1IA$ is K-bounded. The class of all languages representable by $1BIA$ is denoted by $\mathcal{L}(1BIA)$.

It is easy to see that if Γ is bounded, then its growth functions are bounded too. Indeed, there exists some $K \in N^+$ such that

$$h_\Gamma(r) \ \leqslant \ |Q|^{K+1} = O(C) \ , \quad \forall r \in N^+,$$
$$g_\Gamma(r) \ \leqslant \ \frac{|Q|(|Q|^{K+1} - 1)}{|Q| - 1} = O(C) \ , \quad \forall r \in N^+.$$

where $C =$ const.

Corollary 1 (The necessary condition for $\mathcal{L}(1BIA)$)**.** *If* $L \in \mathcal{L}(1BIA)$ *then*

$$H_L(r) = O(C) \ , \quad \forall r \in N^+;$$
$$G_L(r) = O(C) \ , \quad \forall r \in N^+;$$

where $C =$ const*.*

Proof. It follows from Theorem 2 and the growth functions of any $1BIA$ are bounded. □

Example 1. Let $\Sigma = \{a, b\}$ and

$$L_1 \ = \ \{ \ a^n b^n \mid n \in N^+\}.$$

Denote $W = \{a, a^2, \cdots, a^r\}$. Therefore $W \subseteq \Sigma^{\leqslant r}$ and $|W| = r$. We can verify that $a^i \overline{L}_{\leqslant r} a^j \, (mod L)$ with $i \neq j$. It follows that $G_{L_1}(r) \geqslant |W| = r > C$. According to Corollary 1 we get $L_1 \notin \mathcal{L}(1BIA)$.

Example 2. Let Σ be an alphabet, $|\Sigma| = m \geqslant 2$ and

$$L_2 \;=\; \{\; xx^R \mid x \in \Sigma^*\},$$

where x^R is the inverse image of x. We can see that for all $x_1, x_2 \in \Sigma^r$, if $x_1 \neq x_2$ then $x_1 \overline{L}_r x_2\,(modL_2)$. Therefore

$$H_{L_2}(r) = |\Sigma^r| = m^r.$$

Now we assume that $\Gamma \;=\; (\Sigma\,,\,Q\,,\,q_0\,,\,q\,,\,\lambda_0\,,\,\lambda\,,\,F)$ is an 1IA with $|Q| = n$ and $L_2 = L(\Gamma)$. In Section 3 we have shown that for any 1IA Γ with $|Q| = n$, the growth function $h_\Gamma(r) \;\leqslant\; |Q|^{r+1} \;=\; n^{r+1}$.

Applying Theorem 1 and Theorem 2 we get

$$H_{L_2}(r) = m^r \;\leqslant\; h_\Gamma(r) \;\leqslant\; n^{r+1}\;, \quad \forall m, n, r \in N^+.$$

If there is a triplet (m, n, r) satisfying $m^r \;>\; n^{r+1}$ (for example ($m = 10$, $n = 3$, $r = 1$) is a thus triplet) then L_2 on alphabet Σ, $|\Sigma| = m$ is not representable by any $1IA$ with $|Q| \;\leqslant\; n$. At the time $t = r$, the iterative array shall be overfull.

5. The second representative theorem for 1IA

Let $L \subseteq \Sigma^*$. We now define the other equivalent relations $R_r\,(modL)$ and $R_{\leq r}\,(modL)$. They are dual relations to the relations $L_r\,(modL)$ and $L_{\leq r}\,(modL)$ in Section 4.

$$uR_rv\,(modL) \Leftrightarrow \forall w \in \Sigma^r : uw \in L \leftrightarrow vw \in L\;, \quad \forall u, v \in \Sigma^*;$$

$$uR_{\leqslant r}v\,(modL) \Leftrightarrow \forall w \in \Sigma^{\leqslant r} : uw \in L \leftrightarrow vw \in L\;, \quad \forall u, v \in \Sigma^*.$$

The relation $R_{\leqslant r}\,(modL)$ was first considered by S. N. Cole in [2]. We can see that they are equivalent relations. So we define

$$I_L(r) \;=\; Rank\,R_r\,(modL);$$

$$J_L(r) \;=\; Rank\,R_{\leqslant r}\,(modL).$$

It is easy to prove that $\forall L \subseteq \Sigma^*$, we have:

$$1 \leqslant I_L(r) \leqslant 2^{|\Sigma^r|} = O(2^{m^r}) = O(C^{Exp(r)})\;, \quad \forall r \in N^+;$$

$$1 \leqslant J_L(r) \leqslant 2^{|\Sigma^{\leqslant r}|} = O(2^{\frac{m(m^r-1)}{m-1}}) = O(C^{Exp(r)})\;, \quad \forall r \in N^+.$$

where $C =$ const and $Exp(r)$ denote an exponential function of r.

The functions $I_L(r)$ and $J_L(r)$ are considered to be other complexity functions of language L, and we get also an other representative theorem for 1IA.

Theorem 3 (The second representative theorem for 1IA)**.** *Let Γ be an 1IA and $L = L(\Gamma)$. Then*

$$I_L(r) \;\leq\; h_\Gamma(r)\,, \quad \forall r \in N^+;$$

$$J_L(r) \;\leq\; g_\Gamma(r)\,, \quad \forall r \in N^+.$$

Proof. Let $\Gamma = (\Sigma, Q, q_0, q, \lambda_0, \lambda, F)$ be an 1IA and $L = L(\Gamma)$. We shall prove that, for example $J_L(r) \leq g_\Gamma(r)$, $\forall r \in N^+$. We assume the contrary, i. e. there exists an $r \in N^+$ such that $J_L(r) > g_\Gamma(r)$. Then we shall prove that there are two words $u, v \in \Sigma^*$ such that $u\overline{R}_{\leqslant r} v\,(modL)$, but $C_0(u)$ and $C_0(v)$, where $C_0(w) = M(w, c_0)$, $w \in \Sigma^*$, are non r-distinguished in the sense

$$M(\omega, C_0(u)) = M(\omega, C_0(v)), \quad \forall \omega \in \Sigma^{\leqslant r};$$

(Note that here we have not $C_0(u) = C_0(v)$, because $u, v \notin \Sigma^{\leqslant r}$, in general). Indeed, we set $J_(r) = p > g_\Gamma(r)$. The relation $R_{\leqslant r}\,(modL)$ devises Σ^* into p equivalent classes. Let $\tau_1, \cdots, \tau_p$ be representatives of these classes. With each word $\omega \in \Sigma^{\leqslant r}$, we consider the following set:

$$\{M(\omega, C_0(\tau_1)), \cdots, M(\omega, C_0(\tau_p))\},$$

This set has p components. But on the other hand,

$$\{M(\omega, C_0(\tau_1)), \cdots, M(\omega, C_0(\tau_p))\} \subseteq \mathcal{C}_{\Gamma, \leqslant r},$$

with $|\mathcal{C}_{\Gamma, \leqslant r}| = g_\Gamma(r) < p$. Therefore we have

$$|\{M(\omega, C_0(\tau_1)), \cdots, M(\omega, C_0(\tau_p))\}| < p.$$

So there are two words $u = \tau_i$, $v = \tau_j$ such that

$$M(\omega, C_0(u)) = M(\omega, C_0(v)).$$

Since ω is any word in $\Sigma^{\leqslant r}$, we obtain

$$M(\omega, C_0(u)) = M(\omega, C_0(v)), \quad \forall \omega \in \Sigma^{\leqslant r}.$$

It mean u, v are two non r-distinguished words.

Now we continue to prove Theorem 3. From $M(\omega, C_0(u)) = M(\omega, C_0(v))$, $\forall \omega \in \Sigma^{\leqslant r}$, we obtain

$$M(u\omega, c_0) = M(v\omega, c_0), \quad \forall \omega \in \Sigma^{\leqslant r};$$

$$N(u\omega, c_0) = N(v\omega, c_0), \quad \forall \omega \in \Sigma^{\leqslant r};$$

$$u\omega \in L \longleftrightarrow v\omega \in L, \quad \forall \omega \in \Sigma^{\leqslant r};$$

$$uR_{\leqslant r} v\,(modL)$$

It contradicts the hypothesis $u\overline{R}_{\leqslant r} v\,(modL)$. Therefore, we get

$$J_L(r) \leqslant g_\Gamma(r), \quad \forall r \in N^+.$$

By an analogous argument, we have $I_L(r) \leqslant h_\Gamma(r)$, $\forall r \in N^+$. □

Corollary 2 (The Cole's necessary condition for $\mathcal{L}(1IA)$)**.** (see [2]). *If $L \in \mathcal{L}(1IA)$ then*

$$I_L(r) = O(C^{P_1(r)}), \quad \forall r \in N^+,$$

$$J_L(r) = O(C^{P_1(r)}), \quad \forall r \in N^+;$$

where $C = \text{const}$ and $P_1(r)$ is a polynominal of degree 1.

Proof. It follows from Theorem 1 and Theorem 3. □

Example 3. Let Σ be an alphabet, $|\Sigma| = m \geqslant 2$ and

$$L_3 \,=\, \Sigma^* D,$$

where D is the set of all symmetric words of length more than 3 in Σ^*. In [2] S. N. Cole had shown that $J_{L_3} \geqslant 2^{2^{(r-3)/2}} = O(C^{Exp(r)}) > O(C^{P_1(r)})$, with r large enough. By Corollary 2, it follows that $L_3 \notin \mathcal{L}(1IA)$.

Example 4. Let Σ be an alphabet, $|\Sigma| = m \geqslant 2$ and

$$L_{4,l} \,=\, \{\tau_1\tau_2\cdots\tau_n\tau_0 |\ \tau_i \in \Sigma^*; l(\tau_i) = l; \exists \tau_i = \tau_0\};$$

For each subset $A = \{\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_k}\} \subseteq \Sigma^l$ we associate it with a word

$$U_A = \tau_{i_1}\tau_{i_2}\ldots\tau_{i_k}$$

We can verify that:

$$\forall w \in \Sigma^l \,:\, (U_A w \in L_{4,l} \Leftrightarrow w \in A).$$

Now if we choose $r = l$ and l is large enough, then we have

$$I_{L_{4,l}}(l) \,\geqslant\, 2^{|\Sigma^l|} = O(2^{m^l}) = O(C^{Exp(l)}) > O(C^{P_1(l)})$$

According to Corollary 2 we obtain $L_{4,l} \notin \mathcal{L}(1IA)$, with l large enough.

6. One-dimensional iterative arrays of finite automata with a time-variant structure

A natural way to generalize the notion of an iterative array of finite automata is to allow the structure of the component automata to be time-variant.

Formally, an *one-dimensional iterative array of finite automata with a time-variant structure* (abbreviated 1TVIA) is given by a list

$$\Delta \,=\, (\Sigma\,,\, Q\,,\, q_0\,,\, q\,,\, \lambda_{0,t}\,,\, \lambda_t\,,\, F_t),$$

where

Σ, Q, q_0, q are the sames in 1IA;

$\forall t \in N\,,\ \lambda_{0,t} \,:\, \Sigma \times Q \times Q \longrightarrow Q$ is the state transition function of Γ_0 at the time t;

$\forall t \in N\,,\ \lambda_t \,:\, Q \times Q \times Q \longrightarrow Q$ is the state transition function of Γ_n, $n \geqslant 1$ at the time t, with the condition $\lambda_{0,t}(q_0, q, q) = \lambda_t(q, q, q) = q$;

$\forall t \in N\,,\ F_t \subseteq Q$ is the set of final states of Γ_0 at the time t.

Let M_t be a mapping from $\Sigma \times Q^+ \longrightarrow Q^+$ defined by

$$M_t(a, p_0p_1\cdots p_n) = \lambda_{0,t}(a, p_0, p_1)\lambda_t(p_0, p_1, p_2)\cdots\lambda_t(p_{n-1}, p_n, q)\lambda_t(p_n, q, q),$$

with $n \geqslant 0$, $a \in \Sigma$, $p_i \in Q$, $0 \leqslant i \leqslant n$.

The mapping M_t can be extended to a mapping from $\Sigma^* \times Q^+ \longrightarrow Q^+$ by the recursive definition

$$\begin{cases} M_t(\Lambda, u) &= u; \\ M_t(\omega a, u) &= M_t(a, M_{t-1}(\omega, u)); \end{cases}$$

with $a \in \Sigma$, $\omega \in \Sigma^*$ and $u \in Q^+$.

After that we define N_t to be a mapping from $\Sigma^* \times Q^+ \longrightarrow Q$ such that for $\omega \in \Sigma^*$, $u \in Q^+$ and if $M_t(\omega, u) = p'_0 p'_1 \cdots p'_n$ then

$$N_t(\omega, u) = p'_0;$$

$N_t(\omega, u)$ is the state transition function of Γ_0 at the time t.

The language representable by the iterative array Δ is the set

$$L(\Delta) \;=\; \{\, \omega \;\in\; \Sigma^* \,|\, N_0(\omega, q) \in F_{l(\omega)} \,\}.$$

A language which is representable by an 1TVIA, is called an 1TVIA-language. The set of all 1TVIA-languages is denoted by $\mathcal{L}(1TVIA)$.

For $\omega \in \Sigma^*$, we define

$$\mathcal{C}_{\Delta,r} \;=\; \{M_0(\omega, c_0)\,|\,\forall q_0 \in Q, \forall \omega \in \Sigma^r\},$$
$$\mathcal{C}_{\Delta,\leq r} \;=\; \{M_0(\omega, c_0)\,|\,\forall q_0 \in Q, \forall \omega \in \Sigma^{\leqslant r}\}.$$

$\mathcal{C}_{\Delta,r}$ (resp. $\mathcal{C}_{\Delta,\leq r}$) is the set of all reachable configurations of Δ from any initial configuration c_0 and with any input $\omega \in \Sigma^r$ (resp. $\omega \in \Sigma^{\leqslant r}$).

The growth functions of iterative array of finite automata with a time-variant structure Δ are

$$h_\Delta(r) \;=\; |\mathcal{C}_{\Delta,r}|,$$
$$g_\Delta(r) \;=\; |\mathcal{C}_{\Delta,\leqslant r}|.$$

By the analogous argument in proof of Theorem 1, we get the upper bounds of the growth functions for any 1TVIA:

$$h_\Delta(r) \;=\; O(C^{P_1(r)}), \quad \forall r \in N^+,$$
$$g_\Delta(r) \;=\; O(C^{P_1(r)}), \quad \forall r \in N^+.$$

where $C = const$ and $P_1(r)$ is a polynominal of degree 1.

Theorem 4 (The first representative theorem for 1TVIA)**.** *Let Δ be an 1TVIA and $L = L(\Delta)$. We have*

$$H_L(r) \,\leq\, h_\Delta(r), \quad \forall r \in N^+.$$

Proof. Let $\Delta \;=\; (\Sigma\,, Q\,, q_0\,, q\,, \lambda_{0,t}\,, \lambda_t\,, F_t)$ be an 1TVIA and $L = L(\Delta)$. To prove $H_L(r) \,\leqslant\, h_\Delta(r)\,,\;\; \forall r \in N^+$, we assume the contrary, i. e., there exists an $r \in N^+$ such that $H_L(r) \,>\, h_\Delta(r)$. Therefore, there are two words $u, v \in \Sigma^r$ such that $u\overline{L_r}v$ (mod L), but $C_0(u) = C_0(v)$, where $C_0(\omega) = M_0(\omega, c_0)$, $\omega \in \Sigma^*$. It follows that

$$M_{l(u)}(\omega, C_0(u)) \;=\; M_{l(v)}(\omega, C_0(v))\;, \quad \forall \omega \in \Sigma^*;$$
$$M_0(\omega u, c_0) \;=\; M_0(\omega v, c_0)\;, \quad \forall \omega \in \Sigma^*.$$
$$N_0(\omega u, c_0) \;=\; N_0(\omega v, c_0)\;, \quad \forall \omega \in \Sigma^*;$$
$$\omega u \in F_{r+l(\omega)} \longleftrightarrow \omega v \in F_{r+l(\omega)}\;, \quad \forall \omega \in \Sigma^*;$$
$$\omega u \in L \longleftrightarrow \omega v \in L\;, \quad \forall \omega \in \Sigma^*;$$
$$uL_r v\,(modL)$$

It contradicts the hypothesis $u\overline{L_r}v$ (mod L). Therefore we get $H_L(r) \,\leqslant\, h_\Delta(r)\,$, $\forall r \in N^+$. ◻

Example 5. Let Σ be an alphabet, $|\Sigma| = m \geqslant 2$ and

$$L_2 \;=\; \{\; xx^R \mid x \in \Sigma^*\},$$

where x^R is the inverse image of x.

In Example 2 we have shown that $H_{L_2}(r) = |\Sigma^r| = m^r$ and if there exists a triplet (m, n, r) satisfying the condition $m^r \;\geqslant\; n^{r+1}$, then L_2 is not representable by any 1IA with $|Q| \leqslant n$.

Now we continue to prove that if we have the above condition then L_2 is not representable by even any 1TVIA with $|Q| \leqslant n$.

Indeed, for any 1TVIA Δ we have $h_\Delta(r) = |Q|^{r+1} = n^{r+1}$. If $L_2 = L(\Delta)$ then according to Theorem 4, we get

$$H_{L_2}(r) = m^r \leqslant h_\Delta(r) = n^{r+1}.$$

It contradicts the condition $m^r > n^{r+1}$. Our confirmation is proved.

Now we consider the other equivalent relations that are also suitable to 1TVIA. Let $L \subseteq \Sigma^*$ and $i, r \in N^+$. We define the relations $R_{i,r}\,(modL)$ and $R_{i,\leq r}\,(modL)$ as follows:

$$uR_{i,r}v\,(modL) \Leftrightarrow \forall w \in \Sigma^r : uw \in L \leftrightarrow vw \in L\,, \quad \forall u, v \in \Sigma^i;$$

$$uR_{i,\leqslant r}v\,(modL) \Leftrightarrow \forall w \in \Sigma^{\leqslant r} : uw \in L \leftrightarrow vw \in L\,, \quad \forall u, v \in \Sigma^i.$$

We can see that they are equivalent relations. So we define:

$$K_L(i,r) \;=\; Rank\, R_{i,r}\,(modL);$$

$$F_L(i,r) \;=\; Rank\, R_{i,\leq r}\,(modL).$$

It is easy to prove that $\forall L \subseteq \Sigma^*$, $\forall i, r \in N^+$, we have:

$$1 \leqslant K_L(i,r) \leqslant 2^{|\Sigma^r|} = O(2^{m^r}) = O(C^{Exp(r)})\,, \quad \forall r \in N^+;$$

$$1 \leqslant F_L(i,r) \leqslant 2^{|\Sigma^{\leq r}|} = O(2^{\frac{m(m^r-1)}{m-1}}) = O(C^{Exp(r)})\,, \quad \forall r \in N^+.$$

where $C = \text{const}$ and $Exp(r)$ denote an exponential function of r.

Theorem 5 (The second representative theorem for 1TVIA)**.** *Let Δ be an 1TVIA and $L = L(\Delta)$. Then*

$$K_L(i,r) \;\leqslant\; h_\Delta(r)\,, \quad \forall i, r \in N^+;$$

$$F_L(i,r) \;\leqslant\; g_\Delta(r)\,, \quad \forall i, r \in N^+.$$

Proof. Let $\Delta \;=\; (\Sigma\,, Q\,, q_0\,, q\,, \lambda_{0,t}\,, \lambda_t\,, F_t)$ be an 1TVIA and $L = L(\Delta)$. We shall prove that, for example $F_L(i,r) \;\leqslant\; g_\Delta(r)$, $\;\forall i, r \in N^+$. We assume the contrary, i. e. there are $i, r \in N^+$ such that $F_L(i,r) \;>\; g_\Delta(r)$. By analogous argument in the proof of Theorem 3, it follows that there are two words $u, v \in \Sigma^i$ such that $u\overline{R}_{i,\leqslant r}v\,(modL)$, but $C_0(u)$ and $C_0(v)$, where $C_0(w) = M_0(w, c_0)$, $w \in \Sigma^*$, are non r-distinguished in the sense

$$M_{l(u)}(\omega, C_0(u)) \;=\; M_{l(v)}(\omega, C_0(v))\;, \quad \forall \omega \in \Sigma^{\leqslant r};$$

From this, we have:

$$M_0(u\omega, c_0) = M_0(v\omega, c_0), \quad \forall \omega \in \Sigma^{\leqslant r};$$
$$N_0(u\omega, c_0) = N_0(v\omega, c_0), \quad \forall \omega \in \Sigma^{\leqslant r};$$
$$u\omega \in L \longleftrightarrow v\omega \in L, \quad \forall \omega \in \Sigma^{\leqslant r};$$
$$uR_{i,\leq r}v\,(modL)$$

It conflicts with the hypothesis $u\overline{R}_{i,\leqslant r}v\,(modL)$. Therefore we get

$$F_L(i,r) \leqslant g_\Delta(r), \quad \forall i, r \in N^+.$$

By an analogous argument, we have $K_L(i,r) \leqslant h_\Delta(r)$, $\forall i, r \in N^+$. □

Corollary 3 (The necessary condition for $\mathcal{L}(1TVIA)$)**.** *If $L \in \mathcal{L}(1TVIA)$ then*

$$K_L(i,r) = O(C^{P_1(r)}), \quad \forall i, r \in N^+;$$
$$F_L(i,r) = O(C^{P_1(r)}), \quad \forall i, r \in N^+;$$

where $C = \text{const}$ and $P_1(r)$ is a polynominal of degree 1.

Proof. It follows from Theorem 5 and the upper bounds of $h_\Delta(r)$ and $g_\Delta(r)$. □

Example 6. Let $|\Sigma| = m \geqslant 2$ and $r = \text{const}$. We define

$$L_{6,r} = \{\tau_1\tau_2\ldots\tau_n\tau_0|\ \tau_i \in \Sigma^*; l(\tau_i) = r; n = m^r; \exists\tau_i = \tau_0\};$$
$$L_6 = \bigcup_{r\geq 0} L_{6,r}.$$

For each subset $A = \{\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_k}\} \subseteq \Sigma^r$ we associate it with a word U_A,

$$U_A = \tau_{i_1}\tau_{i_2}\ldots\tau_{i_k} \underbrace{\tau_{i_k}\ldots\tau_{i_k}}_{(m^r-k)-times}.$$

It follows $l(U_A) = rm^r = \text{const}$.

We choose $i = rm^r$. We can verify that

$$\forall w \in \Sigma^k : U_A w \in L_{6,r} \Leftrightarrow w \in A.$$

Since $l(U_A) = rm^r$, therefore we also have:

$$\forall w \in \Sigma^k : U_A w \in L_6 \Leftrightarrow w \in A.$$

So we have $K_{L_6}(i,r) \geqslant 2^{|\Sigma^r|} = 2^{m^r}$, $\forall r \in N^+$, $\forall i = rm^r$. Finally, we obtain:

$$K_{L_6}(rm^r, r) = 2^{m^r} = O(C^{Exp(r)}) > O(C^{P_1(r)}),$$

with r enough large.

According to Corollary 3, we obtain $L_6 \notin \mathcal{L}(1TVIA)$.

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