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SOME FIXED POINT THEOREMS FOR MAPPINGS OF TWO VARIABLES

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Abstract. The existence of fixed points $x = T(x, x)$ and the convergence of implicit iteration $x_n = T(x_n, x_{n-1})$ to the fixed points for generalized nonexpansive type mappings of two variables are investigated.

1. Introduction

The aim of this paper is to study the existence of fixed points $x = T(x, x)$ of mappings of two variables and to investigate the convergence of implicit iteration $x_n = T(x_n, x_{n-1})$ to fixed points of the mapping $T(x, y)$.

The implicit iteration $x_n = T(x_n, x_{n-1})$ was studied by Kurpel in [13]. It contains some iteration methods such as Picard and Seidel method as special cases. In [13], [17] and other works the implicit iteration was applied to nonlinear integral equations, Volterra integral equations, differential equations of parabolic type, linear and nonlinear systems of integral equations, Cauchy problem, boundary value problems of linear itegro-differential equations, eigenvalue problems etc.

In this paper we give some extensions of fixed point theorems of Meir-Keeler and Boyd-Wong contraction and nonlinear contraction theorems for mappings of two variables. The fixed point theorems for nonexpansive and condensing mappings with measure of weak noncompactness are also given. Under our assumptions the implicit iteration converges to the fixed points, while the Picard iteration may not. The paper is a continuation of [1, 2, 3].

2. Main results

Throughout the paper we denote by D a nonemty closed subset of a complete metric space X, T a mapping of $D \times D$ into D and for $x, y, z, t \in D$,

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$m(T(x,y),T(z,t)) =$

$$\max\Big\{d(x,z), d(y,t), d(x,T(x,y)), d(z,T(z,t)), \frac{1}{2}[d(x,T(z,t)) + d(z,T(x,y))]\Big\}.$$

First we extend Meir-Keeler's theorem [15] for mappings of two variables. The Meir-Keeler contractive mappings, and contractive mappings in general, are investigated (see [14], [9],... and references therein). We shall consider mappings which satisfy the following condition: $\forall\varepsilon > 0$, $\exists\delta > 0$,

$$\begin{aligned} &\varepsilon \le m(T(x,y),T(z,t)) < \varepsilon + \delta \Longrightarrow \\ (1) \quad &d(T(x,y),T(z,t)) < \varepsilon \text{ if } (x,y) \ne (z,t) \text{ and } x \ne y \text{ or/and } z \ne t; \text{ and} \\ &d(T(x,y),T(z,t)) \le \varepsilon \text{ otherwise.} \end{aligned}$$

Note that condition (1) implies

$$\begin{aligned} (2) \quad &d(T(x,y),T(z,t)) \le m(T(x,y),T(z,t)) \text{ and the inequality} \\ &\text{holds strictly if } (x,y) \ne (z,t) \text{ and } x \ne y \text{ or/and } z \ne t. \end{aligned}$$

Indeed, if $(x,y) \ne (z,t)$ and $x \ne y$ or $z \ne t$, then putting $m(T(x,y),T(z,t)) = \varepsilon > 0$ we get from (1)

$$d(T(x,y),T(z,t)) < m(T(x,y),T(z,t)).$$

Otherwise, again by (1) we see that

$$d(T(x,y),T(z,t)) \le \varepsilon \le m(T(x,y),T(z,t)),$$

so (2) holds.

Theorem 2.1. *Let T be a continuous mapping of $D \times D$ into D and let (1) hold $\forall x,y,z,t \in D$. Then T has a fixed point. Moreover, for any $x_0 \in D$ the sequence $x_n = T(x_n, x_{n-1})$ is well-defined for all $n \ge 1$ and converges to a fixed point of T.*

Proof. For an arbitrary fixed $v \in D$ define a mapping T_v such that $T_v(x) = T(x,v)$. Then by condition (1) T_v satisfies the condition $\forall\varepsilon > 0$, $\exists\delta > 0$,

$$\begin{aligned} &\varepsilon \le \max\Big\{d(x,z), d(x,T_v(x)), d(z,T_v(z)), \frac{1}{2}[d(x,T_v(z)) + d(z,T_v(x))]\Big\} < \varepsilon + \delta \\ &\Longrightarrow d(T_v(x),T_v(z)) < \varepsilon, \end{aligned}$$

since if $x = z$ then $d(T_v(x),T_v(z)) = 0 < \varepsilon$, and if $x \ne z$ then $(x,v) \ne (z,v)$ and $x \ne v$ or $z \ne v$, hence by (1) $d(T_v(x),T_v(z)) < \varepsilon$. From [18, Theorem 1] there exists a unique $\overline{x}$ such that $\overline{x} = T_v(\overline{x}) = T(\overline{x},v)$. Thus for any $x_0 \in D$ the sequence $x_n = T(x_n,x_{n-1})$ is well-defined.

We may assume that $x_{n+1} \ne x_n \;\; \forall n$. Denote $a_n = d(x_n,x_{n-1})$. Then by (2)

$$a_{n+1} < \max\Big\{d(x_{n+1},x_n), d(x_n,x_{n-1}), 0, 0, \frac{1}{2}[d(x_{n+1},x_n) + d(x_n,x_{n+1})]\Big\}.$$

Hence $\{a_n\}$ is decreasing and there exists $a = \lim\limits_n a_n$. Suppose $a > 0$. Set $a = \varepsilon$ then there esists $\delta > 0$ and N such that $\forall n \ge N : \varepsilon \le a_n < \varepsilon + \delta$. So by

(1) $a_{n+1} = d(T(x_{n+1}, x_n), T(x_n, x_{n-1})) < \varepsilon$ since $x_{n+1} \neq x_n$. The contradiction shows that $a = 0$.

We will show that $\{x_n\}$ is a Cauchy sequence. Suppose not. Then there exists $\varepsilon > 0$ such that $\forall N,\ \exists n > m > N : d(x_m, x_n) \geq 2\varepsilon$. We choose δ from (1). Let $\alpha = \min\{\varepsilon, \delta\}$. Since $\lim_n a_n = 0$, there exists N such that $a_n < \dfrac{\alpha}{8}$ $\forall n \geq N$. As in [15] we can assert the existence of l, $m \leq l \leq n$, such that

$$\varepsilon + \frac{\alpha}{4} \leq d(x_m, x_l) \leq \varepsilon + \frac{3\alpha}{8}\cdot$$

On the other hand,

$$\begin{aligned} d(x_m, x_l) &= d(T(x_m, x_{m-1}), T(x_l, x_{l-1})) \\ &< \max\Big\{d(x_m, x_l), d(x_{m-1}, x_{l-1}), 0, 0, \frac{1}{2}[d(x_m, x_l) + d(x_l, x_m)]\Big\} \\ &= d(x_{m-1}, x_{l-1}). \end{aligned}$$

Hence

$$\begin{aligned} \varepsilon &\leq d(x_m, x_l) - 2a_N \leq m(T(x_m, x_{m-1}), T(x_l, x_{l-1})) \\ &\leq d(x_m, x_l) + 2a_N < \varepsilon + \frac{3\alpha}{8} + \frac{\alpha}{4} < \varepsilon + \delta. \end{aligned}$$

From (1) we obtain

$$d(x_m, x_l) = d(T(x_m, x_{m-1}), T(x_l, x_{l-1})) < \varepsilon,$$

which contradicts $d(x_m, x_l) \geq \varepsilon + \dfrac{\alpha}{4}$. Thus $\{x_n\}$ is a Cauchy sequence. Since X is complete, D is closed and T is continuous, we have $x_n \to u \in D$ and $u = T(u, u)$. □

We note that under the assumptions of Theorem 2.1, if we put $T_0(x) = T(x, x)$ then (1) becomes: $\forall \varepsilon > 0,\ \exists \delta > 0$ such that

$$\begin{aligned} \varepsilon &\leq \max\Big\{d(x, z), d(x, T_0(x)), d(z, T_0(z)), \frac{1}{2}[d(x, T_0(z)) + d(z, T_0(x))]\Big\} \\ &< \varepsilon + \delta \Longrightarrow d(T_0(x), T_0(z)) \leq \varepsilon, \end{aligned} \tag{3}$$

so the Picard iteration $x_n = T(x_{n-1}, x_{n-1})$ may not converge to a fixed point of T. Furthermore, (3) alone does not imply the existence of fixed points of T_0.

In the sequel we give the extensions of Boyd-Wong contraction theorem and Boyd-Wong nonlinear contraction theorem [4, Theorem 1].

Theorem 2.2. *Let T be a continuous mapping of $D \times D$ into D satisfying*

$$d(T(x, y), T(z, t)) \leq \alpha(r, c) m(T(x, y), T(z, t)), \tag{4}$$

$\forall x, y, z, t \in D$, $(x, y) \neq (z, t)$, where $r = \max\{d(x, z), d(y, t)\}$, $c = \max\{d(x, y), d(z, t)\}$, $\alpha(r, c) : (0, \infty) \times [0, \infty) \to [0, 1]$ is a function upper semicontinuous (u.s.c) from the right in r and nonincreasing in c. Moreover, assume that

$$0 \leq \alpha(r, c) < 1 \ \text{ if } c > 0; \ \ 0 \leq \alpha(r, 0) \leq 1 \ \text{ if } c = 0.$$

Besides, $\forall\varepsilon>0\ \ \exists c_0(\varepsilon)$ *such that* $\forall c: 0<c\le c_0$ *and* $\forall b\in[\varepsilon,\varepsilon+c)$ *the following inequality holds*

$$3c+\alpha(b,c)\varepsilon<\varepsilon. \tag{5}$$

Then the conclusion of Theorem 2.1 holds.

Proof. First we prove that the equation $y=T(y,v)$ has a solution for each $v\in D$. Then $x_n=T(x_n,x_{n-1})$ are solvable $\forall n,\forall x_0\in D$. Define

$$y_n=T(y_{n-1},v), y_0\in D, c_n=\max\{d(y_n,v),d(y_{n-1},v)\}, s=\inf_n c_n.$$

If $s=0$ then there exists a subsequence $\{c_{n_j}\}\to 0$ or a number n_0 such that $c_{n_0}=0$. In the first case, $y_{n_j}\to v$, $y_{n_j-1}\to v$, so $v=T(v,v)$. In the other case, $y_{n_0}=y_{n_0-1}=v$, hence we also have $v=T(v,v)$. So, in both cases, v is a solution of equation $y=T(y,v)$.

Suppose that $s>0$. We will show that $\{y_n\}$ is a Cauchy sequence, hence its limit is a solution of $y=T(y,v)$. Assume $y_{n+1}\ne y_n\ \forall n$ and define $d_n=d(y_n,y_{n-1})$. Since $\alpha(r,c)$ is nonincreasing in c and $d(y_{n-1},y_{n+1})\le d_n+d_{n+1}$ we obtain from (4)

$$\begin{aligned} d(y_{n+1},y_n)&\le\alpha(d_n,s)\max\Big\{d_n,0,d_{n+1},d_n,\frac{1}{2}[0+d(y_{n-1},y_{n+1})]\Big\}\\ &<\max\{d_n,d_{n+1}\}=d_n. \end{aligned}$$

Hence $\{d_n\}$ is decreasing and there exists $d=\lim\limits_n\ d_n$. Suppose $d>0$. Then, since $d_{n+1}\le\alpha(d_n,s)d_n$ and $\alpha(r,s)$ is u.s.c from the right, we get $d\le\alpha(d,s)d<d$, a contradiction. So $d=0$.

Suppose $\{y_n\}$ is not a Cauchy sequence. Then $\exists\varepsilon>0\ \ \forall n\ \exists p_n>q_n>n$ such that

$$d(y_{p_n},y_{q_n})\ge\varepsilon,\ \ d(y_{p_n-1},y_{q_n})<\varepsilon.$$

Denote $e_n=d(y_{p_n+1},y_{q_n+1})$, $b_n=d(y_{p_n},y_{q_n})$. Then

$$\varepsilon\le b_n\le d(y_{p_n},y_{p_n-1})+d(y_{p_n-1},y_{q_n})<\varepsilon+d_{p_n},$$

so $\{b_n\}$ tends to ε from the right. Besides, for n sufficiently large,

$$\max\{d(y_{p_n},v),d(y_{q_n},v)\}\ge d(y_{p_n},v)\ge\max\{d(y_{p_n},v),d(y_{p_n-1},v)\}-d_{p_n}\ge s/2.$$

Therefore,

$$\begin{aligned} e_n&\le\alpha(b_n,s/2)\max\Big\{b_n,0,d_{p_n+1},d_{q_n+1},\frac{1}{2}[d(y_{p_n},y_{q_n+1})+d(y_{q_n},y_{p_n+1})]\Big\}\\ &\le\alpha(b_n,s/2)(b_n+d_{p_n+1}+d_{q_n+1}). \end{aligned}$$

Hence

$$\varepsilon\le\alpha\Big(b_n,\frac{s}{2}\Big)(b_n+d_{p_n+1}+d_{q_n+1})+2d_{q_n+1}.$$

Letting $n\to\infty$ we get $\varepsilon\le\alpha\Big(\varepsilon,\dfrac{s}{2}\Big)\varepsilon<\varepsilon$. This contradiction shows that $\{y_n\}$ is a Cauchy sequence. So $x_n=T(x_n,x_{n-1})$ are well-defined.

It remains to show that $\{x_n\}$ is a Cauchy sequence. Assume $x_{n+1} \neq x_n \forall n$. Denote $a_n = d(x_n, x_{n-1})$. It is easy to see from (4) that $a_{n+1} < a_n$, hence there exists $a = \lim_n a_n$. Suppose $a > 0$. Then from $a_{n+1} \leq \alpha(a_n, a)a_n$ and passing to the limit as $n \to \infty$ we get $a \leq \alpha(a, a)a < a$, a contradiction. This implies $a = 0$.

Suppose $\{x_n\}$ is not a Cauchy sequence. Then $\exists \varepsilon > 0 \ \forall n \ \exists p_n > q_n > n$ such that

$$d(x_{p_n}, x_{q_n}) \geq \varepsilon, \ \ d(x_{p_n-1}, x_{q_n}) < \varepsilon.$$

Denote $e'_n = d(x_{p_n+1}, x_{q_n+1})$, $b'_n = d(x_{p_n}, x_{q_n})$. Then

$$\varepsilon \leq b'_n < \varepsilon + a_{p_n} \leq \varepsilon + a_{q_n+1} \tag{6}$$

and $\varepsilon \leq e'_n + a_{p_n+1} + a_{q_n+1} \leq e'_n + 2a_{q_n+1}$. By (4),

$$e'_n \leq \alpha(b'_n, a_{q_n+1}) \max\left\{e'_n, b'_n, 0, 0, \frac{1}{2}[e'_n + e'_n]\right\},$$

hence $e'_n < b'_n$ and therefore $e'_n \leq \alpha(b'_n, a_{q_n+1})b'_n$. It follows that

$$\varepsilon \leq \alpha(b'_n, a_{q_n+1})b'_n + 2a_{q_n+1} \leq \alpha(b'_n, a_{q_n+1})\varepsilon + 3a_{q_n+1}. \tag{7}$$

By (5), (6), (7) and since $\lim_n a_n = 0$, for n sufficiently large, we get $\varepsilon < \varepsilon$, a contradiction. So $\{x_n\}$ is a Cauchy sequence. The proof of Theorem 2.2 is complete. □

Example. We give an example of the function $\alpha(r, c)$ of Theorem 2.2. Define

$$\alpha(r, c) = \begin{cases} 1 - \sqrt{c}/(r+1) & \text{if } 0 < c \leq 1, \\ 1 - 1/(r+1) & \text{if } c > 1. \end{cases}$$

When c_0 is small, (5) can be rewritten as

$$\varepsilon[1 - \sqrt{c_0}/(\varepsilon + c_0 + 1)] + 3c_0 < \varepsilon,$$

which is true if $3c_0 \leq \varepsilon\sqrt{c_0}/(2\varepsilon + 1)$. This inequality holds when $3\sqrt{c_0} \leq \varepsilon/2$ and $\varepsilon \leq 1/2$. Thus, if we choose $c_0 = \varepsilon^2/36$, then (5) will hold.

We also notice that in Theorem 2.2 the Picard iteration may not converge.

The following theorem will be stated without proof because its proof is similar and simpler than that of Theorem 2.2.

Theorem 2.3. *Let T be a continuous mapping of $D \times D$ into D satisfying*

$$d(T(x, y), T(z, t)) \leq \alpha(r)m(T(x, y), T(z, t)), \tag{8}$$

$\forall x, y, z, t \in D, (x, y) \neq (z, t)$, where $r = \max\{d(x, z), d(y, t)\}, \alpha : (0, \infty) \to [0, 1)$ is a function upper semicontinuous from the right. Then T has a unique fixed point u. Moreover, for any $x_0 \in D$, the sequence $x_n = T(x_n, x_{n-1})$ is well-defined and converges to u.

Theorem 2.4. *Let T be a continuous mapping of $D \times D$ into D satisfying*

$$d(T(x,y),T(z,t)) \leq \psi(m(T(x,y),T(z,t)),c), \tag{9}$$

$\forall x,y,z,t \in D$. Here c is as in Theorem 2.2, $\psi(r,c) : [0,\infty)\times[0,\infty) \to [0,\infty)$ is a function upper semicontinuous in r and nonincreasing in c. Moreover, for $r > 0$, $0 < \psi(r,c) < r$ if $c > 0$, $0 < \psi(r,0) \leq r$ if $c = 0$. Assume that $\forall \varepsilon > 0$ $\exists c_0(\varepsilon)$ such that $\forall c : 0 < c \leq c_0$ $\forall b \in [\varepsilon, \varepsilon + c)$ the following condition is satisfied

$$2c + \psi(b,c) < \varepsilon. \tag{10}$$

Then the conclusion of Theorem 2.1 holds.

Proof. Define y_n, c_n, s as in Theorem 2.2. If $s = 0$ then as before we see that $y = T(y,v)$ has v as a solution. Suppose that $s > 0$. By the similar argument as in Theorem 2.2 one can show that $\{y_n\}$ is a Cauchy sequence. So $x_n = T(x_n, x_{n-1})$ are well-defined.

We shall prove that $\{x_n\}$ is a Cauchy sequence. Assume $x_{n+1} \neq x_n$ $\forall n$. Define $a_n = d(x_n, x_{n-1})$ we have

$$\begin{aligned} a_{n+1} &\leq \psi(\max\{a_{n+1}, a_n, 0, 0, \frac{1}{2}[d(x_{n+1},x_n) + d(x_n,x_{n+1})]\}, c) \\ &< \max\{a_{n+1}, a_n\}. \end{aligned}$$

So $\{a_n\}$ is decreasing and there exists $a = \lim_n a_n$. Since $a_{n+1} \leq \psi(a_n, a)$ then $a = 0$.

Suppose that $\{x_n\}$ is not a Cauchy sequence. Then $\exists \varepsilon > 0$ $\forall n$ $\exists p_n > q_n > n$ such that

$$d(x_{p_n}, x_{q_n}) \geq \varepsilon, \;\; d(x_{p_n - 1}, x_{q_n}) < \varepsilon.$$

Denote $e_n = d(x_{p_n+1}, x_{q_n+1})$, $b'_n = d(x_{p_n}, x_{q_n})$. Then $\varepsilon \leq b'_n < \varepsilon + a_{p_n} \leq \varepsilon + a_{q_n+1}$, and by (9)

$$e_n \leq \psi(\max\{e_n, b'_n, 0, 0, \frac{1}{2}[e_n + e_n]\}, a_{q_n+1}) < \max\{e_n, b'_n\}.$$

Hence $e_n \leq \psi(b'_n, a_{q_n+1})$, and then $\varepsilon \leq 2a_{q_n+1} + \psi(b'_n, a_{q_n+1}) < \varepsilon$ by (10). The contradiction shows that $\{x_n\}$ is a Cauchy sequence and the conclusion follows. □

Example. We give an example of the function $\psi(r,c)$ of Theorem 2.4. Define

$$\psi(r,c) = r/(1+\sqrt{c}), \;\; r, c \geq 0.$$

Condition (10) is satisfied if for a small $\varepsilon > 0$ there exists c_0 such that

$$2c_0(1+\sqrt{c_0}) < \varepsilon(1+\sqrt{c_0}) - \varepsilon - c_0 = \varepsilon\sqrt{c_0} - c_0, \;\text{ or }\; \sqrt{c_0}(3+2\sqrt{c_0}) < \varepsilon.$$

Choose $c_0 = \varepsilon^4$ then the last inequality becomes $3\varepsilon^2 + 2\varepsilon^4 < \varepsilon$, or equivalently $\varepsilon^3 < (1-3\varepsilon)/2$, which holds when $\varepsilon \leq 1/6$. So (10) holds for all $c \leq c_0 = \varepsilon^4$.

The following theorem will be given without proof. In this theorem one does not need the continuity of T.

Theorem 2.5. *Let T be a mapping of $D \times D$ into D satisfying*

$$d(T(x,y),T(z,t)) \leq \psi(m(T(x,y),T(z,t))), \tag{11}$$

$\forall x,y,z,t \in D$, where $\psi : [0,\infty) \to [0,\infty)$ is an upper semicontinuous function with $0 < \psi(r) < r$ for all $r > 0$. Then the conclusion of Theorem 2.3 holds.

In the sequel we consider condensing mappings with measure of weak noncompactness in Banach spaces. The concept of measure of weak noncompactness was introduced by De Blasi in [5].

Let X be a Banach space, K^ω the family of all weak compact subsets of X, B the closed unit ball of X and U a nonempty bounded subset of X. Following De Blasi, we call a measure of weak noncompactness of U the following number

$$\omega(U) = \inf\{t > 0 : \exists C \in K^\omega \text{ such that } U \subset C + tB\}.$$

It was shown in [5] that $\omega(U) = 0$ if and only if U is weakly precompact and $\omega(B) = 1$ if X is not reflexive. The measure $\omega(.)$ has many properties as the Kuratowski and Hausdorff measures. Fixed point theorems for weakly continuous mappings $T : D \to D$ were established in [6], where D is a closed bounded convex subset of X, T maps bounded sets into bounded sets and satisfies $\omega(T(U)) \leq \lambda\omega(U)$, $\forall U \subset D$, $0 \leq \lambda < 1$, or $\omega(T(U)) < \omega(U)$ for any $U \subset D$ with $\omega(U) > 0$.

The following theorem is the weak noncompactness version of [11, Theorem 2].

Theorem 2.6. *Let D be a nonempty closed bounded convex subset of a Banach space X, $T : D \times D \to D$ a weakly continuous mapping which satisfies the nonexpansive condition*

$$\|T(x,y) - T(z,t)\| \leq \max\{\|x-z\|, \|y-t\|\}; \text{ and} \tag{12}$$
$$\text{the inequality holds strictly when } \|x-z\| \neq \|y-t\|$$

$\forall x,y,z,t \in D$. Suppose further that

$$\omega(T(U,V)) < \max\{\omega(U), \omega(V)\} \tag{13}$$

for subsets $U, V \subset D$ such that $\omega(U \setminus V) > 0$. Then T has a fixed point.

Proof. Let $x_0 \in D$, $a \in (0,1)$, and $\lambda_n \in [a,1)$ such that there exists $\lambda = \lim_n \lambda_n$, $\lambda \in [a,1)$. Defined the sequence $\{x_n\}$ as follows

$$x_n = \lambda_n x_{n-1} + (1-\lambda_n)\overline{x}_n, \tag{14}$$

where $\overline{x}_n = T(\overline{x}_n, x_{n-1})$, $n \geq 1$. By (13) the mapping $T_v(.) = T(., v)$ is condensing in the weak topology of X, T_v is weakly continuous and maps bounded sets into bounded sets, so from [6, Theorem 6] T_v has a fixed point, for any $v \in D$. Then $\overline{x}_n$, and so x_n, are well-defined.

By [11, Proposition 1] we have $\lim_n \|x_n - \overline{x}_{n+1}\| = 0$. As in [2], $\omega(\{x_n\}) \leq \omega(\{\overline{x}_n\})$. We show that $\omega(\{\overline{x}_n\}) = 0$. In order to prove this assertion we follow the proof informed by Professor W. A. Kirk. Because D is convex, there are an uncountable of points in every neighborhood of x_n in D. Since the sequence $\{\overline{x}_n\}$ is countable, it is possible to choose $x'_n \in D$ such that $\overline{x}_n \neq x'_m$ $\forall m$ and also so that $\|x_n - x'_n\| \leq 1/n$ (and therefore $\|T(\overline{x}_n, x_{n-1}) - T(\overline{x}_n, x'_{n-1})\| \leq 1/(n-1)$ by the contractive condition).

By the well-known properties of $\omega(.)$ [5] we can see that

$$\omega(\{x'_n\}) = \omega(\{x'_n - x_n + x_n\}) \leq \omega(\{x'_n - x_n\}) + \omega(\{x_n\}) = \omega(\{x_n\})$$

since $\omega(\{x'_n - x_n\}) = 0$ and similarly $\omega(\{x_n\}) \leq \omega(\{x'_n\})$. Therefore,

$$\omega(\{x_n\}) = \omega(\{x'_n\}),$$

and analogously $\omega(\{T(\overline{x}_n, x_{n-1})\}) = \omega(\{T(\overline{x}_n, x'_{n-1})\})$.

Consequently, if $\omega(\{\overline{x}_n\}) > 0$, then

$$\begin{aligned}\omega(\{\overline{x}_n\}) &= \omega(\{T(\overline{x}_n, x_{n-1})\}) \\ &= \omega(\{T(\overline{x}_n, x'_{n-1})\}) \\ &\leq \omega(T(\{\overline{x}_n\}, \{x'_n\})) \\ &< \max\{\omega(\{\overline{x}_n\}), \omega(\{x'_n\})\} \\ &= \max\{\omega(\{\overline{x}_n\}), \omega(\{x_n\})\} \\ &\leq \omega(\{\overline{x}_n\}).\end{aligned}$$

This contradiction shows that $\omega(\{\overline{x}_n\}) = 0$. Hence $\omega(\{x_n\}) = \omega(\{\overline{x}_n\}) = 0$.

Thus $\{x_n\}$ and $\{\overline{x}_n\}$ are weakly precompact. Therefore, there exist subsequences $\{x_{n_j}\}$ and $\{\overline{x}_{n_j+1}\}$ which converges weakly to u and $\overline{u}$ respectively. Since $\lim_n \|x_n - \overline{x}_{n+1}\| = 0$ we get $u = \overline{u}$. By the weak continuity of T we see that u is a fixed point of T. □

We say that a Banach space X satisfies Opial's condition if for any sequence $\{x_n\}$ in X which converges weakly to x_0, we have

$$\liminf \|x_n - x_0\| < \liminf \|x_n - x\| \ \forall x \neq x_0, x \in X.$$

It is known [8] that for any Banach space X, the existence of a weakly sequentially continuous duality map (which holds for Hilbert spaces and the spaces $l_p, 1 < p < \infty$) implies that X satisfies Opial's condition which in turn implies that every weakly compact convex subset of X has normal structure, but none of the converse implications hold.

Corollary 2.7. *Let all conditions of Theorem 2.6 hold. Suppose further that X satisfies Opial's condition. Then the sequence $\{x_n\}$ defined by* (14) *converges weakly to a fixed point of T.*

Proof. By Theorem 2.6 the sequence $\{x_n\}$ is weakly precompact and every weak cluster point of $\{x_n\}$ is a fixed point of T. Suppose that there exist two distinct weak cluster points of $\{x_n\}$, say p_1 and p_2, and two weakly convergent subsequences $\{x_{n_j}\} \to p_1$, $\{x_{n_i}\} \to p_2$.

For any p_i $(i = 1, 2)$ we get from contractive condition (12) that

$$\|\overline{x}_n - p_i\| = \|T(\overline{x}_n, x_{n-1}) - T(p_i, p_i)\| \leq \|x_{n-1} - p_i\|,$$

hence

$$\|x_n - p_i\| = \|\lambda_n(x_{n-1} - p_i) + (1 - \lambda_n)(\overline{x}_n - p_i)\| \leq \|x_{n-1} - p_i\|.$$

Then there exists $\lim\limits_{n} \|x_n - p_i\|$. Opial's condition implies that

$$\lim \|x_n - p_1\| = \liminf \|x_{n_j} - p_1\| < \liminf \|x_{n_j} - p_2\| = \lim \|x_n - p_2\|,$$

and similarly,

$$\lim \|x_n - p_2\| = \liminf \|x_{n_i} - p_2\| < \liminf \|x_{n_i} - p_1\| = \lim \|x_n - p_1\|.$$

This contradiction shows that exactly one weak cluster point p of $\{x_n\}$ exists, hence $\{x_n\}$ converges weakly to p. □

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