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# COMMUTATIVE GROUP ALGEBRAS OF $p^{\omega+n}$ - PROJECTIVE ABELIAN GROUPS

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Abstract. Suppose $G$ is an abelian group and $R$ is a unitary commutative ring of prime characteristic $p$. The first main result is that the $p^{\omega+1}$-projective $p$-group $G$ is a direct factor of the group of normed units $V(RG)$ and $V(RG)/G$ is totally projective provided $R$ is perfect. The second main result is that the complete set of invariants for the $R$-algebra $RG$ consists of $G$, in the cases when $G$ is splitting or $G$ is with torsion-free rank one and in both situations the torsion part of $G$ is a $p^{\omega+1}$-projective $p$-group. These claims strengthen a theorem due to Beers-Richman-Walker.

## 1. Introduction

Let $R$ be a commutative ring with identity of prime characteristic $p$ and $G$ a multiplicatively written abelian group. A problem of some interest and importance, in which we concentrate, is that of deducing information about $G$ from the group algebra $RG$ over $R$ (often called the Problem of Invariants). Of particular interest are the conditions under which $RG$ determines $G$ up to an isomorphism, or equivalently, conditions under which the isomorphism of $RG$ and $RH$ as $R$-algebras for any group $H$ implies an isomorphism between $G$ and $H$ (see, for instance, cf. [4, 11, 14]).

Throughout the rest of this paper, $F$ will denote a field of characteristic $p \neq 0$ and $F_p$ the field with $p$-elements (i.e. a simple field of characteristic $p$). Let $G_0$ denote the maximal torsion subgroup of $G$, and $G_p$ the $p$-primary part of $G$ with socle $G[p]$. Moreover, $V(RG)$ denotes the group of all normed units in $RG$, and $S(RG)$ is its Sylow $p$-subgroup, i.e. its $p$-torsion component.

The principal known results concerning this theme are the following two theorems of Beers-Richman-Walker.

**Theorem A** ([1]). *If $G$ and $H$ are abelian groups and $F_pG \cong F_pH$ as $F_p$-algebras, then $G[p]$ and $H[p]$ are isometric.*

Since $G[p]$ as a valuated vector space over the simple field $F_p$ serves to classify every $p$-group $G$ that belongs to the class of $p^{\omega+1}$-projective groups ([16, 17]),

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Beers-Richman and Walker have argued the following extra attainment consequence.

**Theorem B** ([1]). *Let $G$ and $H$ be $p^{\omega+1}$-projective $p$-groups. Then the $F_p$-isomorphism $F_pH \cong F_pG$ implies $H \cong G$.*

In the present research exploration, we shall generalize the latter statement and moreover we shall study the behaviour of the $p^{\omega+n}$-projective $p$-groups, introduced in [23, 24] by Nunke, in $RG$. Our main results are listed in the next sections. In the first one, we deal with the characterization of certain normalized units in group rings. We use successfully the established results to derive in the second section two independent theorems on the direct factor. Referring to the preceding claims, we obtain in the last third section two types of isomorphism classifications pertaining to the modular group algebras of $p^{\omega+n}$-projective abelian $p$-groups.

## 2. Descriptions of unit groups

For simplicity of the notation, the symbol $\otimes_C$ plainly designates direct sums of cyclic groups. A most part of the notions and terminology from the abelian group theory is the same as in [15]. In addition, we shall let $U(R)$ denote the unit group of a ring $R$.

The following two lemmas are well-known and documented, and are included here only for the sake of completeness and for the convenience of the reader (see [6, 10, 13]).

**Lemma 1.** *For each ordinal $\nu$ the following formulas are fulfilled:*

$$U^{p^\nu}(R) = U(R^{p^\nu});$$
$$U^{p^\nu}(RG) = U(R^{p^\nu}G^{p^\nu});$$
$$V^{p^\nu}(RG) = V(R^{p^\nu}G^{p^\nu});$$
$$S^{p^\nu}(RG) = S(R^{p^\nu}G^{p^\nu}).$$

**Lemma 2.** *Let $R$ be a ring with no nilpotents. Then $S(RG) = 1$ if and only if $G_p = 1$.*

Denote by $N(R)$ the nilradical (a Baer's radical) of a ring $R$. If $H$ is a subgroup of $G$, the letter $I(RG; H)$ will denote the relative augmentation ideal of $RG$ with respect to $H$.

The following technical matter is crucial.

**Lemma 3.** *The following isomorphism dependences hold:*

(i) $[1 + I(RG; H)]/[1 + I(RG; B)] \cong 1 + I(R(G/B); H/B)$, *where $B \leq H \leq G$ and $H$ is $p$-torsion.*

(ii) $V(RG)/[1 + I(RG; B)] \cong V(R(G/B))$ *hence* $S(RG)/[1 + I(RG; B)] \cong S(R(G/B))$, *where $B \leq G$ and $B$ is $p$-torsion.*

(iii) $S(RG) = 1 + I(RG; G_p)$, *when $N(R) = 0$.*

*Proof.* (i) Clearly, $1 + I(RG; H)$ is a multiplicative $p$-group since $I(RG; H)$ is a nil-ideal. Similarly for $1 + I(RG; B)$. It is a plain technical matter to see that the natural map $G \to G/B$ induces an $R$-algebra epimorphism $RG \to R(G/B)$ with kernel $I(RG; B)$. Analogously for $H \to H/B$. Thus they induce a group surjection $1+I(RG; H) \to 1+I(R(G/B); H/B)$ with kernel $1+I(RG; B)$, which gives (i).

(ii) Follows by the same arguments as in (i), using the fact that $(A/B)_p = A_p/B$ holds for $B \leq A_p$ and any abelian groups $A$ and $B$.

(iii) Follows by application of (ii) and Lemma 2 at $B = G_p$. □

We shall continue with the following assertion documented in [6] (see [5] or [12, 13] too).

**Proposition 1.** *Suppose $H$ is a $p$-primary pure subgroup of $G$. Then $1+I(RG; H)$ is $\oplus_C$ if and only if $H$ is $\oplus_C$.*

We can now enlarge the above proposition to a statement announced in [5], which is our goal here, namely:

**Theorem 1.** *Let $H$ be a $p$-primary pure subgroup of $G$. Then $1 + I(RG; H)$ is $p^{\omega+n}$-projective if and only if $H$ is $p^{\omega+n}$-projective.*

*Proof.* First, we note that the following useful affirmation is valid (cf. [23] or [2, 3]).

**Criterion (Nunke [23]).** *An abelian $p$-group $G$ is $p^{\omega+n}$-projective ($n \in N_0 = N \cup \{0\}$) if and only if there exists a subgroup $B \subseteq G[p^n]$ such that $G/B = \oplus_C$.*

An immediate consequence is that an arbitrary subgroup of a $p^{\omega+n}$-projective $p$-group is with the same property. Moreover, each $\oplus_C$ is $p^{\omega+n}$-projective, and even more, it is $p^{\alpha}$-projective for all $\alpha \geq \omega$.

Now, we will use the cited criterion to achieve our aim.

By the above commentary, $H$ is $p^{\omega+n}$-projective provided that so is $1+I(RG; H)$.

Now, we treat the more difficult converse question. For this purpose, let $H$ be $p^{\omega+n}$-projective. Certainly, then there is $B \subseteq H[p^n]$ so that $H/B$ is $\oplus_C$.

Furthermore,

$$1 + I(RG; B) \subseteq 1 + I(RG; H[p^n]) \subseteq (1 + I(RG; H))[p^n]$$

and owing to (i) we conclude

$$[1 + I(RG; H)]/[1 + I(RG; B)] \cong 1 + I(R(G/B); H/B).$$

Since $H/B$ is pure in $G/B$, then Proposition 1 and the Nunke's criterion yield the result. □

The next statement is helpful for applications (see [8] and [5]).

**Corollary 1.** *Suppose $G$ is $p$-torsion. Then $V(RG)$ is $p^{\omega+n}$-projective if and only if so is $G$. Moreover, if $G$ is arbitrary and $N(R) = 0$, $S(RG)$ is $p^{\omega+n}$-projective if and only if so is $G_p$.*

*Proof.* Setting $H = G$ or $H = G_p$, we observe that Theorem 1 and (iii) are applicable. This completes the proof. □

A new result of this aspect, however, is the following supplement to the above theorem, namely:

**Proposition 2.** *Suppose $G$ is a $p^{\omega+n}$-projective $p$-group. Then $V(RG)/G$ is $p^{\omega+n}$-projective.*

Before proving this claim, we need a few conventions.

**Lemma 4.** *Assume $1 \in L \leq R$ and $A, B \leq G$. Then*

$$(GV(RA)) \cap V(LB) = BV(L(A \cap B)).$$

*Proof.* Given $x$ in the left-hand side. Hence

$$x = \sum_i \alpha_i b_i = g \sum r_i a_i,$$

where $\alpha_i \in L$, $b_i \in B$; $g \in G$, $r_i \in R$, $a_i \in A$. On the other hand, the canonical forms yield $\alpha_i = r_i$ and $b_i = ga_i$ for each natural number $i$. Thus $a_i a_j^{-1} = b_i b_j^{-1}$ for every positive integers $i$ and $j$. Consequently,

$$x = b_1 \sum_i \alpha_i b_i b_1^{-1} \in BV(L(A \cap B)).$$

Indeed $\sum\limits_i \alpha_i b_i b_1^{-1}$ is in $L(A \cap B)$ and besides it is a normed unit because so is $\sum\limits_i \alpha_i b_i$. This verifies that the right-hand side contains the left-hand one. The reverse is elementary. The lemma is proved. □

**Proof of Proposition 2.** In fact, consuming the Nunke's criterion, there is $B \subseteq G[p^n]$ with $G/B = \oplus_C$, whence employing the Kulikov's criterion appeared in [15], $G = \bigcup\limits_{i<\omega} G_i$, $G_i \subseteq G_{i+1}$ and $G_i \cap G^{p^i} \subseteq B \subseteq G_i$. Furthermore, $GV(RB)/G \subseteq (V(RG)/G)[p^n]$ and

$$V(RG)/GV(RB) = \bigcup_{i<\omega} [GV(RG_i)/GV(RB)].$$

Moreover, according to the modular law and to Lemma 1 and Lemma 4, we compute

$$\begin{aligned}[GV(RG_i)] \cap [V(R^{p^i} G^{p^i})GV(RB)] &= GB(RB)[(GV(RG_i)) \cap V(R^{p^i} G^{p^i})] \\ &= GV(RB)V(R^{p^i}(G_i \cap G^{p^i})) = GV(RB).\end{aligned}$$

Hence, again by virtue of the criterion of Kulikov [15],

$$V(RG)/G/GV(RB)/G \cong V(RG)/GV(RB)$$

is $\oplus_C$ and complying with the Nunke's criterion as well, we thus complete the proof. □

The next statement is stronger than Proposition 2.

**Proposition 3.** *The factor-group* $[1+I(RG;H)]/H$ *is* $p^{\omega+n}$*-projective, provided* $H$ *is a pure* $p^{\omega+n}$*-projective* $p$*-subgroup of* $G$*. In particular, if* $G_p$ *is* $p^{\omega+n}$*-projective and* $N(R)=0$ *then* $S(RG)/G_p$ *is* $p^{\omega+n}$*-projective.*

*Proof.* On the way of this formulation, we develop machinery which will be used to prove the claim. In fact, in view of the above stated Nunke's and Kulikov's criteria, $H=\cup_{k<\omega}H_k$, $H_k\leq H_{k+1}$, and

$$H_k\cap G^{p^k}\subseteq B\subseteq H_k\cap H[p^n]$$

for all $k<\omega$ and some group $B$. We observe that

$$H[1+I(RG;B)]/H\subseteq(1+I(RG;H)/H)[p^n],$$

$$(1+I(RG;H))/H/H[1+I(RG;B)]/H\cong(1+I(RG;H))/H[1+I(RG;B)].$$

Besides, we find that

$$(1+I(RG;H))/H[1+I(RG;B)]=\bigcup_{k<\omega}[H[1+I(RG;H_k)]/H[1+I(RG;B)]],$$

where the latter is an ascending sequence of subgroups. Moreover, conforming with the intersection ratios obtained in [10, 12], we see that

$$\begin{aligned}&[H(1+I(RG;H_k))]\cap[(1+I(R^{p^k}G^{p^k};H^{p^k}))H(1+I(RG;B))]\\&=H(1+I(RG;B)).[(H(1+I(RG;H_k)))\cap(1+I(R^{p^k}G^{p^k};H^{p^k}))]\\&=H(1+I(RG;B)).(1+I(R^{p^k}G^{p^k};H_k\cap G^{p^k}))=H(1+I(RG;B)).\end{aligned}$$

Thereby, in virtue of the consecutive application of the just applied Kulikov's and Nunke's criteria, we detect that $(1+I(RG;H))/H$ is a $p^{\omega+n}$-projective abelian $p$-group, as desired.

If $N(R)=0$, then $S(RG)=1+I(RG;G_p)$ by Lemma 3 (see, for instance, also cf. [10]), hence setting $H=G_p$ we deduce the final part. The proof is completed. □

## 3. Direct factors of unit groups

Exploiting a lemma of May (cf. [20, 21]), it is a routine exercise to check that the $p$-group $G$ is a direct factor of $V(RG)$ if and only if

$$G\times V(R(\coprod_{i<\omega}G))\cong V(R(\coprod_{i<\omega}G)).$$

This fact will be freely used below without further comments.

We now quote one of our main results in this section, which extends a similar result for $\oplus_C$, namely:

**Theorem 2.** (Direct Factor). *Suppose* $G$ *is a* $p^{\omega+1}$*-projective abelian* $p$*-group. Then* $G$ *is a direct factor of* $V(RG)$ *with* $p^{\omega+1}$*-projective complement.*

*Proof.* Put $G' = \coprod_{i<\omega} G_i$, where $G_i = G$ for all $i < \omega$. Observe that

$$(G \times V(RG'))[p] = G[p] \times V(RG')[p].$$

Apparently, we can write

$$G[p] = \coprod_{i \in I} \langle g_i \rangle \quad \text{and} \quad V(RG')[p] = \coprod_{j \in J} \langle v_j \rangle,$$

where $g_i$ and $v_j$ are elements of order $p$, and $I$ and $J$ are index sets. We observe that $|G[p]| = |I| \geq \aleph_0$ or otherwise $|G[p]| = p^{|I|} < \aleph_0$. By the same token, $|V(RG')[p]| = |J| \geq \aleph_0$ or in the remaining case $|V(RG')[p]| = p^{|J|} < \aleph_0$. Next, consider the outer direct product

$$G[p] \times V(RG')[p] = \coprod_{i \in I} \langle g_i \rangle \times \coprod_{j \in J} \langle v_j \rangle.$$

Because of the fact that $G'$ is infinite and $V(RG')[p] \supseteq V(RG'; G'[p])$ is one also, we easily conclude that $|G[p] \times V(RG')[p]| = |V(RG')[p]|$, since $|I| \leq |J|$. We also exclude the case $|V(RG')[p]| < \aleph_0$ that is obviously impossible. Since every two cyclic groups of order $p$ are isomorphic, then we elementarily obtain that

$$G[p] \times V(RG')[p] \cong V(RG')[p].$$

On the other hand the isotypity of $G'$ in $V(RG')$ means that all heights will be computed in $G'$. The last isomorphism between the investigated socles can be chosen to be height preserving bearing in mind the fact that if $g_i$ is an element of $G[p]$ of order $p$ and of $p$-height as calculated in $G$ equal to $\alpha$, then there exists an index $j \in J$ so that $v_j$ is a generating element of a cyclic group in $V(RG')[p]$ of order $p$ and of height $\alpha$. But as we have observed $|I| \leq |J|$ and picking the component isometries between the cyclic members, we get the wanted isometry between the explored socles, whence the groups $G \times V(RG')$ and $V(RG')$ have isometric socles, as promised.

On the other hand $G$ is $p^{\omega+1}$-projective and hence so is $G'$ consulting with [15]. Thus by virtue of Corollary 1 and [16, 17] we derive $G \times V(RG') \cong V(RG')$. Consequently, $G$ is a direct factor of $V(RG)$. The complementary factor is also $p^{\omega+1}$-projective since by Corollary 1 the same is $V(RG)$. The proof is finished. □

**Remark.** The evidence of the previous attainment was based on the assertion that the abelian groups $G \times V(RG')$ and $V(RG')$ have isometric socles. To say that they are isometric means that there is an isomorphism of the socles which preserves heights in the full group. We indicate that $(G \times V(RG'))[p] \neq V(RG')[p]$. Such an equation is nonsense since the two socles containing different kinds of elements are not equal; they are isomorphic but not equal.

We also point out the facts that

$$\coprod_{i<\omega} G[p] \cong (G \times \coprod_{i<\omega} G)[p]$$

$$V(RG')[p] = G'[p] \times M,$$

whence

$$G[p] \times V(RG')[p] = G[p] \times G'[p] \times M \cong \coprod_{i<\omega} G[p] \times M = V(RG')[p],$$

do not give our wanted isomorphism since the above decomposition raises a rather arbitrary vector space complement, called $M$. The socles are vector spaces over the integers modulo $p$, and certainly any subspace has a complementary subspace. But in the category of valuated vector spaces, a vector space decomposition may not be a decomposition in the category.

For a classical example, consider a group $A$ generated by two independent elements $a$ and $b$ of respective orders $p$ and $p^2$, that is

$$A = \langle a, b \rangle = \langle a \rangle \times \langle b \rangle$$

such that $o(a) = p$ and $o(b) = p^2$. Specifically, one can decompose $A[p]$ as $\langle a \rangle \times \langle b^p \rangle$ or as $\langle a \rangle \times \langle ab^p \rangle$, i.e.

$$A[p] = \langle a \rangle \times \langle b^p \rangle = \langle a \rangle \times \langle ab^p \rangle.$$

The first is a coproduct of valuated vector spaces, while the second one is not. This is so, because $a^i \in A \setminus A^p \ \forall 1 \le i \le p-1$, $b^p \in A^p \setminus A^{p^2}$; $A^{p^2} = 1$ and $a^{p-1}.ab^p = b^p \in \langle a \rangle \times \langle ab^p \rangle$. Thus

$$\text{height}_A(a^{p-1}.ab^p) = \text{height}_A(b^p) = 1 > \min\left\{\text{height}_A(a^{p-1}), \text{height}_A(ab^p)\right\} =$$
$$= \min\left\{\text{height}_A(a^{p-1}), \text{height}_A(a)\right\} = 0.$$

The choice of a complement for $\langle a \rangle$ affects height computations in the coproduct.

Now, we shall expand the direct factor theorem listed above in the following light. Well, the achievement can be formulated as follows.

**Theorem 3.** (Direct Factor). *Suppose $G$ is a $p^{\omega+1}$-projective abelian $p$-torsion group and $R^p = R^{p^2}$. Then $V(RG)/G$ is totally projective and so $G$ is a direct factor of $V(RG)$ with totally projective complement.*

*Proof.* Since $V(RG)/G$ is totally projective if and only if

$$V(RG)/G/(V(RG)/G)^{p^\alpha} \cong V(RG)/GV(R^pG^{p^\alpha})$$

is $p^\alpha$-projective for all $\alpha \le \omega + 1$, then owing to Proposition 2 and to the fact we shall show in the sequel that $V(RG)/GV(R^pG^{p^\omega})$ is $\oplus_C$, we will be done. In fact, because $G$ is $p^{\omega+1}$-projective, there exists $B \le G[p]$ such that $G/B = \bigcup_{k<\omega} (G_k/B)$ with $G_k \subseteq G_{k+1}$ and $(G_k/B) \cap (G/B)^{p^k} = B$. Thereby $G = \bigcup_{k<\omega} G_k$ with $G_k \cap G^{p^k} \subseteq B \subseteq G_k \cap G[p]$ and $G^{p^\omega} \subseteq B$. That is why it is clear that $G_k^{p^{k+1}} = 1$. Besides, if $g \in G_k$ and $\text{order}(g) > p$, then $\text{height}(g) < k$. Let $R$ be perfect, that is, $R^p = R$. Then

$$V(RG)/GV(RG^{p^\omega}) = \bigcup_{k<\omega} [V(RG_k)GV(RG^{p^\omega})/GV(RG^{p^\omega})].$$

Next, we shall select height-finite subgroups $T_k$ with the properties

$$V(RG)/GV(RG^{p^\omega}) = \bigcup_{k<\omega} T_k$$

and $T_k \subseteq T_{k+1}$. These groups may be constructed in the following manner:

$$T_k = M_k GV(RG^{p^\omega})/GV(RG^{p^\omega})$$

where $M_k$ will be chosen as special subgroups of $V(RG)$ with finite height spectrum of finite heights as computed in $V(RG)$ and such that $M_k \subseteq M_{k+1}$. Indeed, take

$$M_k = \langle \alpha_1^{(k)} + \alpha_2^{(k)} g_2^{(k)} + \cdots + \alpha_m^{(k)} g_m^{(k)} | \alpha_1^{(k)}, \ldots, \alpha_m^{(k)} \in R,$$
$$\alpha_1^{(k)} + \cdots + \alpha_m^{(k)} = 1;$$
$$g_2^{(k)}, \ldots, g_m^{(k)} \in G_k : \langle g_2^{(k)}, \ldots, g_m^{(k)} \rangle \subseteq (G \setminus G^{p^k}) \cup G^{p^\omega} \rangle.$$

The last inclusion is equivalent to

$$\langle g_2^{(k)}, \ldots, g_m^{(k)} \rangle \cap G^{p^k} \subseteq G^{p^\omega}$$

and it insures the property that $g_2^{(k)}, ..., g_m^{(k)}$ together with the products of all their nontrivial degrees have in $G$ heights $< k$ or $\geq \omega$ (see [12], too). Observe that $G_k \subseteq M_k \subseteq M_{k+1}$ and $M_k \subseteq V(RG_k)$. Evidently every element in $M_k$ has the form $x_1^{\varepsilon_1} \cdots x_t^{\varepsilon_t}$, where $x_i$ are of the above kind and $0 \leq \varepsilon_i \leq$ order $(x_i)$ $(1 \leq i \leq t < \omega)$. Certainly, every element of $M_k$ can be written as $c_k v_k$, where $c_k \in G_k$ and $v_k$ has in the canonical form basis member 1 and either has a basis member with height $< k$ or all of its basis members are in heights $\geq \omega$. So, it is a routine technical matter to verify that $v_k$ has height bounded at $k$ or $\geq \omega$ in $V(RG)$ for each $k < \omega$.

Moreover, given nonidentity $y_k \in T_k$, whence $y_k = a_k GV(RG^{p^\omega})$ where $a_k \in M_k \setminus GV(RG^{p^\omega})$ possesses the above described properties. But by [12], we have that $GV(RG^{p^\omega}) = GV^{p^\omega}(RG)$ is nice in $V(RG)$ and therefore height$(y_k)$ = height$(a_k b_k)$ when $b_k \in G$ because $a_k b_k$ does not lie in $V(RG^{p^\omega})$. It is not hard to see that height$(a_k) \leq$ height$(y_k)$ and consequently height$(b_k) \geq$ height$(a_k)$. Otherwise height$(y_k)$ = height$(b_k) <$ height$(a_k)$, which is false. Finally, by what we have shown above, it is easy to observe that height$(y_k)$ = height$(a_k)$ according to the present form of $a_k$, or more especially owing to the fact that we may select $g_1^{(k)} = 1$. Thus $T_k$ are height-finite subgroups with $T_k \subseteq T_{k+1}$. Moreover, we easily infer that

$$\bigcup_{k<\omega} T_k = V(RG)/GV(RG^{p^\omega}).$$

This completes the first half.

Now, suppose that $R^p = R^{p^2}$. Since $G$ is $p^{\omega+1}$-projective, the same holds for $G^p$ and thus by what we have just proved

$$V(R^p G^p)/G^p \cong (V(RG)/G)^p$$

is totally projective, i.e. via [24] so is the quotient group $V(RG)/G$. This concludes the proof in general after all. □

## 4. Isomorphisms of group algebras

We are now ready to prove the following generalization of the last half of Theorem B and of [8, Proposition 1].

**Proposition 4.** (Isomorphism). *Let $G_p$ be a $p^{\omega+n}$-projective group. Then $F_pH \cong F_pG$ as $F_p$-algebras for any group $H$ implies $H_p \cong G_p$. Moreover, if $G_p$ is totally projective of length $\omega + n$, then the $F$-isomorphism $FH \cong FG$ for some group $H$ yields $H_p \cong G_p$.*

*Proof.* Since $F_pH \cong F_pG$ guarantees $S(F_pG) \cong S(F_pH)$, then we can apply Corollary 1 to obtain $H_p$ is $p^{\omega+n}$-projective. But adapting the technique described in [1] and analogous arguments, we can deduce that $G[p^n] = G_p[p^n]$ may be retrieved from $F_pG$ as a valuated vector space, hence $F_pH \cong F_pG$ implies that $H_p[p^n]$ and $G_p[p^n]$ are isometric as filtered vector spaces and so by virtue of [16] or [17], $G_p \cong H_p$, as claimed.

For the second part, owing to [10],

$$F(G/G_p^{p^{\omega+k}}) \cong F(H/H_p^{p^{\omega+k}})$$

for $0 \leq k \leq n$. On the other hand,

$$G_p/G_p^{p^{\omega+k}} = (G/G_p^{p^{\omega+k}})_p$$

is $p^{\omega+k}$-projective by [15]. Therefore, because of the isomorphism

$$S(F(G/G_p^{p^{\omega+k}})) \cong S(F(H/H_p^{p^{\omega+k}})),$$

the application of Corollary 1 ensures that

$$(H/H_p^{p^{\omega+k}})_p = H_p/H_p^{p^{\omega+k}}$$

is $p^{\omega+k}$-projective, i.e. $H_p$ is totally projective of length $\omega + n$. As a final step, since the Ulm-Kaplansky cardinal functions of $G_p$ and $H_p$ are known by May (cf. [19], [1]) to be equal, we extract via [15] that $G_p \cong H_p$, and thus we are done. The proposition is verified. □

**Remark.** The second half of the above proposition partially settles a question posed by W. May in [21] (see also [10]).

**Corollary 2.** [8]. *Suppose $G$ is a $p^{\omega+n}$-projective $p$-group. Then $F_pH \cong F_pG$ as $F_p$-algebras for some group $H$ if and only if $H \cong G$.*

We recall now that

$$s_q(F_p) = \{i \in N_0 = N \cup \{0\} : F_p(\varepsilon_i) \neq F_p(\varepsilon_{i+1})\}$$

where $\varepsilon_i$ is a primitive $q^i$-th root of unity. This set is called a spectrum of $F_p$ with respect to $q$ ($q$ a prime) and was introduced by T. Mollov. We come now to the main affirmation in this direction.

**Theorem 4.** (Invariants). *Let $G$ be a splitting abelian group so that $G_0/G_p$ is finite and $G_p$ is $p^{\omega+1}$-projective. Then $F_pH \cong F_pG$ as $F_p$-algebras for some group $H$ if and only if the following hold:*

*(1) $H$ is splitting abelian;*

*(2) $H_p \cong G_p$;*

*(3) $H/H_0 \cong G/G_0$;*

*(4) $|H_0/H_p| = |G_0/G_p|$;*

*(5) $|(H_0)^{q^i}/H_p| = |(G_0)^{q^i}/G_p|$ for all primes $q \neq p$ and all $i \in s_q(F_p)$.*

*Proof.* "Necessity". And so, (2) is guaranteed by virtue of Proposition 3, and (3) follows from [19]. Further, owing to [19] or [10], $F_p(G/G_p) \cong F_p(H/H_p)$ and so we can apply our algorithm in [9] to get that

$$F_p((G/G_p)_0) = F_p(G_0/G_p) \cong F_p(H_0/H_p) = F_p((H/H_p)_0).$$

Hence, by application of a result of T. Mollov (cf. [7, 9] for example), we conclude (4) and (5). Now we shall show that (1) is true. Indeed, $G_p$ is a direct factor of $G$, hence obviously $V(F_pG_p)$ is a direct factor of $V(F_pG)$. Thus $V(F_pG) \cong V(F_pG_p) \times M$ for some subgroup $M$. But (2) and the hypothesis ensure $V(F_pH) \cong V(F_pH_p) \times M$. After this, Theorems 2 or 3, both combined with a mild modification of ([20], Lemma 2), lead us to the fact that $H_p$ is a direct factor of $V(F_pH)$, whence even of $H$ as a subgroup. On the other hand, (4) assures that $H_0/H_p$ is finite and thus it is a direct factor of $H/H_p$ as its pure subgroup utilizing [15]. Finally,

$$H \cong H_p \times H/H_p \cong H_p \times H_0/H_p \times H/H_0 \cong H_0 \times H/H_0$$

and so $H$ is splitting, as stated.

"Sufficiency". Write

$$G \cong G_0 \times G/G_0 \cong G_p \times G_0/G_p \times G/G_0$$

and by a reason of symmetry $H \cong H_p \times H_0/H_p \times H/H_0$. Consequently

$$F_pG \cong F_pG_p \otimes_{F_p} F_p(G_0/G_p) \otimes_{F_p} F_p(G/G_0)$$

and by the same token

$$F_pH \cong F_pH_p \otimes_{F_p} F_p(H_0/H_p) \otimes_{F_p} F_p(H/H_0).$$

It is easily seen that (2) and (3) imply $F_pG_p \cong F_pH_p$ and $F_p(G/G_0) \cong F_p(H/H_0)$, respectively. After this, (4) and (5) in view of the cited above result of Mollov yield $F_p(G_0/G_p) \cong F_p(H_0/H_p)$. Finally, we deduce that $F_pG \cong F_pH$, as desired. The proof is complete after all. □

We obtain an immediate consequence.

**Corollary 3.** (Isomorphism). *Suppose $G$ is splitting whose $G_0$ is a $p^{\omega+1}$-projective $p$-group. Then $F_pH \cong F_pG$ as $F_p$-algebras for any group $H$ if and only if $H \cong G$.*

**Remark.** The last two statements improve the corresponding assertions for $\oplus_C$ ([4, 9]).

After this, we consider the question pertaining to the isomorphism of commutative group algebras of rank one mixed abelian groups. But first and foremost we state outline a group - theoretical fact, which plays a key role and which is formulated only for a complete information; it was proved in [18] (see [22] and [25] as well).

**Criterion** (Fuchs -Toubassi). *Let $G$ and $H$ be two abelian groups of torsion-free rank one and let $G_0$ and $H_0$ be $p^{\omega+1}$-projective p-groups. Then $G \cong H$ if and only if the following are true:*

(i) *there is a height-preserving isomorphism $\phi_p : G[p] \to H[p]$;*

(ii) *$\phi_p$ preserves limits, i.e. any sequence $\{g_{0i}\}_{i\in I} \in G_0[p]$ converges in the p-adic topology to an element $g \in G$ of infinite order if and only if $\{\phi_p(g_{0i})\}_{i\in I} \in H_0[p]$ converges in the p-adic topology to an element $h \in H$ of infinite order, and such that for every integer $k \geq 1$ the convergence of a sequence $\{g_{0i}\}_{i\in I} \in G_0[p]$ to some $g \in G$ with*

$$g^{p^j} = x^{p^k}.a(a \in G_0[p]; x \in G, o(x) = \infty; j, k \in N_0)$$

*implies that $\{\phi_p(g_{0i})\}_{i\in I} \in H_0[p]$ converges to some $h \in H$ so that*

$$h^{p^j} = y^{p^k}.\phi_p(a)(y \in H, o(y) = \infty)$$

*and vice versa - here again, the p-adic topologies are meant.*

We proceed by proving now the central affirmation in this direction.

**Theorem 5.** (Isomorphism). *Suppose $G$ is an abelian group of torsion-free rank one such that $G_0$ is $p^{\omega+1}$-projective p-primary. Then $F_pH \cong F_pG$ as $F_p$-algebras for any group $H$ if and only if $H \cong G$.*

*Proof.* Clearly, $H$ is $p$-mixed of torsion-free rank one (see also cf. [19]). Next, an appeal to Proposition 3 gives that $H_0$ is $p^{\omega+1}$-projective and even more, that $G_0 \cong H_0$.

The first condition (i) holds true in view of Theorem A.

We may assume without loss of generality that $F_pG = F_pH$. From [21, 10] it follows that

$$V(F_pG) = GS(F_pG) = HS(F_pH) = V(F_pH).$$

Further, suppose for each $k \geq 1$ is fulfilled $g \in g_{0i}G^{p^k}$ such that $g^{p^j} = x^{p^k}.a$, where the notions are as to the foregoing formulated in the Group Criterion. Then by what we have already shown above along with Theorem A tell us that there is $h \in H$ of infinite order so that $h \in h_{0i}H^{p^k}$ where $h_{0i} = \phi_p(g_{0i})$ and such that $h^{p^j} = y^{p^k}.\phi_p(a)$. Taking into account that $y$ is with infinite order, we infer that point (ii) is really satisfied.

Finally, the above criterion riches us that $G \cong H$, as asserted. The proof is finished in all generality. □

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