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UNIFORM NORMAL STRUCTURE AND FIXED POINTS OF NONEXPANSIVE MAPS IN A GENERAL TOPOLOGICAL SPACE (X, τ) WITH A τ-SYMMETRIC FUNCTION

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ABSTRACT. The main purpose of this paper is to define the concept of uniform normal structure and give some new fixed point theorems of nonexpansive maps in a general topological space (X, τ) with a τ-symmetric function.

1. INTRODUCTION

Let (X, d) be a metric space. A self-mapping T of X is said to be nonexpansive if for each $x, y \in X$, $d(Tx, Ty) \leqslant d(x, y)$. Although such mappings are nartural extension of the contraction mappings, it was clear from the outset that the study of fixed points of nonexpansive mappings required techniques which go far beyond the purely metric approach. On the one hand, it is well known that fixed point theory for mappings of this class has its origin in 1965 existence theorems when M. S. Brodskii and D. P. Mil'man introduced a geometric property, called normal structure, for subsets of Banach spaces. This property was inroduced into fixed point theory by W. A. Kirk in Banach spaces and since then a number of abstract results were discovered, along with important discoveries related both to the structure of the fixed point sets and to techniques for approximating fixed points. On the one hand, there are attempts to generalize certain existence fixed point theorems to metric spaces. In 1969, Kijima and Takahashi [1] gave a metric formulation of Kirk's theorem [2]. But their definition of convex metric spaces is rather restraining. However, many results in metric spaces were developed after Penot's formulation [4]. In [7], Khamsi extended the concept of uniform normal structure and established some important fixed point theorems of nonexpansive maps in metric spaces. On the other hand, it has been observed that the distance function used in metric fixed point theorems proofs need not satisfy the triangular inequality nor $d(x, x) = 0$ for all $x \in X$. Motivated by this idea, Hicks and Rhoades [8] established several important common fixed point theorems for general contractive self-mappings of a symmetrizable (resp. semimetrizable) topological spaces. Recall that a symmetric function on a set X is a nonnegative real valued function d defined on $X \times X$ satisfying

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1) $d(x,y)=0$ if and only if $x=y$,

2) $d(x,y)=d(y,x)$.

A symmetric function d on a set X is a semi-metric if for each $x\in X$ and each $\varepsilon>0$, $B_d(x,\varepsilon)=\{y\in X : d(x,y)\leqslant\varepsilon\}$ is a neighborhood of x in the topology $t(d)$ defined as follows

$$\tau=\{U\subseteq X|\ \forall\, x\in U,\ B_d(x,\varepsilon)\subset U,\ \text{for some } \varepsilon>0\}.$$

A topological space X is said to be symmetrizable (semi-metrizable) if its topology is induced by a symmetric (semi-metric) on X. Moreover, Hicks and Rhoades [8] proved that very general probabilistic structures admit a compatible symmetric or semi-metric. For further details on semi-metric spaces (resp. probabilistic metric spaces), see, for example, [11] (resp. [10]).

In this paper, we follows some ideas in [3], [4], [6], [7] to establish a generalization of Khamsis's fixed point theorem [7]. Let (X,τ) be a topological space. In Section 2 we define a new notion called τ-symmetric function which extends the usual notion of symmetric function and define the concept of p-normal structure. Then we give some new fixed point theorems of nonexpansive maps in general topological space (X,τ) by introducing the notion of a τ-symmetric function $p : X\times X\longrightarrow\mathbb{R}^+$. An application to symmetrizable topological spaces has been made.

2. τ-symmetric function and the existence of fixed points

Let (X,τ) be a topological space and $p : X\times X\longrightarrow\mathbb{R}^+$ be a function. For any $\varepsilon>0$ and any $x\in X$, let $B_p(x,\varepsilon)=\{y\in X : p(x,y)<\varepsilon\}$ and $B'_p(x,\varepsilon)=\{y\in X : p(x,y)\leqslant\varepsilon\}$.

Definition 2.1. The function p is said to be a τ-symmetric if

(τ_1) For all $x,y\in X$, $p(x,y)=p(y,x)$,

(τ_2) For each $x\in X$ and any neighborhood V of x, there exists $\varepsilon>0$ with $B_p(x,\varepsilon)\subset V$.

Example 2.1.

1. Let $X=\{0;1;3\}$ and $\tau=\{\emptyset;X;\{0;1\}\}$. Consider the function $p : X\times X\to\mathbb{R}^+$ defined by

$$p(x,y)=\begin{cases} y & \text{for } x\neq 1\\ \frac{1}{2}y & \text{for } x=1\end{cases}$$

We have, $p(1,3)=\dfrac{3}{2}\neq p(3,1)=1$. Thus p is not symmetric. Moreover, we have

$$p(0,3)=3>p(0,1)+p(1,3)=\frac{5}{2}$$

which implies that p does not satisfying the triangular inequality. However, the function p is a τ-distance.

2. Let $X = \mathbb{R}^+$ and $\tau = \{X, \emptyset\}$. It is well known that the space (X, τ) is not metrisable. Consider the function p defined on $X \times X$ by $p(x, y) = (x - y)^2$ for all $x, y \in X$. It is easy to see that p is a τ-symmetric function.

3. A symmetric function on a set X is a nonnegative real valued function d defined on $X \times X$ which satisfies

1) $d(x, y) = 0$ if and only if $x = y$,

2) $d(x, y) = d(y, x)$.

Each symmetric function d on a nonempty set X is a τ-symmetric on X where the topology τ is defined as follows: $U \in \tau$ if for each $x \in U$ one has $B_d(x, \varepsilon) \subset U$ for some $\varepsilon > 0$.

4. Let $X = [0, +\infty[$ and $d(x, y) = |x - y|$ the usual metric. Consider the function $p : X \times X \longrightarrow \mathbb{R}^+$ defined by

$$p(x, y) = e^{|x-y|}, \quad \forall\, x, y \in X.$$

It is easy to see that p is a τ-symmetric function on X, where τ is the usual topology since $B_p(x, \varepsilon) \subset B_d(x, \varepsilon)$ for all $x \in X$ and $\varepsilon > 0$. Moreover, (X, p) is not a symmetric space since for all $x \in X$, $p(x, x) = 1$.

Lemma 2.1. *Let (X, τ) be a topological space with a τ-symmetric p.*

(a) *Let (x_n) be an arbitrary sequence in X and (α_n) be a sequence in $\mathbb{R}^+$ converging to 0 such that $p(x_n, x) \leqslant \alpha_n$ for all $n \in \mathbb{N}$. Then (x_n) converges to x with respect to the topology τ.*

(b) *If τ is a Hausdorff topology, then*

(b_1) *$p(x, y) = 0$ implies $x = y$,*

(b_2) *for any sequence (x_n) in X, the conditions $\lim_{n\to\infty} p(x, x_n) = 0$ and $\lim_{n\to\infty} p(x_n, y) = 0$ imply $x = y$.*

Proof. (a) Let V be a neighborhood of x. Since $\lim_{n\to\infty} p(x, x_n) = 0$, there exists $N \in \mathbb{N}$ such that for every $n \geqslant N$, $x_n \in V$. Therefore $\lim_{n\to\infty} x_n = x$ with respect to τ.

(b_1) Since $p(x, y) = 0$, $p(x, y) < \varepsilon$ for all $\varepsilon > 0$. Let V be a neighborhood of x. Then there exists $\varepsilon > 0$ such that $B_p(x, \varepsilon) \subset V$, which implies that $y \in V$. Since V is arbitrary, we conclude that $y = x$.

(b_2) From (a), $\lim_{n\to\infty} p(x, x_n) = 0$ and $\lim_{n\to\infty} p(y, x_n) = 0$ imply $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$ with respect to the Hausdorff topology τ. Thus $x = y$. □

Let us recall that each family (A_n) of closed nonempty subsets of a complete metric space (X, d) such that $\lim_{n\to\infty} \delta(A_n) = 0$, where $\delta(A) = \sup\{d(x, y) : x, y \in A\}$, has a nonempty intersection. It will be helpful in the sequel to generalize this result to our setting.

Definition 2.2. Let (X, τ) be a topological space with a τ-symmetric p.

1. we say that a nonempty subset A of X is p-closed iff

$$\overline{A}^p = \{x \in X : p(x, A) = 0\} \subset A,$$

where $p(x, A) = \inf\{p(x, y) | y \in A\}$.

2. A sequence in X is said p-Cauchy sequence if it satisfies the usual metric condition. There are several concepts of completeness in this setting:

(i) X is S-complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim\limits_{n\to\infty} p(x_n, x) = 0$;

(ii) X is p-Cauchy complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim\limits_{n\to\infty} x_n = x$ with respect to the topology τ.

3. We say that X is sequentially p-compact if each sequence (x_n) of X has a p-convergent subsequence $(x_{n'})$, i.e., there exists $x \in X$ with $\lim\limits_{n'\to\infty} p(x, x_{n'}) = 0$.

Remark 2.1. Let (X, τ) be a topological space with a τ-symmetric p and let (x_n) be a p-Cauchy sequence. If X is S-complete, then there exists $x \in X$ such that $\lim\limits_{n\to\infty} p(x_n, x) = 0$. Lemma 2.1(a) then gives $\lim\limits_{n\to\infty} x_n = x$ with respect to the topology τ. Therefore S-completeness implies p-Cauchy completeness. Moreover, it is easy to see that sequential p-compactness implies that (X, τ) is sequentially compact.

Lemma 2.2. *Let (X, τ) be a Hausdorff topological space with a τ-symmetric p. Suppose that for each $x \in X$, the function $p(x, .) : X \longrightarrow \mathbb{R}^+$ is lower semi-continuous. Then for each $x \in X$, $B'_p(x, r)$ is p-closed.*

Proof. Let $y \in \overline{B'_p(x, r)^p}$. Then $p(y, B'_p(x, r)) = 0$ and therefore, for all $n \in \mathbb{N}^*$, there exists a sequence (y_n) in $B'_p(x, r)$ such that $\lim\limits_{n\to\infty} p(y, y_n) = 0$, which implies that $\lim\limits_{n\to\infty} y_n = y$ with respect to the topology τ (Lemma 2.1(a)). Since $p(x, y_n) \leqslant r$ and $p(x, .)$ is lower semi-continuous, by letting n to infty we get $p(x, y) \leqslant r$. Hence $y \in B'_p(x, r)$ and therefore $B'_p(x, r)$ is p-closed. □

Proposition 2.1. *Let (X, τ) be a Hausdorff topological space with a τ-symmetric p. Suppose that X is S-complete and p-bounded. Let (A_n) be a family of p-closed nonempty subsets of X such that $\lim\limits_{n\to\infty} \delta_p(A_n) = 0$. Then $\bigcap\limits_{n\in\mathbb{N}} A_n = \{a\}$ for some $a \in X$.*

Proof. As in the metric case, we can show that there exists $a \in X$ with $a \in A_n$ for all $n \in \mathbb{N}$. Lemma 2.1(b_1) then guarantees the uniqueness of a. □

Definition 2.3. Let $\mathcal{F}$ be a nonempty family of subsets of X. We say that $\mathcal{F}$ defines a convexity structure on X if and only if it is stable by intersection.

Example 2.2. Let (X, τ) be a Hausdorff topological space with a τ-symmetric p. An admissible subset of X is any intersection of balls. Let us denote the family of admissible subsets of X by $\mathcal{A}(X)$. It is obvious that $\mathcal{A}(X)$ define a convexity structure on X. In this work, we suppose that any other convexity structure $\mathcal{F}$ on X, contains $\mathcal{A}(X)$.

Remark 2.2. In view of Lemma 2.2, if for each x in X the function $p(x, .) : X \longrightarrow \mathbb{R}^+$ is lower semi-continuous, then each admissible subset of X is p-closed.

Definition 2.4. Let (X, τ) be a topological space with a τ-symmetric p. For a subset A of X, we write

$$r_{p,x}(A) = \sup_{y \in A} p(x, y), \tag{1}$$

$$r_p(A) = \inf_{x \in A} r_{p,x}(A), \tag{2}$$

$$\delta_p(A) = \sup_{x \in A} r_{p,x}(A), \tag{3}$$

$$cov(A) = \bigcap_{B'_p \in \mathcal{F}} B'_p, \tag{4}$$

$$co(A) = \bigcap_{f \in A} B'_p(f, r_{p,f}(A)), \tag{5}$$

where $\mathcal{F}$ is the family of balls containing A. Clearly, a subset A of X is admissible if and only if $A = cov(A)$.

Definition 2.5. We say that X has uniform p-normal structure if there exists a convexity structure $(\mathcal{F})$ on X and a nondecreasing function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

(ϕ_1) $\phi(0) = 0$,

(ϕ_2) For every $t \in]0, +\infty[$, $\lim_{n \to \infty} \phi^n(t) = 0$,

(ϕ_3) $r_p(A) \leqslant \phi(\delta_p(A))$, for every $A \in \mathcal{F}$ not reduced to a single point.

Example 2.3. Let (X, d) be a metric space. It is clear that d is a τ-symmetric where τ is the topology induced by the metric d. Recall that X is said to have uniform normal structure if there exists a convexity structure $\mathcal{F}$ on X such that $r_d(A) \leqslant \alpha \delta_d(A)$, for a fixed constant $\alpha \in (0, 1)$, for any nonempty $A \in \mathcal{F}$, which is d-bounded and not reduced to a single point. Let $\phi : X \longrightarrow \mathbb{R}^+$ be the function defined by $\phi(t) = \alpha t$, for all $t \in \mathbb{R}^+$. It is clear that this function ϕ satisfies (ϕ_1)-(ϕ_3). Hence, (X, d) has uniform d-normal structure.

In [3], Kirk proved the following lemma in metric spaces. The proof of the lemma in our setting is essentially the same as in [8].

Lemma 2.3. *Let (X, τ) be a topological space with a τ-symmetric p. Assume that X is p-bounded and has uniform p-normal structure. Let T be a nonexpansive self-mapping of X. If $D \in \mathcal{A}(X)$ is T-invariant set, then there exists a nonempty admissible subset D^* of D, which is T-invariant, such that*

$$\delta_p(D^*) \leqslant \frac{1}{2}(\phi(\delta_p(D)) + r_p(D)).$$

Proof. Set $r = \dfrac{1}{2}(\phi(\delta_p(D)) + r_p(D))$. We can assume that $\delta_p(D) > 0$, otherwise we can take $D^* = D$ since $\phi(0) = 0$. Since X has uniform p-normal structure, we have $r_p(D) \leqslant \phi(\delta_p(D))$. Therefore, the set $A = \{f \in D : D \subset B'_p(f,r)\}$ is nonempty subset of X. Moreover, $A = \bigcap\limits_{f\in D} B'_p(f,r) \cap D$, which implies that A is admissible. Clearly, there is no reason for A to be T-invariant. Put $\vartheta = \{M \in \mathcal{A}(X) : A \subset M, T(M) \subset M\}$ and $L = \bigcap\limits_{M\in\vartheta} M$. Note that ϑ is nonempty since $X \in \vartheta$. It is clear that L is T-invariant, admissible subset of X which contains A. Let $C = A \cup T(L)$. Observe that $co(C) = L$. Indeed, since $C \subset L$ and $L \in \mathcal{A}(X)$, we have $co(C) \subset L$. From this we obtain $T(co(C)) \subset T(L) \subset C$, hence $C \in \mathcal{A}(X)$, and therefore $L \subset co(C) \subset L$. This gives the desired equality. Define $D^* = \{f \in L : L \subset B'_p(f,r)\}$. We claim that D^* is the desired set. Observe that D^* is nonempty since it contains A. Using the same argument we can prove that D^* is an admissible subset of X. On the other hand, it is clear that $\delta_p(D^*) \leqslant r$. To complete the proof, we have to show that D^* is T-invariant. Let $f \in D^*$. By definition of D^*, we have $L \subset B'_p(f,r)$. Since T is nonexpansive, we have $T(L) \subset B'_p(T(f),r) \subset B'_p(T(f),r)$. Let $g \in A$. Then $L \subset B'_p(g,r)$. But $T(f) \in L$, so that $T(f) \in B_p(g,r)$, which is equivalent to $g \in B'_p(T(f),r)$. Therefore $A \subset B'_p(T(f),r)$. Since $C = A \cup T(L)$, we deduce that $C \subset B'_p(T(f),r)$. Thus, we have $co(C) = L \subset B'_p(T(f),r)$. From the definition of D^* it follows that $T(f) \in D^*$. In other words, D^* is T-invariant. □

Now we are able to prove the main result

Theorem 2.1. *Let (X,τ) be a topological space with a τ-symmetric p such that for each x in X, the function $p(x,.)$ is lower semicontinuous. Assume that X is S-complete, p-bounded and has uniform p-normal structure. Let T be a nonexpansive self-mapping of X. Then T has a fixed point.*

Proof. Since X has uniform p-normal structure, there exists a nondecreasing function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying (ϕ_1)-(ϕ_3). Moreover, we can apply Lemma 2.3 and Remark 2.2 to deduce the existence of a decreasing sequence (A_n) of p-closed nonempty admissible subsets of X such that A_n is T-invariant and $\delta_p(A_{n+1}) \leqslant \dfrac{1}{2}(\phi(\delta_p(A_n)) + r_p(A_n))$. This implies that

$$\delta_p(A_n) \leqslant \phi^n(\delta_p(X)),$$

which in turn implies $\lim\limits_{n\to\infty} \delta_p(A_n) = 0$. Now we apply Proposition 2.1 to prove that $\cap A_n$ is reduced to a single point, which is the desired fixed point of T. □

In the metric space setting, we have the following result.

Corollary 2.1. *Let (X,d) be a bounded complete metric space. Assume that X has uniform d-normal structure. Let T be a nonexpansive self-mapping of X. Then T has a fixed point.*

According to the Example 2.2, the Corollary 2.1 is a generalization of the following result of [7].

Corollary 2.2. *Let (X, d) be a bounded complete metric space. Assume that X has uniform normal structure. Let T be a nonexpansive self-mapping of X. Then T has a fixed point.*

In [8], the authors established some common fixed point theorems for general contractive maps in symmetric spaces and proved that very general probabilistic structures admit a compatible symmetric or semi-metric. Now we apply our main result to the symmetric spaces setting.

Corollary 2.3. *Let (X, d) be a symmetric space such that for each x in X the function $d(x, .)$ is lower semicontinuous. Assume that X is S-complete, d-bounded and has uniform d-normal structure. Let T be a nonexpansive self-mapping of X. Then T has a fixed point.*

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