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ON MAXIMAL SUBGROUPS IN DIVISION RINGS

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Abstract. In this work we discuss three conjectures, raised in [3] on maximal subgroups in division rings. Recently M. Mahdavi-Hezavehi gave a negative answer to Conjecture 3 (see below) by constructing a counterexample in the division ring of real quaternions. Here, we show that the other conjectures have a positive answer for division rings of quaternions of characteristics different from 2. The description of all solvable maximal subgroups of the division ring of real quaternions is also given.

1. Introduction

In [3] there are the following conjectures:

Conjecture 1. *Let D be a division ring with the center F and M a maximal subgroup of D^*. Then $Z(M) = M \cap F$, where $Z(M)$ is the center of M.*

Conjecture 2. *Let D be a division ring and M a nilpotent maximal subgroup of D^*. Then D is commutative.*

Conjecture 3. *Let D be a division ring and M a solvable maximal subgroup of D^*. Then D is commutative.*

We will establish some statement which is equivalent to Conjecture 1 in the case of infinite center F of D. Using this fact, we prove that for division rings of quaternions of characteristics different from 2 the conjectures 1 and 2 have positive answers.

In [5], M. Mahdavi-Hezavehi constructed the solvable maximal subgroup $M_H := \mathbb{C}^* \cup \mathbb{C}^* j$ of the multiplicative group of the division ring of real quaternions H, which gave a negative answer to Conjecture 3. Here, we proved that every solvable maximal subgroup of H^* is conjugate with M_H in H^*.

Throughout this paper the following notations will be used: D denotes a division ring with the center F. We say that a division ring D is *algebraic over its center F* if every element of D is algebraic over F. If R is a ring with identity 1 then R^* denotes the group of all units in R. For nonempty subset $A \subseteq R, C_R(A)$ denotes the centralizer of A in R, i.e.

$$C_R(A) := \{x \in R | xa = ax \text{ for all } a \in A\}.$$

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If G is a group, then for any nonempty subset $S \subseteq G$, $\langle S \rangle$ denotes the subgroup of G generated by S. For $x, y \in G$, and a subgroup H of G, we set

$$y^x := xyx^{-1} \quad \text{and} \quad H^x := xHx^{-1}.$$

For any field extension $K \subseteq L$, denote by $Gal(L/K)$ the Galois group of this extension, i. e. the group of all automorphisms of L which fix every element of K.

2. Conjecture 1 for division rings with infinite center

Let D be a division ring with infinite center F and M a maximal subgroup of D^* with center $Z(M)$. Denote by $P(Z(M))$ the minimal subfield of D containing $Z(M)$. We shall prove that $F \subseteq P(Z(M))$ and that Conjecture 1 has a positive answer if and only if $F = P(Z(M))$ for every maximal subgroup M of D^*. Moreover, we shall prove that if D is algebraic over F, then the Galois group $Gal(P(Z(M))/F)$ is trivial and there are no proper intermediate subfields of the field extension $F \subseteq P(Z(M))$.

For further use we note the following simple results from group theory. Note that lemmas 1, 2 are well-known, so we state them without proofs.

Lemma 1. *Let G be a group with center $Z(G)$. If M_1 and M_2 are distinct maximal subgroups of G, then $C_G(M_1) \cap C_G(M_2) = Z(G)$.*

Lemma 2. *Let G be a group and M a normal maximal subgroup of G. Then there exist an element $b \in G$ and a prime number p such that*

$$G = M\langle b \rangle, \quad b^p \in M \text{ and } b^k \notin M, \quad \forall k = 1, 2, \dots, p-1.$$

Corollary 1. *Let G be a group with center $Z(G)$ and M a maximal subgroup of G. Then either $Z(G) \subseteq M$ or there exist an element $b \in Z(G)$ and a prime number p such that*

$$G = M\langle b \rangle, \quad b^p \in M \text{ and } b^k \notin M, \quad \forall k = 1, 2, \dots, p-1.$$

Lemma 3. *Let D be a division ring and H a subgroup of the multiplicative group D^* of D. Then either $-H \cap H = \emptyset$ or $-H = H$.*

Proof. If $-H \cap H \neq \emptyset$, then there exists some element $h \in H$ such that $-h \in H$. It follows $-1 = (-h)h^{-1} \in H$. So $-y = (-1)y \in H$ for any $y \in H$. This means that $-H \subseteq H$. Clearly, $\forall h \in H$, $h = -(-h) \in -H$. So it follows $H = -H$. □

Lemma 4. *Let D be a division ring and M a maximal subgroup of D^*. Then either $-1 \in M$ or $-M \cup M = D^*$.*

Proof. Put $H := M\langle -1 \rangle = -M \cup M$. Then H is the subgroup of D^* containing M. Since M is maximal, $H = M$ or $H = D^*$. □

Lemma 5. *Let D be a division ring with center F and M a maximal subgroup of D^*. Then, either $Z(M) \subseteq F$ or $M \cup \{0\}$ is a division subring of D.*

Proof. Put $K := M \cup \{0\}$. Then $K \subseteq C_D(Z(M))$. Therefore $K^* \subseteq (C_D(Z(M)))^*$. Since $K^* = M$ is maximal in D^*, either $(C_D(Z(M)))^* = D^*$ or $(C_D(Z(M)))^* = K^*$. It follows that either $Z(M) \subseteq F$ or K is a division subring of D. $\square$

For our use we need also the following result:

Theorem A. (see [**2**, Cor. (13.24), p.225]) *Let D be a division ring and L a division subring of D. If $[D^* : L^*] < \infty$, then either D is a finite field or $L = D$.*

From Theorem A it follows the following corollaries:

Corollary 2. *Let D be a division ring with center $F \neq D$. Then $[D^* : F^*] = \infty$.*

Corollary 3. *There is no extension of infinite fields $F \subsetneq K$ with $[K^* : F^*] < \infty$.*

Lemma 6. *Let D be a noncommutative division ring and L a division subring of D such that L^* is the maximal subgroup of D^*. Then L^* is self-normalized in D^*.*

Proof. Since L^* is maximal in D^*, either $N_{D^*}(L^*) = L^*$ or $N_{D^*}(L^*) = D^*$. If L^* is normal in D^*, then it is of finite index in D^*. By Theorem A, either D is a field or $D = L$, which is imposible. Hence L^* is self-nomalized in D^*. $\square$

Lemma 7. *Let D be a division ring with infinite center F and M a maximal subgroup of D^* containing -1. Then either $F^* \subseteq M$ or there exists an element $u \in Z(M)$ such that $F^* = Z(M)\langle 1+u\rangle$.*

Proof. Since M is a maximal subgroup of D^*, by Corollary 1 either $F^* \subseteq M$ or there exist some element $b \in F^*$ and a prime number p such that $D^* = M\langle b\rangle$ with $b^p \in M$ and $b^k \notin M$ for all $k = 1, 2, \dots, p-1$. It follows $b^p \in Z(M)$ and $b^k \notin Z(M)$ for all $k = 1, 2, \dots, p-1$.

Suppose that the last assertion occurs. We have to find an element $u \in Z(M)$ such that $F^* = Z(M)\langle 1+u\rangle$.

Put $K := Z(M) \cup \{0\}$. We claim that $F = K\langle b\rangle$. Let us consider an arbitrary element $x \in F^*$. Since $D^* = M\langle b\rangle$, there exist some elements $h \in M$ and $c \in \langle b\rangle$ such that $x = hc$. Clearly $c \in F^*$, so $h = xc^{-1} \in F^*$. It follows $h \in F^* \cap M \subseteq K^*$, so $x = hc \in K^*\langle b\rangle$. We have shown that $F^* \subseteq K^*\langle b\rangle$, hence $F \subseteq K\langle b\rangle$.

Now, we have to show $K\langle b\rangle \subseteq F$. Since $b \in F$, it remains only to show $K^* \subseteq F^*$. Suppose that y is an arbitrary element in K^* and z is an arbitrary element in D^*. Writing $z = md$ with $m \in M$ and $d \in \langle b\rangle \subseteq F^*$, we have

$$yz = y(md) = (ym)d = (my)d = d(my) = (dm)y = (md)y = zy.$$

Therefore $y \in F^*$, hence $K\langle b\rangle \subseteq F$.

Next, we prove the following:

$$\exists u \in K, \quad 1+u \notin K. \tag{1}$$

In fact, if $1+u \in K$ for every $u \in K$, then using the fact that $-1 \in K$, it is easy to verify that K is a field. Since $F = K\langle b\rangle$ with $b^p \in K$ and $b^k \notin K$ for all

$k = 1, 2, \dots, p-1$, it follows $[F^* : K^*] < \infty$. But in view of Corollary 3 this is impossible. Hence (1) holds.

Now, we have $u \in K \subseteq C_D(M)$, so $1 + u \in C_D(M)$. Since M is maximal in D^*, it can be shown that $1 + u \in F \setminus M$. It follows $D = M\langle 1 + u\rangle$, hence $F^* = Z(M)\langle 1 + u\rangle$ and the proof of the lemma is now completed. □

Lemma 8. *Let D be a division ring with infinite center F and M a maximal subgroup of D^*. Then $F \subseteq P(Z(M))$, where $P(Z(M))$ is the minimal subfield of D containing $Z(M)$.*

Proof. In view of Lemma 4, either $-1 \in M$ or $-M \cup M = D^*$. If $-M \cup M = D^*$ then $F^* = -Z(M) \cup Z(M)$, so $F = P(Z(M))$. Otherwise, suppose $-1 \in M$. By Lemma 7, either $F^* \subseteq M$ or there exists an element $u \in Z(M)$ such that $F = Z(M)\langle 1+u\rangle$. If $F^* \subseteq M$ then $F^* \subseteq Z(M) \subseteq P(Z(M))$. If $F = Z(M)\langle 1+u\rangle$ for some $u \in Z(M)$ then $F = P(Z(M))$. Therefore $F \subseteq P(Z(M))$ in any case. □

We are now ready to prove the following:

Proposition 1. *Let D be a division ring with infinite center F and M a maximal subgroup of D^*. Then $Z(M) = M \cap F$ if and only if $P(Z(M)) = F$.*

Proof. Suppose $Z(M) = M \cap F$. Then $Z(M) \subseteq F$ and it follows $P(Z(M)) \subseteq F$. So in the connection with Lemma 8 we have $P(Z(M)) = F$.

Conversely, if $P(Z(M)) = F$ then $Z(M) \subseteq F$ and hence $Z(M) = M \cap F$. □

Clearly, Proposition 1 gives us an other way to attack Conjecture 1 for division rings with infinite center.

3. Minimal subfield containing the center of a maximal subgroup

Let D be a division ring with center F and M a maximal subgroup of D^*. In Paragraph 2 we have defined the minimal subfield $P(Z(M))$ containing the center $Z(M)$ of M. Moreover, if F is infinite then we have the field extension $F \subseteq P(Z(M))$ (see Lemma 8). In this section, we study in detail this field extension in the case when D is noncommutative division ring which is algebraic over its center F. In our study we need two important results on simple rings. For the convenience of the readers we list them here.

Centralizer Theorem. (see [**1**, p. 42]) *Let B be a simple subring of a simple ring A, $K := Z(A) \subseteq Z(B)$ and $[B : K] < \infty$. Then*

(i) *$C_A(C_A(B)) = B$.*

(ii) *If $[A : K] < \infty$ then $[A : K] = [B : K][C_A(B) : K]$.*

Note that this theorem is in fact only a part of Centralizer Theorem in [1]. Note also the part (i) is often referred as the Double Centralizer Theorem.

Skolem-Noether Theorem. (see [**1**, p. 39]) *Let A, B be simple rings, $K := Z(B) \subseteq Z(A)$ and $[A : K] < \infty$. If $f, g : A \longrightarrow B$ are K-algebra homomorphisms, then there exists a unit $b \in B^*$ such that $g(a) = bf(a)b^{-1}$ for all $a \in A$.*

Lemma 9. *Let D be a noncommutative division ring which is algebraic over its center F and M a maximal subgroup of D^*. Then we have the algebraic field extension $F \subseteq P(Z(M))$. Moreover, this extension have no proper intermediate subfields.*

Proof. If F is finite then F is algebraic over its prime subfield P, hence D is algebraic over the finite subfield P. By Jacobson Theorem (see [**2**, p. 219]) D is commutative, a contradiction. Thus, F must be infinite. Therefore, by Lemma 8 we have the field extension $F \subseteq P(Z(M))$.

Suppose $F \neq P(Z(M))$. We have to show that the extension $F \subseteq P(Z(M))$ has no proper intermediate subfield. Consider an arbitrary element $a \in P(Z(M)) \setminus F$. Since $a \in P(Z(M)) \subseteq C_D(M)$,

$$M \subseteq (C_D(a))^* = (C_D(F(a)))^*.$$

By assumpsion M is maximal in D^* and $a \notin F$, therefore $M = (C_D(F(a)))^*$.

Putting $L := M \cup \{0\}$, we have $L = C_D(F(a))$. By Lemma 6, L^* is self-normalized. It follows that $C_D(L) = P(Z(M))$. Since a is algebraic over F, $[F(a) : F] < \infty$. By applying the Double Centralizer Theorem we have

$$P(Z(M)) = C_D(L) = C_D(C_D(F(a))) = F(a).$$

The proof is now completed. □

Now, let us consider a division ring D with infinite center F and suppose that D^* has at least one maximal subgroup. Then

$$S := \{P(Z(M)) | M \text{ is maximal in } D^*\} \neq \emptyset.$$

By Lemma 8 we have the field extension $F \subseteq P(Z(M))$ for every maximal subgroup M of D^*. By Proposition 1, Conjecture 1 is true for D if and only if $F = P(Z(M))$ for every maximal subgroup M. Putting

$$P_S := \cap\{P(Z(M)) | M \text{ is maximal in } D^*\},$$

we have $F \subseteq P_S$. Clearly, if $F \neq P_S$ then Conjecture 1 is false. However, as we can see in the following, we always have $F = P_S$. To see this, we first need some results:

Lemma 10. *Let D be a division ring with center F and M a normal maximal subgroup of D^*. Then $Z(M) = M \cap F$.*

Proof. Suppose $Z(M) \neq M \cap F$. Then $Z(M) \nsubseteq F$. By Lemma 5, M is the multiplicative group of some division subring of D. Then by Lemma 6, M is self-normalized. But, by assumption M is normal in D^*, hence we have a contradiction. □

Lemma 11. *Let D be a division ring with infinite center F and M_1, M_2 distinct maximal subgroups of D^*. Then*

$$P(Z(M_1)) \cap P(Z(M_2)) = F.$$

Proof. By Lemma 8 we have

$$F \subseteq P(Z(M_1)) \cap P(Z(M_2)).$$

Conversely, clearly

$$P(Z(M_1))^* \cap P(Z(M_2))^* \subseteq C_D(M_1)^* \cap C_D(M_2)^*.$$

By Lemma 1, $C_D(M_1)^* \cap C_D(M_2)^* = F^*$ and the proof is now finished. □

Now, it is easy to obtain that $P_S = F$.

Proposition 2. *Let D be a division ring with infinite center F. Then $P_S = F$.*

Proof. If D^* has distinct maximal subgroups then by Lemma 11 the conclusion holds. Suppose that D has a unique maximal subgroup M. Then M is normal in D^*. By Lemma 10, $Z(M) = M \cap F$ and clearly it follows $P(Z(M)) = F$. □

Corollary 4. *Let D be a noncommutative division ring with infinite center F and M a maximal subgroup of D^*. Then for every $x \in D^* \setminus M$ we have*

$$P(Z(M)) \cap P(Z(M^x)) = F.$$

Proof. If M is normal in D^* then by Lemma 10, $Z(M) = M \cap F$. Hence, by Proposition 1, $P(Z(M)) = F$. If M is self-normalized then M and M^x are distinct maximal subgroups of D^*. By Lemma 11, $P(Z(M)) \cap P(Z(M^x)) = F$. □

Now, we are ready to prove the main result of this section.

Theorem 1. *Let D be a noncommutative division ring which is algebraic over its center F and M a maximal subgroup of D^*. Then $F \subseteq P(Z(M))$. Moreover, the following properties hold:*

(i) *There are no proper intermediate subfields of the field extension $F \subseteq P(Z(M))$,*

(ii) *$Gal(P(Z(M))/F) = \{Id_{P(Z(M))}\}$.*

Proof. In view of Lemma 9, it remains to prove the assertion (ii). Put $K := P(Z(M))$ and suppose that $K \neq F$. Consider an arbitrary element $a \in K \setminus F$. By (i), $K = F(a)$, and it follows $[K : F] = [F(a) : F] < \infty$.

Suppose $Gal(K/F) \neq \{Id_K\}$. Then there exists some $\tau \in Gal(K/F)$ such that $\tau \neq Id_K$. Consider the following homomorphisms of F-algebras:

$$\psi : K \longrightarrow D \text{ and } \varphi : K \longrightarrow D$$

with $\psi(x) = \tau(x)$ and $\varphi(x) = x$ for all $x \in K$. By Skolem-Noether Theorem there exists some element $\omega \in D^*$ such that

$$\varphi(x) = \omega\psi(x)\omega^{-1}, \quad \forall x \in K.$$

It follows $a = \omega\tau(a)\omega^{-1}$ or $a\omega = \omega\tau(a)$.

Suppose $\omega \in M$. Since $a \in K \subseteq C_D(M), \omega a = a\omega = \omega\tau(a)$. Therefore $\tau(a) = a$ and it follows $\tau = Id_K$, a contradiction. Thus, $\omega \in D^* \setminus M$. By Corollary 4, we have

$$\begin{aligned} F &= P(Z(M)) \cap P(Z(M^\omega)) \\ &= P(Z(M)) \cap P((Z(M))^\omega) \\ &= P(Z(M)) \cap (P(Z(M)))^\omega \\ &= K \cap K^\omega. \end{aligned}$$

On the other hand, $a = \tau(a)^\omega \in K^\omega$, hence $a \in K \cap K^\omega = F$ and we obtain again a contradiction. The proof is now completed. □

4. Maximal subgroups in division rings of quaternions

In this section, using the results obtained in the preceding section, we show that Conjectures 1 and 2 are true for division rings of quaternions in the case of characteristics different from 2. Concerning Conjecture 3, recently [5] M. Mahdavi-Hezavehi gave a negative answer by showing that the subgroup $M_H := \mathbb{C}^* \cup \mathbb{C}^* j$ is solvable maximal in H^*, where H is the division ring of real quaternions. Here, in addition, we shall prove that every solvable maximal subgroup of H^* is conjugate with M_H in H^*.

Definition 1. Let D be a noncommutative division ring with center F. We say that D is *a division ring of quaternions* over F if $[D : F] = 4$.

Theorems 2 and 3 below show that the Conjectures 1 and 2 are true for division rings of quaternions of characteristics different from 2.

Theorem 2. *Let D be a division ring of quaternions over its center F and $char D \neq 2$. Then $Z(M) = M \cap F$ for every maximal subgroup M of D^*.*

Proof. Put $K := P(Z(M))$. By Theorem 1, $Gal(K/F) = \{Id_K\}$. Suppose $F \neq K$. Then by Centralizer Theorem we have

$$[D : F] = [K : F][C_D(K) : F] = 4.$$

Therefore $[K : F] = 2$, so K is normal extension over F. Moreover, since $char F \neq 2$, it follows K is Galois over F. Hence

$$|Gal(K/F)| = [K : F] = 2$$

and we are in a contradiction. □

Theorem 3. *Let D be a division ring of quaternions over its center F and $char D \neq 2$. Then, there are no nilpotent maximal subgroups in D^*.*

Proof. Let M be a nilpotent maximal subgroup of D^*. By Theorem 7 in [3], M is the multiplicative group of some subfield K of D. By Theorem 2 we have

$$K = Z(K) = K \cap F \subseteq F.$$

Hence $D = F$, a contradiction. $\square$

Now, we finish our discussion by describing all solvable maximal subgroups of H^*. In fact, we shall show that every solvable maximal subgroup of H^* is conjugate with $M_H := \mathbb{C}^* \cup \mathbb{C}^* j$ in H^*.

For convenience we restate the following result of M. Mahdavi-Hezavehi:

Theorem C. [5, Cor. 4] *Let D be a division ring which is finite dimensional over its center F and M a nonabelian maximal subgroup of D^*. Then M is solvable if and only if there exists some maximal subfield K of D, satisfying the following conditions:*

(i) *K^* is the normal subgroup of M;*

(ii) *K is the Galois extension over F;*

(iii) *$M/K^* \simeq Gal(K/F)$ is solvable group.*

Proposition 3. *Let H be a division ring of real quaternions and M be a solvable maximal subgroup of H^*. Then there exist the elements $\omega, \theta \in H^*$ such that:*

(i) *$\omega^2 = \theta^2 = -1, \omega\theta = -\theta\omega$;*

(ii) *$M = \mathbb{R}(\omega)^* \cup \mathbb{R}(\omega)^*\theta$, where $\mathbb{R}$ denotes the field of real numbers.*

Proof. Let M be a solvable maximal subgroup of H^*. By Theorem 3, M is nonabelian. Therefore, in view of Theorem C, there exists some maximal subfield K of H such that K^* is normal in M, K is Galois over $\mathbb{R}$ and $M/K^* \simeq Gal(K/\mathbb{R})$. It follows that

$$[M : K^*] = |M/K^*| = |Gal(K/\mathbb{R})| = [K : \mathbb{R}] = 2.$$

Suppose $\tau \in Gal(K/\mathbb{R}), \tau \neq Id_K$. By Skolem-Noether Theorem, there exists an element $a \in H^*$ such that

$$\tau(x) = axa^{-1}, \forall x \in K.$$

If $a \in H^* \setminus M$, then $a \in N_{H^*}(K^*) \setminus M$. It follows that M is the proper subgroup of $N_{H^*}(K^*)$. Since M is maximal in $H^*, N_{H^*}(K^*) = H^*$ or $K^* \triangleleft H^*$. By Cartan-Brauer-Hua Theorem (see [2, p. 222]), $K = \mathbb{R}$ or $K = H$, which is impossible. Thus, $a \in M$. Since $\tau \neq Id_K$, it follows $a \notin K$, hence $a \in M \setminus K^*$. Clearly, we can write $K = \mathbb{R}(\omega)$ with $\omega \in K \setminus \mathbb{R}$ and $\omega^2 = -1$. Then $\tau(\omega) = -\omega$ and it follows $a\omega = -\omega a$. Therefore

$$a^2\omega = a(a\omega) = a(-\omega a) = -(a\omega)a = -(-\omega a)a = \omega a^2,$$

and we have $a^2 \in C_H(\omega) = \mathbb{R}(\omega) = K$. Moreover, since $a\omega \neq \omega a$, it follows that

$$a^2 \in \mathbb{R}(a) \cap \mathbb{R}(\omega) = \mathbb{R}.$$

Writing $a^2 = -s^2$ for some positive number s, we have for $\theta := as^{-1} \in M \setminus K^*$,

$$\theta^2 = \omega^2 = -1 \quad \text{and} \quad \omega\theta = -\theta\omega.$$

Since $[M : \mathbb{R}(\omega)^*] = 2$, it follows $M = \mathbb{R}(\omega)^* \cup \mathbb{R}(\omega)^*\theta$. $\square$

Now, we show that the converse of Proposition 3 is true.

Proposition 4. *Let ω and θ be elements of H^* such that $\omega^2 = \theta^2 = -1$ and $\omega\theta = -\theta\omega$. Then $M := \mathbb{R}(\omega)^* \cup \mathbb{R}(\omega)^*\theta$ is the solvable maximal subgroup of H^*.*

Proof. Putting $a := \omega, b := \theta$ and $c := \omega\theta$, we have $a^2 = b^2 = c^2 = -1$, $ab = c$, $bc = a$, and $ca = b$. It can be shown that $\{1, a, b, c\}$ is the basis of H over $\mathbb{R}$. So, we may define the following automorphism of the $\mathbb{R}$-algebra H:

$$f : H \longrightarrow H,$$

with $f(1) = 1$, $f(i) = a$, $f(j) = b$, $f(k) = c$. Then $M = f(M_H)$ is the solvable maximal subgroup of H^*. □

Note that, by Skolem-Noether Theorem, the map f in Proposition 4 is the inner automorphism. As a consequence we get the following result:

Theorem 4. *Let H be the division ring of real quaternions. Then every solvable maximal subgroup of H^* is conjugated with $M_H = \mathbb{C}^* \cup \mathbb{C}^* j$.*

References

[1] P. Draxl, *Skew fields*, London Mathematical Society, Lecture Note Series **81** (1983), Cambridge University Press.

[2] T. Y. Lam, *A First Course in Non-Commutative Rings*, GTM **131** (1991), Springer-Verlag.

[3] S. Akbari, M. Mahdavi-Hezavehi, M. G. Mahmudi, *Maximal subgroups of $GL_1(D)$*, J. Algebra **217** (1999), 422-433.

[4] M. Mahdavi-Hezavehi, M. G. Mahmudi, S. Yasamin, *Finitely generated subnormal subgroups of $GL_n(D)$ are central*, J. Algebra **225** (2000), 517-521.

[5] M. Mahdavi-Hezavehi, *Free subgroups in maximal subgroups of $GL_1(D)$*, J. Algebra **241**, No 2 (2001), 720-730.

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