ACTA MATHEMATICA VIETNAMICA 251
Volume 29, Number 3, 2004, pp. 251-257

NORMALITY OF A FAMILY OF BANACH-VALUED HOLOMORPHIC MAPS

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Abstract. The aim of this paper is to give conditions under which a family of holomorphic maps from the unit disc $\triangle$ in $\mathbb{C}$ with values in a Banach analytic manifold X is non-normal.

1. Introduction

Let X,Y be complex spaces. By $\mathrm{Hol}(Y, X)$ we denote the space of holomorphic maps from Y to X with the open-compact topology. It is a natural problem to study the behavior of a given family $\mathcal{F} \subset \mathrm{Hol}(Y, X)$ with respect to this topology (see [K], [GR], [Wu]). First we recall the notion about a normal family. A family $\mathcal{F} \subset \mathrm{Hol}(Y, X)$ is called *normal* if for every sequence $\{f_n\} \subset \mathcal{F}$ one of the following conditions is satisfied:

(a) There is a subsequence $\{f_{n_k}\}$ which converges in $\mathrm{Hol}(Y, X)$.

(b) There is a compactly divergent subsequence $\{f_{n_k}\}$, i.e. for every compact subsets $K \subset X$, $L \subset Y$ there exists k_0 such that for every $k \geq k_0$ we have

$$f_{n_k}(L) \cap K = \emptyset.$$

It is proved in [AK] (see also [TTH]) that if Y is a domain in $\mathbb{C}^n$ and X is a complex manifold with a complete Hermitian metric, then a family $\mathcal{F} \subset \mathrm{Hol}(Y, X)$ is not normal if and only if there exists a sequence $\{p_j\} \subset Y$ with $p_j \to p_0 \in Y$, a sequence $\{f_j\} \subset \mathcal{F}$ and a sequence $\{\rho_j\} \subset \mathbb{R}$ with $\rho_j > 0$ such that the sequence

$$g_j(\lambda) = f_j(p_j + \rho_j \lambda), \lambda \in \mathbb{C}^n$$

satisfies one of the following conditions:

(1) $\{g_j\}$ is compactly divergent.

(2) $\{g_j\}$ converges in $\mathrm{Hol}(\mathbb{C}^n, X)$ to a non constant map $g \in \mathrm{Hol}(\mathbb{C}^n, X)$.

The main goal of this note is to find an analogue of this theorem in the case where X is a Banach analytic manifold and Y is the unit disc $\triangle$ in $\mathbb{C}$. To understand the difficulty in dealing with the case where X is of infinite dimension we consider the following example.

Assume that E is an infinite dimension Banach space and

$$\mathbb{B}(0, 1) = \{z \in E : \|z\| < 1\}$$

Received August 15, 2003; in revised form September 13, 2004.

the unit ball in E. Choose a sequence $\{x_n\} \subset \mathbb{B}(0,1)$ such that

$$\inf_{n\neq m} \|x_n - x_m\| > 0.$$

Consider the sequence of holomorphic maps $h_n : \triangle \longrightarrow \mathbb{B}(0,1)$ given by

$$h_n(\lambda) = \lambda x_n.$$

Since $h_n(0) = 0$ for all $n \geq 1$ and if $0 < |\lambda| < 1$ we have

$$\|h_n(\lambda) - h_m(\lambda)\| = |\lambda|\|x_n - x_m\| \nrightarrow 0$$

as $n, m \to \infty$, the sequence $\{h_n\}$ does not contain any subsequence which is convergent or compactly divergent. We notice, however, that the sequence $\{h_n\}$ is compactly divergent on $\triangle \setminus \{0\}$. This is in sharp contrast to the case of finite dimension since it is even true that $\mathrm{Hol}(\triangle, X)$ is normal for every complete hyperbolic domain X (in the sense of Kobayashi) in $\mathbb{C}^n$.

In order to obtain a correct generalization of this result for infinite dimension, the definition of normality should be modified appropriately. A reasonable choice is to replace (b) by

(b'): There exists a discrete subset S of $\triangle$ such that $\{f_n\}$ contains a compactly divergent subsequence on $\triangle \setminus S$.

Indeed, it is shown in [LNP] that $\mathrm{Hol}(\triangle, X)$ is normal in our sense for every complete hyperbolic domain in a Banach analytic manifold. It is also proved in this paper that in the case of finite dimension our definition of normality coincides with the usual one.

The main result of this paper, Theorem 2.1, is an analogue to the above mentioned theorem of Alardro and Krant in the special case $Y = \triangle$. It is an interesting and challenging problem to find a complete generalization of this result in the case of infinite dimension. The techniques of the proof are inspired of our previous work [BHK] and of the paper [AK]. Finally, we would like to mention that the normality of a family of holomorphic mappings defined on a given domain in a Banach space was studied also in [KiKr]. In that paper the family $\mathrm{Hol}(\triangle, \mathbb{B})$ is not normal in the usual sense, where $\mathbb{B}$ is the unit ball in the Hilbert space l_2.

In this note we use the notion of Banach analytic manifolds presented in the books of P. Mazet [Ma] and J. P. Ramis [Ra].

The following definition of normality is motivated from the above example. Let X be a Banach analytic manifold and $\mathcal{F} \subset \mathrm{Hol}(\triangle, X)$. We say that $\mathcal{F}$ is normal if for every sequence $\{f_n\} \subset \mathcal{F}$, one of the following two conditions holds:

i) There exists a subsequence $\{f_{n_k}\}$ which is convergent in $\mathrm{Hol}(\triangle, X)$.

ii) There exists a discrete subset S of $\triangle$ and a subsequence $\{f_{n_k}\}$ such that $\left\{f_{n_k}|_{\triangle\setminus S}\right\}$ is compactly divergent, i.e. for every compact subsets $K \subset \triangle \setminus S$ and $L \subset X$, there exists k_0 such that

$$f_{n_k}(K) \cap L = \emptyset$$

for $k \geq k_0$.

2. Main results

The aim of this note is to prove the following theorem.

Theorem 2.1. . *Let X be a Banach analytic manifold and $\mathcal{F} \subset \mathrm{Hol}(\triangle, X)$. Then $\mathcal{F}$ is not normal if and only if there exist a compactly divergent sequence $\{f_n\}_{n\geq 1} \subset \mathcal{F}$, a sequence $\{p_n\} \subset \triangle$ which is convergent to $p_0 \in \triangle$ and a positive number sequence $\{\rho_n\} \downarrow 0$ so that the sequence $\{g_n(\lambda) = f_n(p_n + \rho_n\lambda) : \lambda \in \mathbb{C}\}$ satisfies one of the following conditions:*

1) There exists a subsequence $\{\widetilde{g}_n\}_{n\geq 1}$ of the sequence $\{g_n\}$ such that $\{\widetilde{g}_n\}$ is compactly divergent outside a discrete subset $S \subset \mathbb{C}$.

2) There exists a subsequence $\{\widetilde{g}_n\}_{n\geq 1}$ of the sequence $\{g_n\}$ which is convergent in $\mathrm{Hol}(\mathbb{C}, X)$ to a non-constant holomorphic map $g : \mathbb{C} \to X$.

Proof. From the non-normality of $\mathcal{F}$ it follows that there exists a sequence $\{f_n\} \subset \mathcal{F}$ such that $\{f_n\}$ does not contain neither any compactly divergent subsequence on $\triangle$ nor any convergent subsequence in $\mathrm{Hol}(\triangle, X)$. Hence we can find a subsequence $\{h_n\} \subset \{f_n\}$ and a sequence $\{\lambda_n\}$ which is convergent to $a \in \triangle$ so that the sequence $\{h_n(\lambda_n)\}$ is also convergent. We consider the following cases.

Case 1. Assume that the sequence of derivatives $\{h_n^{'}\}$ of the sequence $\{h_n\}$ is bounded on every compact subset of $\triangle$. Put

$$Z = \{\lambda \in \triangle : \exists \lim_n h_n(\lambda)\}.$$

Using the boundedness of $\{h_n^{'}\}$ on every compact subset of $\triangle$ we claim that Z is closed.

First we assume that $Z^{'} \neq \emptyset$ where $Z^{'}$ is the set of limit points of Z. Take an arbitrary point $\lambda_1 \in Z^{'}$. Then there exists $x_1 = \lim_n h_n(\lambda_1)$. Let U be a coordinate neighbourhood of x_1 which is biholomorphic to a ball of a Banach space E. Since $\{h_n\}$ is equicontinuous on every compact subset of $\triangle$ we can find $r > 0$ with $\{h_n(\lambda)\} \subset U$ for all $\lambda \in \triangle(\lambda_1, r) = \{\lambda \in \triangle : |\lambda - \lambda_1| < r\}$. Since λ_1 is a limit point of Z then Vitali theorem (Theorem 2.1 in [AN]) implies that the sequence $\{h_n\}$ is convergent in $\mathrm{Hol}(\triangle(\lambda_1, r), E)$. Hence $\triangle(\lambda_1, r) \subset Z$ and, consequently, $\mathrm{Int}\, Z \neq \emptyset$. Repeating the above argument we claim that $\mathrm{Int}\, Z$ is closed in $\triangle$ which implies $\mathrm{Int}\, Z = \triangle$. Now from the equicontinuity of the sequence $\{h_n\}$ on each compact subset of $\triangle$ by the Arzela-Ascoli theorem we infer that the sequence $\{h_n\}$ contains a convergent subsequence $\{h_{n_k}\}$ in $\mathrm{Hol}(\triangle, X)$. This contradicts the assumption on the sequence $\{f_n\}$.

Now consider the case $Z' = \emptyset$. Let

$$\mathcal{A} = \{(U, h_n^U) : U \text{ is open in } \triangle \text{ and } \{h_n^U\} \subset \{h_n\} \text{ which is compactly divergent on } U\}.$$

Endowed $\mathcal{A}$ with the order relation defined as follows: $U_1 < U_2$ if $U_1 \subset U_2$ and $\{h_n^{U_2}\} \subset \{h_n^{U_1}\}$. First we check that $\mathcal{A} \neq \emptyset$. It suffices to show that

$\exists\, \epsilon > 0\ \exists\, \{h_n\}_{n\in B} \subset \{h_n\}$ such that $\{h_n\}_{n\in B}$ is compactly divergent on

$$\triangle^*(a,\epsilon) = \{\lambda \in \triangle : 0 < |\lambda - a| < \epsilon\}.$$

where a is the point at the beginning of this proof. Suppose that this is not true. Using the same argument at the beginning of Case 1, we can find a subsequence $B_k \subset \mathbb{N}$ and $\lambda_k \in \triangle^*\big(a, \frac{1}{k}\big)$ for each k such that $\{h_n(\lambda_k)\}_{n\in \mathrm{B}_k}$ is convergent. Moreover, we can assume that $\mathrm{B}_{k+1} \subset \mathrm{B}_k$. Now by the diagonal process we claim that there exists a subsequence $\{h_n\}_{n\in\mathrm{B}} \subset \{h_n\}$ such that $a \in \Big(Z(\{h_n\}_{n\in\mathrm{B}})\Big)'$. Again using the same argument we deduce that there exists a subsequence $\{h_n\}_{n\in\mathrm{C}} \subset \{h_n\}_{n\in\mathrm{B}}$ which is convergent in $\mathrm{Hol}(\triangle, X)$, a contradiction.

Now let $\Big\{(U_\alpha, h_n^{U_\alpha})\Big\}$ be a linearly ordered subset of $\mathcal{A}$. By the Lindelöfness of $\triangle$ there exists a sequence

$$U_1 < U_2 < \cdots < U_n < \cdots$$

such that

$$U = \bigcup_{\alpha\in I} U_\alpha = \bigcup_{j=1}^{\infty} U_{\alpha_j}$$

and for every $\alpha \in I$ there exists $j \geq 1$ with $U_\alpha \subset U_{\alpha_j}$. Using the diagonal process we can find a subsequence $\{q_n\} \subset \{h_n\}$ such that it is compactly divergent on U. Hence $(U, q_n) \in \mathcal{A}$. We notice that (U, q_n) just is a majorant element of the family $\Big\{(U_\alpha, h_n^{U_\alpha})\Big\}_{\alpha\in I}$. By Zorn theorem $\mathcal{A}$ has a maximal element (Ω, h_n^{Ω}).

Let

$$T = \Big\{\lambda \in \triangle \setminus \Omega : \exists \text{ a subsequence } \{h_{n_k}^{\Omega}\} \subset \{h_n\} \text{ such that } \{h_{n_k}^{\Omega}(\lambda)\} \text{ is convergent in } X\Big\}.$$

For each $\lambda \in T$, by repeating the arguments used in the above proof, we can find $\varepsilon > 0$ and a subsequence $\{l_n\} \subset \{h_n^{\Omega}\}$ such that $\{l_n\}$ is compactly divergent on $\triangle^*(\lambda, \epsilon)$. Then $\{l_n\}$ is compactly divergent on $\triangle^*(\lambda,\epsilon)\cup\Omega$. By the maximality of the element (Ω, h_n^{Ω}) it follows that $\triangle^*(\lambda, \epsilon) \subset \Omega$. Hence T is discrete and $\{h_n^{\Omega}\}$ is compactly divergent on $\triangle \setminus T$. This contradicts the fact that the sequence $\{f_n\}$ does not contain any compactly divergent subsequence.

Case 2. Assume that there exists a compact subset L of $\triangle$ such that

$$\sup\{\|h_n'(\lambda)\| : n \geq 1,\ \lambda \in L\} = +\infty.$$

Then we can find sequences $\{\widetilde{p}_k\} \subset L$, $\{t_k\} \subset \mathbb{C}$ with $|t_k| = 1$ such that

$$\|h_{n_k}'(\widetilde{p}_k)(t_k)\| \geq k^2.$$

Moreover, we can assume that $\{\widetilde{p}_k\}$ converges to $\widetilde{p}_0 \in L$ with $|\widetilde{p}_k - \widetilde{p}_0| < \dfrac{1}{2k}$ for all $k \geq 1$. Without loss of generality we suppose that $\widetilde{p}_0 = 0$. By $\{l_k\}$ we denote the sequence $\{h_{n_k}\}$. Let

$$M_k = \max\Big\{(1-|z|^2k^2)\|l_k'(z)(t)\| : |t|=1, |z| \le \frac{1}{k}\Big\} \tag{1}$$

Choose p_k and t_k^0 with $|p_k| < \frac{1}{k}$, $|t_k^0| = 1$ such that the right-hand side of (1) attains the maximum value at the pair (p_k, t_k^0). From $|\widetilde{p}_k| < \frac{1}{2k}$ we deduce that

$$M_k \ge (1-|\widetilde{p}_k|^2k^2)\|l_k'(\widetilde{p}_k)(t_k)\| \ge (1-\frac{1}{(2k)^2}k^2)k^2 = \frac{3}{4}k^2 \to +\infty$$

as $k \to +\infty$. Let

$$\rho_k = \frac{1}{\|l_k'(p_k)(t_k^0)\|} = \frac{1-|p_k|^2k^2}{M_k} \le \frac{1}{M_k}\,.$$

Then $\rho_k \to 0$ as $k \to +\infty$. Put

$$R_k = \frac{\frac{1}{k} - |p_k|}{\rho_k}\,.$$

Then we obtain

$$|p_k + \rho_k\xi| \le |p_k| + |\rho_k\xi| \le |p_k| + \rho_k\frac{1}{\rho_k}\Big(\frac{1}{k} - |p_k|\Big) = \frac{1}{k}$$

for every $\xi \in \mathbb{C}$ with $|\xi| \le R_k$, where

$$\begin{aligned} R_k &= \frac{\frac{1}{k} - |p_k|}{\rho_k} = \frac{1-k|p_k|}{k\rho_k} = \frac{1-|p_k|^2k^2}{k\rho_k(1+|p_k|k)} \\ &= \frac{M_k}{k(1+|p_k|k)} \ge \frac{\frac{3}{4}k^2}{2k} = \frac{3}{8}k \to +\infty \end{aligned}$$

as $k \to +\infty$.

We define $g_k = l_k(p_k + \rho_k z)$ for $k \ge 1$, $z \in \mathbb{C}$. Let $|z| < R_k$. As above we have $|p_k + \rho_k z| \le \frac{1}{k}$. Thus

$$M_k \ge (1-|p_k+\rho_k z|^2k^2)\|l_k'(p_k+\rho_k z)(t)\|$$

for all $t \in \mathbb{C}$, $|t| = 1$.

Fix $R > 0$ and choose k sufficiently large such that $R < R_k$. If $|z| \leq R$ then for all $t \in \mathbb{C}$, $|t| = 1$ we have

$$\begin{aligned}
\|g_k^{'}(z)(t)\| &= \rho_k \|l_k^{'}(p_k + \rho_k z)(t)\| \leq \rho_k \frac{M_k}{1 - |p_k + \rho_k z|^2 k^2} \\
&\leq \frac{1 - |p_k|^2 k^2}{1 - |p_k + \rho_k z|^2 k^2} = \frac{1 + |p_k| k}{1 + |p_k + \rho_k z| k} \, \frac{1 - |p_k| k}{1 - |p_k + \rho_k z| k} \\
&\leq 2 \frac{1 - |p_k| k}{1 - |p_k| k - \rho_k |z| k} = \frac{2}{1 - \dfrac{\rho_k |z| k}{1 - |p_k| k}} = \frac{2}{1 - \dfrac{|z|}{R_k}} \\
&\leq \frac{2}{1 - \dfrac{R}{R_k}} \to 2
\end{aligned}$$

as $k \to +\infty$.

Hence in every disc of $\mathbb{C}$ the family $\{g_k\}$ is defined and holomorphic with sufficiently large k and $\{g_k^{'}\}$ are bounded on every compact subset of $\mathbb{C}$.

Assume that the sequence $\{g_k\}$ is not compactly divergent. Then as in the proof at the beginning of the case 1, we can find a sequence $\{\lambda_k\} \subset \mathbb{C}$ such that $\{\lambda_k\} \to \lambda_0, \lambda_0 \in \mathbb{C}$ and a subsequence $\{q_k\}$ of the sequence $\{g_k\}$ such that $\{q_k(\lambda_k)\}$ converges to $p \in X$. Let

$$Z = \{\lambda \in \mathbb{C} : \exists \lim_k q_k(\lambda)\}.$$

Then as in the above proofs $Z \neq \emptyset$. Now we again consider the following cases.

Assume that, as above, $Z^{'} \neq \emptyset$. Then using the same arguments at the beginning of Case 1 we can find a subsequence $\{\widetilde{g}_k\}$ of the sequence $\{g_k\}$ such that $\{\widetilde{g}_k\}$ is uniformly convergent on every compact subset of $\mathbb{C}$ to a holomorphic map $g : \mathbb{C} \to X$. Thus

$$\|\widetilde{g}_k^{'}(0)(t_k^0)\| = \rho_k \|l_k^{'}(p_k)(t_k^0)\| = \frac{\rho_k M_k}{1 - |p_k|^2 k^2} = 1.$$

Since $|t_k^0| = 1$ for all $k \geq 1$, by passing to a suitable subsequence of the sequence $\{\widetilde{g}_k\}$ we can find a point $\widetilde{t}$ with $|\widetilde{t}| = 1$ and $\|g^{'}(0)(\widetilde{t})\| = 1$. Hence g is not a constant map.

Assume that $Z^{'} = \emptyset$. Then by repeating the same arguments of the case 1, we can find a subsequence $\{\widetilde{g}_k\}$ of the sequence $\{g_k\}$ and a discrete subset $S \subset \mathbb{C}$ such that the $\{\widetilde{g}_k\}$ is compactly divergent on $\mathbb{C} \setminus S$.

From the definition in Sec. 1 and the hypothesis we can see that the sufficiency part of the theorem is obvious.

The proof of Theorem 2.1 is now complete. □

Remark. We do not know whether Theorem 2.1 is true or not if $\triangle$ is replaced by an arbitrary domain D in $\mathbb{C}^n$.

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