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SOME STABILITY RESULTS FOR THE SEMI-AFFINE VARIATIONAL INEQUALITY PROBLEM

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Abstract. This paper establishes two theorems on the stability of a semi-affine variational inequality problem and gives a characterization of the stability of the generalized linear complementarity problem when the defining matrix is copositive plus on the constraint cone. One of our results gives an affirmative answer to the second open problem stated by M. S. Gowda and T. S. Seidman in their paper "Generalized linear complementarity problems" (Math. Programming Vol. 46, 1990, 329–340).

1. Introduction

Let $\Delta \subset R^n$ be a nonempty, closed, convex set, $M \in R^{n\times n}$ a matrix and $q \in R^n$ a vector. The *semi-affine variational inequality problem* defined by (M, q, Δ), which is denoted by $\mathrm{VI}(M, q, \Delta)$, is the problem of finding a vector x such that

$$x \in \Delta, \quad \langle Mx + q, y - x\rangle \geq 0 \quad \text{for all} \quad y \in \Delta. \tag{1.1}$$

Here $\langle\cdot,\cdot\rangle$ denotes the scalar product in R^n. The solution set of $\mathrm{VI}(M, q, \Delta)$ is denoted by $\mathrm{Sol}(\mathrm{VI}(M, q, \Delta))$. It is well known [1, p. 422] that when Δ is a cone in R^n, (1.1) is equivalent to the problem of finding an x such that

$$x \in \Delta,\ Mx + q \in \Delta^* \quad \text{and} \quad \langle Mx + q, x\rangle = 0,$$

which is denoted by $\mathrm{GLCP}(M, q, \Delta)$ and which is called the *generalized linear complementarity problem.* Here and in the sequel,

$$K^* = \{y \in R^n : \langle y, z\rangle \geq 0 \quad \text{for all} \quad z \in K\}$$

is the positive dual cone of a cone $K \subset R^n$.

Robinson [5] has characterized the stability of $\mathrm{VI}(M, q, \Delta)$ by the nonemptiness and the boundedness of $\mathrm{Sol}(\mathrm{VI}(M, q, \Delta))$ for the case where Δ is a polyhedral convex set and M is a positive semidefinite matrix. Gowda and Pang [1] obtained several sufficient conditions for the stability of $\mathrm{VI}(M, q, \Delta)$. Gowda and Seidman [2] considered the problem $\mathrm{GLCP}(M, q, \Delta)$ and characterized its stability with respect to perturbations of q for the case where M is copositive plus on the cone Δ. In the cited paper [2], Gowda and Seidman stated the following open problem:

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Whether it is true that if M is copositive plus on Δ and the solution set of $\mathrm{GLCP}(M, q, \Delta)$ *is nonempty and bounded, then there exists $\varepsilon > 0$ such that the solution set of* $\mathrm{GLCP}(M', q', \Delta)$ *is nonempty for any $(M', q') \in R^{n\times n} \times R^n$ and $q' \in R^n$ with* $\max\{\|M' - M\|, \|q' - q\|\} < \varepsilon\}$?

Note that Gowda and Pang have obtained a result on the stability of $\mathrm{GLCP}(M, q, \Delta)$ in the case where $\Delta \subset R^n_+$ is a polyhedral convex cone and the matrix M is copositive plus on Δ (see [1, Proposition 7]). This result gives a partial answer to the above open problem.

In this paper we obtain several perturbation results for the problem $\mathrm{VI}(M, q, \Delta)$, where the matrix M is copositive plus on the recession cone of Δ. One of our results provides an affirmative answer to the above open problem. Our main tools are the Hartman-Stampacchia Theorem ([3, Chap. 1]) and some arguments in [4, Chap. 7]. Note that our method of proof is somewhat elementary; it is quite different from those of [1] and [5]. By definition (see [2]), a matrix $M \in R^{n\times n}$ is said to be *copositive plus* on a cone $K \subset R^n$ if

(i) $v \in K$ implies $\langle Mv, v\rangle \geq 0$,

(ii) $(v \in K, \langle Mv, v\rangle = 0)$ implies $(M + M^T)v = 0$,

where superscript ^{T} denotes the matrix transposition. It is a well known fact that if M is positive semidefinite, then M is copositive plus in any cone $K \subset R^n$. Indeed, if $\langle Mx, x\rangle \geq 0$ for any $x \in R^n$, then condition (i) is valid. Moreover, if $\langle Mv, v\rangle = 0$ then v is the minimum point of the convex quadratic program $\min\{\langle (M + M^T)x, x\rangle : x \in R^n\}$. By the Fermat rule, $(M + M^T)v = 0$; thus (ii) is valid. The reader is referred to [4, Chap. 6] for an example of matrices which are copositive plus on R^n_+ and which are not positive semidefinite.

Throughout this paper, for a nonempty, closed, convex set $\Delta \subset R^n$ and a matrix $M \in R^{n\times n}$, the recession cone of Δ is denoted by $0^+\Delta$ and the set $\{Mx : x \in \Delta\}$ is denoted by $M\Delta$. The scalar product and the Euclidean norm in R^n are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. The corresponding matrix norm in $R^{n\times n}$ is also denoted by $\|\cdot\|$. The topological interior of a set $X \subset R^n$ is denoted by $\mathrm{int}X$, and the set of the positive integers is denoted by N.

2. Perturbation results

The main results of this paper can be stated as follows.

Theorem 2.1. *Let Δ be a nonempty, closed, convex cone in R^n, $M \in R^{n\times n}$ copositive plus on Δ. Then, the following properties are equivalent:*

(a) *The solution set of* $\mathrm{GLCP}(M, q, \Delta)$ *is nonempty and bounded.*

(b) $q \in \mathrm{int}((0^+\Delta)^* - M\Delta)$.

(c) *There exists $\varepsilon > 0$ such that for all $M' \in R^{n\times n}$ and $q' \in R^n$ with* $\max\{\|M' - M\|, \|q' - q\|\} < \varepsilon$, *the solution set of* $\mathrm{GLCP}(M', q', \Delta)$ *is nonempty.*

Theorem 2.2. *Let $\Delta \subset R^n$ be a nonempty, closed, convex set, $M \in R^{n\times n}$ and*

$q \in R^n$. If M is positive semidefinite, then the following properties are equivalent:

(a) *The set* $\mathrm{Sol}(\mathrm{VI}(M, q, \Delta))$ *is nonempty and bounded.*

(b) $q \in \mathrm{int}((0^+\Delta)^* - M\Delta)$.

(c) *There exists $\epsilon > 0$ such that for all $M' \in R^{n\times n}$ and $q' \in R^n$ with* $\max\{\|M' - M\|, \|q' - q\|\} < \varepsilon$, *the set* $\mathrm{Sol}(\mathrm{VI}(M', q', \Delta))$ *is nonempty.*

Theorem 2.3. *Let $\Delta \subset R^n$ be a nonempty, closed, convex set, $M \in R^{n\times n}$ and $q \in R^n$. Suppose that:*

(i) *M is copositive plus on $0^+\Delta$;*

(ii) *There exists $\alpha \in R$ such that $\langle Mx, x\rangle \geq \alpha$ for all $x \in \Delta$.*

Then the following properties are equivalent:

(a) *The set* $\mathrm{Sol}(\mathrm{VI}(M, q, \Delta))$ *is nonempty and bounded.*

(b) $q \in \mathrm{int}((0^+\Delta)^* - M\Delta)$.

(c) *There exists $\varepsilon > 0$ such that for all $M' \in R^{n\times n}$ and $q' \in R^n$ with* $\max\{\|M' - M\|, \|q' - q\|\} < \varepsilon$, *the set* $\mathrm{Sol}(\mathrm{VI}(M', q', \Delta))$ *is nonempty.*

Remark 2.1. The equivalence between (a) and (c) in Theorem 2.1 solves affirmatively the open problem stated by Gowda and Seidman [2, Problem 2] which we have recalled in Section 1. The equivalence between (a) and (b) has been established in [2, Theorem 6.1].

Remark 2.2. The implications (a) $\Rightarrow$ (c) and (c) $\Rightarrow$ (b) in Theorem 2.2 have been established in [1, Corollary 3 and the remark following Theorem 4].

Now we proceed to proving the above three theorems.

The following technical lemma was established in [4, Chap. 7] for the case where Δ is a polyhedral convex set.

Lemma 2.1. *Let Δ be a nonempty, closed, convex set in R^n, $M \in R^{n\times n}$ and $q \in R^n$. Then, the inclusion*

$$q \in \mathrm{int}\big((0^+\Delta)^* - M\Delta\big) \tag{2.1}$$

is equivalent to the following property:

$$\forall v \in 0^+\Delta \setminus \{0\},\ \exists x \in \Delta \quad \text{such that} \quad \langle Mx + q, v\rangle > 0. \tag{2.2}$$

Proof. Clearly, if (2.2) is not true then there exists $v \in (0^+\Delta) \setminus \{0\}$ such that

$$\langle Mx + q, v\rangle \leq 0 \leq \langle u, v\rangle \quad \forall x \in \Delta,\ \forall u \in (0^+\Delta)^*. \tag{2.3}$$

This means that v separates $\{q\}$ and $(0^+\Delta)^* - M\Delta$. Then (2.1) does not hold.

Conversely, if (2.1) does not hold then by separation theorem there exists $v \neq 0$ such that

$$\langle q, v\rangle \leq \langle u - Mx, v\rangle \quad \forall u \in (0^+\Delta)^*,\ \forall x \in \Delta.$$

This property implies (2.3), which shows that (2.2) does not hold. □

Proof of Theorem 2.3. (a) ⇒ (b): Let $\mathrm{Sol}(\mathrm{VI}(M, q, \Delta))$ be nonempty and bounded. To obtain a contradiction, suppose that (b) does not hold. Then, by Lemma 2.1 there exists $\bar{v} \in 0^+\Delta \setminus \{0\}$ satisfying

$$\langle Mx + q, \bar{v}\rangle \le 0 \quad \forall x \in \Delta. \tag{2.4}$$

Let $x^0 \in \mathrm{Sol}(\mathrm{VI}(M, q, \Delta))$. For each $t > 0$, $x_t = x^0 + t\bar{v}$ belongs to Δ because $\bar{v} \in 0^+\Delta$. Substituting x_t for x in (2.4) we get

$$\langle Mx^0 + q, \bar{v}\rangle + t\langle M\bar{v}, \bar{v}\rangle \le 0 \quad \forall t > 0.$$

This forces $\langle M\bar{v}, \bar{v}\rangle \le 0$. By assumption (i), $\langle M\bar{v}, \bar{v}\rangle \ge 0$. So we have $\langle M\bar{v}, \bar{v}\rangle = 0$. Then

$$(M + M^T)\bar{v} = 0, \tag{2.5}$$

because M is copositive plus on $0^+\Delta$. Let $y \in \Delta$ be given arbitrarily. Since $x^0 \in \mathrm{Sol}(\mathrm{VI}(M, q, \Delta))$ and $\langle M\bar{v}, \bar{v}\rangle = 0$, by (2.4) and (2.5) we have

$$\begin{aligned}
\langle Mx_t + q, y - x_t\rangle &= \langle Mx^0 + tM\bar{v} + q, y - x^0 - t\bar{v}\rangle \\
&= \langle Mx^0 + q, y - x^0\rangle + t\langle M\bar{v}, y - x^0\rangle \\
&\qquad - t\langle Mx^0 + q, \bar{v}\rangle - t^2\langle M\bar{v}, \bar{v}\rangle \\
&= \langle Mx^0 + q, y - x^0\rangle - t\langle \bar{v}, My + q\rangle \\
&\qquad - t\langle (M + M^T)\bar{v}, x^0\rangle \\
&\ge \langle Mx^0 + q, y - x^0\rangle \\
&\ge 0.
\end{aligned}$$

Since this holds for every $y \in \Delta$, $x_t \in \mathrm{Sol}(\mathrm{VI}(M, q, \Delta))$. Having the last inclusion for all $t > 0$, we conclude that $\mathrm{Sol}(\mathrm{VI}(M, q, \Delta))$ is unbounded, a contradiction. Thus (b) is valid.

(b) ⇒ (c): Suppose that (b) is valid, but (c) does not hold. Then there exists a sequence $\{(M^k, q^k)\} \subset R^{n\times n} \times R^n$ such that $(M^k, q^k) \to (M, q)$ and

$$\mathrm{Sol}(\mathrm{VI}(M^k, q^k, \Delta)) = \emptyset \quad \forall k = 1, 2, \ldots. \tag{2.6}$$

Since Δ is nonempty, closed and convex, there exists $j_0 \in N$ such that

$$\Delta_j := \Delta \cap \{x \in R^n : \|x\| \le j\}$$

is a nonempty, compact and convex for every $j \ge j_0$. By the Hartman-Stampacchia Theorem (see [3, Theorem 3.1]), $\mathrm{Sol}(\mathrm{VI}(M^k, q^k, \Delta_j)) \ne \emptyset$ for $j \ge j_0$. Fix any $x^{k,j} \in \mathrm{Sol}(\mathrm{VI}(M^k, q^k, \Delta_j))$. We have

$$\langle M^k x^{k,j} + q^k, y - x^{k,j}\rangle \ge 0 \quad \forall y \in \Delta_j. \tag{2.7}$$

Note that

$$\|x^{k,j}\| = j \quad \forall j \ge j_0. \tag{2.8}$$

Indeed, if $\|x^{k,j}\| < j$ then there exists $\beta > 0$ such that

$$\bar{B}(x^{k,j}, \beta) := \{x \in R^n : \|x - x^{k,j}\| \le \beta\} \subset \{x \in R^n : \|x\| \le j\}.$$

Hence from (2.7) it follows that

$$\langle M^k x^{k,j} + q^k, y - x^{k,j}\rangle \ge 0 \quad \forall y \in \Delta \cap \bar{B}(x^{k,j}, \beta). \tag{2.9}$$

It is clear that for each $y \in \Delta$ there exists $t = t(y) \in (0,1)$ such that $y(t) := x^{k,j} + t(y - x^{k,j})$ belongs to $\Delta \cap \bar{B}(x^{k,j}, \beta)$. By (2.9),

$$\langle M^k x^{k,j} + q^k), y(t) - x^{k,j}\rangle = t\langle M^k x^{k,j} + q^k, y - x^{k,j}\rangle \ge 0.$$

This implies that $\langle M^k x^{k,j} + q^k, y - x^{k,j}\rangle \ge 0$ for every $y \in \Delta$. Hence $x^{k,j} \in \mathrm{Sol}(\mathrm{VI}(M^k, q^k, \Delta))$, contrary to (2.6). Fixing an index $j \ge j_0$, we note that $\{x^{k,j}\}_{k\in N}$ has a convergent subsequence. By (2.8), without loss of generality, we may suppose that

$$\lim_{k\to\infty} x^{k,j} = x^j \quad \text{for some} \quad x^j \in \Delta \quad \text{with} \quad \|x^j\| = j. \tag{2.10}$$

Letting $k \to \infty$, from (2.7) and (2.10), we get

$$\langle Mx^j + q, y - x^j\rangle \ge 0 \quad \forall y \in \Delta_j. \tag{2.11}$$

On account of (2.10), without loss of generality we can assume that

$$\frac{x^j}{\|x^j\|} \to \bar{v} \in R^n \quad \text{with} \quad \|\bar{v}\| = 1.$$

Since $x^j \in \Delta$ and $\|x^j\|^{-1} \to 0$, by Theorem 8.2 from [6] we have $\bar{v} \in 0^+\Delta \setminus \{0\}$.

It is clear that for each $y \in \Delta$ there exists an index $j_y \ge j_0$ such that $y \in \Delta_j$ for every $j \ge j_y$. From (2.11) we deduce that

$$\langle Mx^j + q, y - x^j\rangle \ge 0 \quad \forall j \ge j_y.$$

Hence

$$\langle Mx^j + q, y\rangle \ge \langle Mx^j, x^j\rangle + \langle q, x^j\rangle \quad \forall j \ge j_y. \tag{2.12}$$

Dividing the last inequality by $\|x^j\|^2$ and letting $j \to \infty$, we get $0 \ge \langle M\bar{v}, \bar{v}\rangle$. By assumption (i), $\langle M\bar{v}, \bar{v}\rangle \ge 0$. So $\langle M\bar{v}, \bar{v}\rangle = 0$. Since M is copositive plus, this equality yields

$$M\bar{v} = -M^T\bar{v}. \tag{2.13}$$

Since $x^j \in \Delta$ for all j, by assumption (ii) we have $\langle Mx^j, x^j\rangle \ge \alpha$ for all j. From (2.12) it follows that

$$\langle Mx^j + q, y\rangle \ge \alpha + \langle q, x^j\rangle \quad \forall j \ge j_y.$$

Dividing the last inequality by $\|x^j\|$ and letting $j \to \infty$, we get $\langle M\bar{v}, y\rangle \ge \langle q, \bar{v}\rangle$. Combining this with (2.13) gives $\langle My + q, \bar{v}\rangle \le 0$. Since the latter holds for every $y \in \Delta$ and since $\bar{v} \in 0^+\Delta \setminus \{0\}$, Lemma 2.1 shows that (b) cannot hold. We have arrived at a contradiction.

(c) $\Rightarrow$ (a): Suppose that (c) holds. Then there exists $\varepsilon > 0$ such that for all $M' \in R^{n\times n}$ and $q' \in R^n$ with $\max\{\|M' - M\|, \|q' - q\|\} < \varepsilon$, the set $\mathrm{Sol}(\mathrm{VI}(M', q', \Delta))$ is nonempty. In particular, $\mathrm{Sol}(\mathrm{VI}(M, q, \Delta)) \neq \emptyset$. If $\mathrm{Sol}(\mathrm{VI}(M, q, \Delta))$ is unbounded, then there exists a sequence

$$\{x^k\} \subset \mathrm{Sol}(\mathrm{VI}(M, q, \Delta))$$

such that $\|x^k\| \to \infty$ and $x^k \neq 0$ for all k. There is no loss of generality in assuming that

$$\|x^k\|^{-1} x^k \to \bar{v} \in R^n \quad \text{with} \quad \|\bar{v}\| = 1.$$

Since $x^k \in \Delta$ and $\|x^k\|^{-1} \to 0$, by Theorem 8.2 from [6] we have $\bar{v} \in 0^+\Delta \setminus \{0\}$. For each k we have

$$\langle Mx^k + q, y - x^k \rangle \geq 0 \quad \forall y \in \Delta.$$

Hence

$$\langle Mx^k + q, y \rangle \geq \langle Mx^k, x^k \rangle + \langle q, x^k \rangle \quad \forall y \in \Delta. \tag{2.14}$$

Fixing any $y \in \Delta$, dividing the last inequality by $\|x^k\|^2$ and letting $k \to \infty$, we obtain $0 \geq \langle M\bar{v}, \bar{v} \rangle$. By assumption (i), this inequality yields

$$\langle M\bar{v}, \bar{v} \rangle = 0 \quad \text{and} \quad M^T\bar{v} = -M\bar{v}. \tag{2.15}$$

Since $x^k \in \Delta$, by assumption (ii) and by (2.14) we have

$$\langle Mx^k + q, y \rangle \geq \alpha + \langle q, x^k \rangle \quad \forall k = 1, 2, \ldots$$

Dividing the last inequality by $\|x^k\|$ and letting $k \to \infty$, we get

$$\langle M\bar{v}, y \rangle \geq \langle q, \bar{v} \rangle.$$

Combining this with (2.15) we can assert that

$$\langle My + q, \bar{v} \rangle \leq 0 \quad \forall y \in \Delta. \tag{2.16}$$

For any $\varepsilon_1 \in (0, \varepsilon)$ and for $q' = q - \varepsilon_1\bar{v}$, we have $\|q' - q\| = \varepsilon_1 < \varepsilon$ and, by (2.16),

$$\langle My + q', \bar{v} \rangle = \langle My + q, \bar{v} \rangle - \varepsilon_1 \langle \bar{v}, \bar{v} \rangle \leq -\varepsilon_1 < 0 \quad \forall y \in \Delta.$$

From this it follows easily that $\mathrm{Sol}(\mathrm{VI}(M, q', \Delta)) = \emptyset$. This contradicts the choice of ε and q'. We have shown that if (c) is valid then (a) must hold. The proof is complete. □

Proof of Theorem 2.1. Since Δ is a convex cone, we have $\Delta = 0^+\Delta$. Note that the assumption (ii) in Theorem 2.3 is satisfied with $\alpha = 0$, because M is copositive plus on Δ. As we have mentioned in Section 1, when Δ is a cone in R^n, $\mathrm{VI}(M, q, \Delta)$ is equivalent to the problem $\mathrm{GLCP}(M, q, \Delta)$. By Theorem 2.3, the properties (a), (b) and (c) are equivalent. □

Proof of Theorem 2.2. If M is positive semidefinite, then M is copositive plus on any closed convex cone in R^n and $\langle Mx, x \rangle \geq 0$ for any $x \in R^n$. So, by Theorem 2.3, the properties (a), (b) and (c) are equivalent. □

We have seen that Theorem 2.3 implies Theorems 2.1 and 2.2.

3. Examples

The following example illustrates the applicability of Theorem 2.3 to concrete variational inequality problems.

Example 3.1. Consider problem $\mathrm{VI}(M, q, \Delta)$, where $n = 2$,

$$M = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \quad q = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\Delta = \left\{x = (x_1, x_2)^T : -1 \le x_1 \le 0,\ x_2 \ge x_1^2\right\}.$$

We have

$$0^+\Delta = \{v = (v_1, v_2)^T : v_1 = 0,\ v_2 \ge 0\},$$

$$\langle Mx, x\rangle = -x_1^2 \ge -1 \quad \forall x \in \Delta, \quad \langle Mv, v\rangle = -v_1^2 = 0 \quad \forall v \in 0^+\Delta,$$

and

$$(M + M^T)v = \begin{pmatrix} -2v_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \forall v \in 0^+\Delta.$$

Hence the assumptions (i) and (ii) in Theorem 2.3 are satisfied. For any $v = (v_1, v_2)^T \in 0^+\Delta \setminus \{0\}$ and $x = (x_1, x_2)^T \in \Delta$, $-1 < x_1 \le 0$, we have

$$\begin{aligned} \langle Mx + q, v\rangle &= (-x_1 + x_2)v_1 + (-x_1 + 1)v_2 \\ &= (-x_1 + 1)v_2 > 0. \end{aligned}$$

Hence $q \in \mathrm{int}\big((0^+\Delta)^* - M\Delta\big)$ by Lemma 2.1. According to Theorem 2.3, there exists $\varepsilon > 0$ such that for all $M' \in R^{n\times n}$ and $q' \in R^n$ with $\max\{\|M' - M\|, \|q' - q\|\} < \varepsilon$, the set $\mathrm{Sol}(\mathrm{VI}(M', q', \Delta))$ is nonempty. Note that Δ is not a polyhedral convex set and M is not a positive semidefinite matrix.

In the next two examples, we consider variational inequality problems $\mathrm{VI}(M, q, \Delta)$ with

$$\Delta = \{x \in R^n : g_i(x) \le 0, i = 1, \dots, m\},$$

where $g_i(x)$, $i = 1, \dots, m$, are differentiable convex functions and the *Slater condition* holds; that is there exists $x^0 \in R^n$ such that $g_i(x^0) < 0$ for $i = 1, \dots, m$. Clearly, $\bar{x} \in \Delta$ is a solution of $\mathrm{VI}(M, q, \Delta)$ if and only if $\bar{x}$ is a solution of the following convex programming problem:

$$\min\{\langle M\bar{x} + q, y\rangle : y \in \Delta\}. \tag{3.1}$$

By [6, Corollary 28.3.1], the following *Lagrange Multiplier Rule* is valid for $\mathrm{VI}(M, q, \Delta)$: Vector $\bar{x} \in R^n$ is a solution of $\mathrm{VI}(M, q, \Delta)$ if and only if there exists $\lambda = (\lambda_1, \dots, \lambda_m) \in R^m$ which, together with $\bar{x}$, satisfy the *Kuhn-Tucker conditions* for (3.1):

$$\begin{cases} M\bar{x} + q + \sum\limits_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) = 0, \\ \lambda_i \ge 0,\ \lambda_i g_i(\bar{x}) = 0,\ g_i(\bar{x}) \le 0,\ i = 1, \dots, m, \end{cases}$$

where $\nabla g_i(\bar{x})$ denotes the gradient of g_i at $\bar{x}$.

Example 3.2. Consider problem VI(M, q, Δ), where $n = 2$, $m = 2$,

$$M = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}, \quad q = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\Delta = \left\{ x = (x_1, x_2)^T : g_1(x) = x_1 \leq 0,\ g_2(x) = x_1^2 - x_2 \leq 0 \right\}.$$

We have

$$0^+\Delta = \{v = (v_1, v_2)^T : v_1 = 0,\ v_2 \geq 0\},$$
$$\langle Mx, x\rangle = -3x_1^2 \quad \forall x \in \Delta, \quad \langle Mv, v\rangle = -3v_1^2 = 0 \quad \forall v \in 0^+\Delta,$$
$$\nabla g_1(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \nabla g_2(x) = \begin{pmatrix} 2x_1 \\ -1 \end{pmatrix},$$

and

$$(M + M^T)v = \begin{pmatrix} -6v_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \forall v \in 0^+\Delta.$$

Hence the assumption (i) in Theorem 2.3 is satisfied, while the assumption (ii) is violated. By the above-mentioned Lagrange Multiplier Rule, $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is a solution of VI(M, q, Δ) if and only if there exists $\lambda = (\lambda_1, \lambda_2)$ such that

$$\begin{cases} -3\bar{x}_1 + \lambda_1 + 2\lambda_2\bar{x}_1 = 0,\ 1 - \lambda_2 = 0, \\ \lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \lambda_1\bar{x}_1 = 0,\ \lambda_2(\bar{x}_1^2 - \bar{x}_2) = 0, \\ \bar{x}_1 \leq 0,\ \bar{x}_1^2 - \bar{x}_2 \leq 0. \end{cases}$$

This gives $\bar{x}_1 = 0$, $\bar{x}_2 = 0$, $\lambda_1 = 0$, $\lambda_2 = 1$, and we have Sol(VI(M, q, Δ)) $= \{(0, 0)^T\}$. Since $0 \in \Delta$ and $\langle q, v\rangle > 0$ for all $v \in 0^+\Delta \setminus \{0\}$, by Lemma 2.1, $q \in \operatorname{int}\big((0^+\Delta)^* - M\Delta\big)$. Take

$$M(\varepsilon) = \begin{bmatrix} -3 & 0 \\ \varepsilon & 0 \end{bmatrix}, \quad q(\varepsilon) = \begin{pmatrix} \varepsilon \\ 1 \end{pmatrix},$$

where $\varepsilon > 0$. We now show that Sol(VI$(M(\varepsilon), q(\varepsilon), \Delta)$) $= \emptyset$. By the Lagrange Multiplier Rule, $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is a solution of VI$(M(\varepsilon), q(\varepsilon), \Delta)$ if and only if there exists $\lambda = (\lambda_1, \lambda_2)$ such that

$$\begin{cases} -3\bar{x}_1 + \varepsilon + \lambda_1 + 2\lambda_2\bar{x}_1 = 0,\ \varepsilon\bar{x}_1 + 1 - \lambda_2 = 0, \\ \lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \lambda_1\bar{x}_1 = 0,\ \lambda_2(\bar{x}_1^2 - \bar{x}_2) = 0, \\ \bar{x}_1 \leq 0,\ \bar{x}_1^2 - \bar{x}_2 \leq 0. \end{cases} \tag{3.2}$$

If $\lambda_1 = \lambda_2 = 0$ then (3.2) gives

$$-3\bar{x}_1 + \varepsilon = 0,\ \varepsilon\bar{x}_1 + 1 = 0,\ \bar{x}_1 \leq 0,$$

a contradiction. If $\lambda_1 = 0, \lambda_2 > 0$ then (3.2) gives

$$\begin{cases} -3\bar{x}_1 + \varepsilon + 2\lambda_2\bar{x}_1 = 0,\ \varepsilon\bar{x}_1 + 1 - \lambda_2 = 0, \\ \bar{x}_1^2 - \bar{x}_2 = 0,\ \bar{x}_1 \leq 0. \end{cases}$$

From this it follows that

$$\varepsilon + \bar{x}_1(2\varepsilon\bar{x}_1 - 1) = 0,\ \bar{x}_1 \leq 0,$$

a contradiction. If $\lambda_1 > 0, \lambda_2 = 0$ then (3.2) gives

$$\lambda_1 + \varepsilon = 0,\ \lambda_2 = 1,\ \ \bar{x}_1 = 0,$$

a contradiction. If $\lambda_1 > 0, \lambda_2 > 0$ then (3.2) gives

$$\lambda_1 + \varepsilon = 0,\ \ \bar{x}_1 = \bar{x}_2 = 0,$$

a contradiction. Thus $\mathrm{Sol}(\mathrm{VI}(M(\varepsilon), q(\varepsilon), \Delta)) = \emptyset$. This example shows that, in Theorem 2.3, one cannot omit the assumption (ii) while keeping other assumptions.

Example 3.3. Consider problem $\mathrm{VI}(M, q, \Delta)$, where $n = 2, m = 2$,

$$M = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad q = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\Delta = \left\{x = (x_1, x_2)^T : g_1(x) = x_1 \le 0,\ g_2(x) = -x_2 \le 0\right\}.$$

We have

$$0^+\Delta = \{v = (v_1, v_2)^T : v_1 \le 0,\ v_2 \ge 0\}.$$

Since

$$\langle Mx, x\rangle = -x_1 x_2 \ge 0 \quad \forall x \in \Delta,$$

the assumption (ii) in Theorem 2.3 is satisfied. As

$$\langle Mv, v\rangle = -v_1 v_2 \ge 0 \quad \forall v \in 0^+\Delta,$$

M is copositive on $0^+\Delta$. However,

$$(M + M^T)v = \begin{pmatrix} -v_2 \\ -v_1 \end{pmatrix} \ne \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \forall v \in 0^+\Delta \setminus \{0\}.$$

In particular, for $v = (0, 1)^T \in 0^+\Delta$ satisfying $\langle Mv, v\rangle = 0$, one does not have $(M + M^T)v = 0$. Thus the assumption (i) in Theorem 2.3 is violated. By the Lagrange Multiplier Rule, $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is a solution of $\mathrm{VI}(M, q, \Delta)$ if and only if there exists $\lambda = (\lambda_1, \lambda_2)$ such that

$$\begin{cases} -\bar{x}_2 + \lambda_1 = 0,\ \ 1 - \lambda_2 = 0, \\ \lambda_1 \ge 0,\ \lambda_2 \ge 0,\ \lambda_1 \bar{x}_1 = 0,\ \lambda_2(-\bar{x}_2) = 0, \\ \bar{x}_1 \le 0,\ -\bar{x}_2 \le 0. \end{cases}$$

Arguing similarly as in Example 3.2, we can show that the above system is equivalent to the following one:

$$\bar{x}_1 \le 0,\ \bar{x}_2 = 0,\ \lambda_1 = 0,\ \lambda_2 = 1.$$

So $\mathrm{Sol}(\mathrm{VI}(M, q, \Delta)) = \{(\bar{x}_1, \bar{x}_2)^T : \bar{x}_1 \le 0, \bar{x}_2 = 0\}$. We have

$$\langle Mx + q, v\rangle = -x_2 v_1 + v_2 \quad \forall v \in 0^+\Delta, \forall x \in \Delta.$$

For $\hat{x} = (0, 1) \in \Delta$, we have

$$\langle M\hat{x} + q, v\rangle = -v_1 + v_2 > 0 \quad \forall v \in 0^+\Delta \setminus \{0\},$$

which, by Lemma 2.1, implies $q \in \operatorname{int}\big((0^+\Delta)^* - M\Delta\big)$. Take

$$q(\varepsilon) = \begin{pmatrix} \varepsilon \\ 1 \end{pmatrix},$$

where $\varepsilon > 0$. We claim that $\mathrm{Sol}(\mathrm{VI}(M, q(\varepsilon), \Delta)) = \emptyset$. Indeed, by the Lagrange Multiplier Rule, $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is a solution of $\mathrm{VI}(M, q(\varepsilon), \Delta)$ if and only if there exists $\lambda = (\lambda_1, \lambda_2)$ such that

$$\begin{cases} -\bar{x}_2 + \varepsilon + \lambda_1 = 0, \;\; 1 - \lambda_2 = 0, \\ \lambda_1 \geq 0, \, \lambda_2 \geq 0, \, \lambda_1 \bar{x}_1 = 0, \, \lambda_2(-\bar{x}_2) = 0, \\ \bar{x}_1 \leq 0, \, -\bar{x}_2 \leq 0. \end{cases}$$

This system implies that

$$\varepsilon + \lambda_1 = 0, \, \lambda_2 = 1, \, \lambda_1 \geq 0, \, \bar{x}_2 = 0, \, \bar{x}_1 \leq 0,$$

which is impossible. Thus $\mathrm{Sol}(\mathrm{VI}(M, q(\varepsilon), \Delta)) = \emptyset$. Note that the set $\mathrm{Sol}(\mathrm{VI}(M, q, \Delta))$ is unbounded. We have seen that the properties (a) and (c) in Theorem 2.3 are not valid, while the property (b) holds. Thus, in general, the assumption (ii) of Theorem 2.3 together with the copositiveness of M on $0^+\Delta$ cannot guarantee the validity of the equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

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