ACTA MATHEMATICA VIETNAMICA
Volume 29, Number 3, 2004, pp. 237-249

DUALITIES AND DIMENSIONS OF IRREDUCIBLE REPRESENTATIONS OF PARABOLIC SUBGROUPS OF LOW DEGREES

NGUYEN DANG HO HAI AND TON THAT TRI

Dedicated to Professor Huynh Mui on the occasion of his sixtieth birthday

Abstract. Let $GL_{n_1,\dots,n_r}$ be a parabolic subgroup of the general linear group GL_n over the prime field $\mathbb{F}_p$ of p elements. A complete set of distinct irreducible modules for $\mathbb{F}_p[GL_{n_1,\dots,n_r}]$ was explicitly constructed in [7]. In this paper, we use this construction to determine the contragredient dual module of each $\mathbb{F}_p[GL_{n_1,\dots,n_r}]$-irreducible module and prove that its dimension can be computed via the dimensions of some $\mathbb{F}_p[GL_{n_i}]$-irreducible modules.

1. Introduction

Let p be a prime number, $\mathbb{F}_p$ the finite field of p elements and GL_n the general linear group of all $n \times n$ invertible matrices over $\mathbb{F}_p$. Let $n_1, \dots, n_r$ be positive integers such that $n_1 + \cdots + n_r = n$. The parabolic subgroup $GL_{n_1,\dots,n_r}$ of GL_n is defined as follows

$$GL_{n_1,\dots,n_r} = \left\{ \begin{pmatrix} B_1 & & * \\ & \ddots & \\ 0 & & B_r \end{pmatrix} \in GL_n : B_i \in GL_{n_i}, 1 \le i \le r \right\}.$$

Let $\mathbb{F}_p[x_1, \dots, x_n]$ be the commutative polynomial algebra in n indeterminants $x_1, \dots, x_n$ over $\mathbb{F}_p$. We have an action of GL_n on $\mathbb{F}_p[x_1, \dots, x_n]$ in the usual way. In other words, $\mathbb{F}_p[x_1, \dots, x_n]$ is thought of as an $\mathbb{F}_p[GL_n]$-module, and hence an $\mathbb{F}_p[G]$-module, for each subgroup G of GL_n. For each $1 \le i \le n$, the i-th Dickson invariant is defined as follows

$$L_i = L_i(x_1, \dots, x_i) = \begin{vmatrix} x_1 & x_2 & \dots & x_i \\ x_1^p & x_2^p & \dots & x_i^p \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{p^{i-1}} & x_2^{p^{i-1}} & \dots & x_i^{p^{i-1}} \end{vmatrix}.$$

Let $\beta = (\beta_1, \dots, \beta_n)$ be a sequence of nonnegative integers and put $L^\beta = \prod_{i=1}^{n} L_i^{\beta_i}$. Denote by $H_\beta(G)$ the $\mathbb{F}_p[G]$-submodule generated by L^β. It is obvious

Received April 16, 2003; in revised form September 6, 2004.

that $H_\beta(G)$ is an $\mathbb{F}_p$-vector space with the generators $\{\sigma L^\beta : \sigma \in G\}$.

Proposition 1.1 ([7, 1.1])**.**

$$\{H_\beta(GL_{n_1,\dots,n_r}) : \beta = (\beta_1,\dots,\beta_n), 0 \le \beta_i \le p-1, 1 \le i \le n, \\ \beta_{n_1}\beta_{n_1+n_2}\cdots\beta_{n_1+\cdots+n_r} \ne 0\}$$

is a complete set of $(p-1)^r p^{n-r}$ *distinct irreducible modules for the algebra* $\mathbb{F}_p[GL_{n_1,\dots,n_r}]$ *and these modules are absolutely irreducible.*

For each $0 \le i \le r$, put $N_i = n_0 + \cdots + n_i$ with $n_0 = 0$. Denote by $\mathbb{F}_p^{(n_1,\dots,n_r)}$ the set of all sequences $(\beta_1,\dots,\beta_n)$ such that $0 \le \beta_j \le p-1$, $1 \le j \le n$ and $\beta_{N_i} \ne p-1$, $1 \le i \le r$. By noting that

$$H_{(\beta_1,\dots,\beta_{N_i-1},p-1,\beta_{N_i+1},\dots,\beta_n)}(GL_{n_1,\dots,n_r}) \cong H_{(\beta_1,\dots,\beta_{N_i-1},0,\beta_{N_i+1},\dots,\beta_n)}(GL_{n_1,\dots,n_r})$$

for $1 \le i \le r$, we can restate the above proposition as follows.

Proposition 1.2. $\{H_\beta(GL_{n_1,\dots,n_r}) : \beta \in \mathbb{F}_p^{(n_1,\dots,n_r)}\}$ *is a complete set of* $(p-1)^r p^{n-r}$ *distinct irreducible modules for the algebra* $\mathbb{F}_p[GL_{n_1,\dots,n_r}]$ *and these modules are absolutely irreducible.*

An immediate consequence of the proposition is the following.

Corollary 1.3. $\{H_\beta(GL_n) : \beta \in \mathbb{F}_p^{(n)}\}$ *is a complete set of* $(p-1)p^{n-1}$ *distinct irreducible modules for the algebra* $\mathbb{F}_p[GL_n]$ *and these modules are absolutely irreducible.*

We recall here the definition of the so-called *contragredient module*. Let G be a finite group, $\mathbb{K}$ an arbitrary field and M a left $\mathbb{K}[G]$-module. The contragredient M^* of M is the left $\mathbb{K}[G]$-module in which the underlying vector space is the dual space M^* of M and with the module operation given by

$$(g\phi)(m) = \phi(g^{-1}m)$$

for $g \in G$, $\phi \in M^*$, $m \in M$. The operation is then extended to all $\mathbb{K}[G]$ by linearity. It is easily verified that M^* is irreducible if and only if so is M.

For each $\beta \in \mathbb{F}_p^{(n_1,\dots,n_r)}$, the contragredient module $H^*_\beta(GL_{n_1,\dots,n_r})$ of $H_\beta(GL_{n_1,\dots,n_r})$ is irreducible. Since $\{H_\beta(GL_{n_1,\dots,n_r}) : \beta \in \mathbb{F}_p^{(n_1,\dots,n_r)}\}$ is a complete set of distinct irreducible modules for the algebra $\mathbb{F}_p[GL_{n_1,\dots,n_r}]$, a natural question arising here is *to determine* $\beta^* \in \mathbb{F}_p^{(n_1,\dots,n_r)}$ *so that* $H^*_\beta(GL_{n_1,\dots,n_r})$ *is isomorphic to* $H_{\beta^*}(GL_{n_1,\dots,n_r})$.

In order to state the results, we need the following notations.

Let β be an element of $\mathbb{F}_p^{(n_1,\dots,n_r)}$. For each $1 \le i \le r$, denote by $\beta(i)$ the sequence $(\beta_{N_{i-1}+1},\dots,\beta_{N_i-1}, \overline{\sum_{k=N_i}^{n}\beta_k}) \in \mathbb{F}_p^{(n_i)}$, where $0 \le \overline{h} < p-1$ is the remainder in the division of h by $p-1$.

Consider the correspondence t from $\mathbb{F}_p^{(n_1,\dots,n_r)}$ to $\mathbb{F}_p^{(n_1)} \times \cdots \times \mathbb{F}_p^{(n_r)}$ given by $\beta \mapsto \big(\beta(1),\dots,\beta(i)\big)$. We can easily check that t is a one-to-one correspondence.

For each $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{F}_p^{(k)}$ with k a positive integer, let

$$\gamma^* = \big(\gamma_{k-1}, \gamma_{k-2}, \dots, \gamma_1, \overline{-(\gamma_1 + \cdots + \gamma_k)}\big).$$

We have then

$$(\gamma^*)^* = \left(\gamma_1, \dots, \gamma_{k-1}, \overline{-\big(\gamma_{k-1} + \cdots + \gamma_1 + \overline{-(\gamma_1 + \cdots + \gamma_k)}\big)}\right) = \gamma.$$

For each $\beta \in \mathbb{F}_p^{(n_1,\dots,n_r)}$, define $\beta^* \in \mathbb{F}_p^{(n_1,\dots,n_r)}$ to be the inverse image of $\big(\beta(1)^*, \dots, \beta(r)^*\big)$ under t, i.e.

$$\beta^* = t^{-1}\big((\beta(1)^*, \dots, \beta(r)^*)\big).$$

Since $\big(\beta(i)^*\big)^* = \beta(i)$ for $1 \le i \le r$, it is clear that $(\beta^*)^* = \beta$. We can explicitly express $\beta^* = (\beta_1^*, \dots, \beta_n^*)$ via $\beta = (\beta_1, \dots, \beta_n)$ as follows

$$\beta_i^* = \begin{cases} \beta_{N_k - i} & \text{if } N_{k-1} + 1 \le i < N_k, \\ \overline{-\sum\limits_{j=N_{k-1}+1}^{N_{k+1}-1} \beta_j} & \text{if } i = N_k. \end{cases}$$

We are now ready to state the results.

Theorem A. *Let $H_\beta^*(GL_{n_1,\dots,n_r})$ be the contragredient module of $H_\beta(GL_{n_1,\dots,n_r})$ for $\beta \in \mathbb{F}_p^{(n_1,\dots,n_r)}$. Then*

$$H_\beta^*(GL_{n_1,\dots,n_r}) \cong H_{\beta^*}(GL_{n_1,\dots,n_r})$$

as $\mathbb{F}_p[GL_{n_1,\dots,n_r}]$-modules.

Theorem B. *For every $\beta \in \mathbb{F}_p^{(n_1,\dots,n_r)}$,*

$$\dim_{\mathbb{F}_p} H_\beta(GL_{n_1,\dots,n_r}) = \prod_{i=1}^{r} \dim_{\mathbb{F}_p} H_{\beta(i)}(GL_{n_i}).$$

For $p = 2$ and $n = 4$, we have

Proposition C. *The dimensions of all irreducible $\mathbb{F}_2[GL_4]$-modules are given as follows*

β	$\dim_{\mathbb{F}_2} H_\beta(GL_4)$
$(0,0,0,0)$	1
$(1,0,0,0)$	4
$(0,1,0,0)$	6
$(0,0,1,0)$	4
$(1,1,0,0)$	20
$(1,0,1,0)$	14
$(0,1,1,0)$	20
$(1,1,1,0)$	64

2. Proof of Theorem A

We first recall some facts on the coefficient space of a $\mathbb{K}[G]$-module M. Suppose M is a $\mathbb{K}[G]$-module of finite dimension. Let $\{m_j : j \in I\}$ be a $\mathbb{K}$-basis of M, we have

$$gm_j = \sum_{i\in I} r_{i,j}(g)m_i \tag{2.1}$$

for $g \in G$, $j \in I$, $r_{i,j}(g) \in \mathbb{K}$. The functions $r_{i,j} : G \longrightarrow \mathbb{K}$ are called coefficient functions of V. Denote by $\mathbb{K}^G$ the space of all mappings from G to $\mathbb{K}$. The $\mathbb{K}$-space spanned by coefficient functions is a subspace of $\mathbb{K}^G$, called the coefficient space of M. It is independent of the choice of the basis $\{m_j\}$. We denote this space by $cf(M) = \sum_{i,j} \mathbb{K}r_{i,j}$.

For each $h \in G$, it follows from (2.1) that

$$(h^{-1}gh)m_j = \sum_{i\in I} r_{i,j}(h^{-1}gh)m_i. \tag{2.2}$$

Acting h on the two sides of (2.2), we get

$$g(hm_j) = \sum_{i\in I} r_{i,j}(h^{-1}gh)(hm_i). \tag{2.3}$$

Since $\{m_j : j \in I\}$ is a $\mathbb{K}$-basis of M, so is $\{hm_j : j \in I\}$. Hence (2.3) shows that if $r \in cf(M)$, then $r^h \in cf(M)$, where $r^h(g) = r(h^{-1}gh)$ for each $g \in G$.

Let M^* be the contragredient module of M and $\{m_j^* : j \in I\}$ the dual $\mathbb{K}$-basis of M^* with respect to the basis $\{m_j : j \in I\}$ of M. By the definition of M^*, (2.1) leads to

$$gm_j^* = \sum_{i\in I} r_{j,i}(g^{-1})m_i^*. \tag{2.4}$$

This equation implies that if $r \in cf(M)$, then $r^* \in cf(M^*)$, where $r^*(g) = r(g^{-1})$ for each $g \in G$.

We summarize the above facts in the following lemma.

Lemma 2.1. *Let M be a $\mathbb{K}[G]$-module of finite dimension, M^* its contragredient module and $r \in cf(M)$. Then*

(i) *$r^h \in cf(M)$ for each $h \in G$,*

(ii) *$r^* \in cf(M^*)$,*

where $r^h(g) = r(h^{-1}gh)$ and $r^*(g) = r(g^{-1})$ for each $g \in G$.

The following lemma holds for an algebraically closed field. Actually, it also holds for a splitting field of an algebra.

Lemma 2.2 ([2, 27.8])**.** *Let $\mathbb{K}$ be a splitting field for an algebra A and $\{M_1, \dots, M_k\}$ a set of pairwise non-isomorphic irreducible A-modules with $\dim_{\mathbb{K}} M_r = n_r$, $1 \le r \le k$. For each r, consider a matrix of coefficient functions $\{f_{i,j}^{(r)} : 1 \le i, j \le$*

$n_r\}$ of M_r. Then $\{f^{(r)}_{i,j} : 1 \le i, j \le n_r, 1 \le r \le k\}$ are linearly independent over $\mathbb{K}$.

We introduce some abbreviated notations for minors of matrix. The minor on the rows $k_1, \dots, k_i$ and the columns $j_1, \dots, j_i$ of a matrix B is denoted by

$$B\begin{pmatrix} k_1 & \dots & k_i \\ j_1 & \dots & j_i \end{pmatrix}.$$

The i-th principal minor

$$B\begin{pmatrix} 1 & \dots & i \\ 1 & \dots & i \end{pmatrix}$$

is briefly denoted by $\det_i B$. The following lemma is entirely analogous to a result in [8].

Lemma 2.3 (cf. [8, 2.3]). *Let $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{F}_p^{(n_1,\dots,n_r)}$ and $B \in GL_{n_1,\dots,n_r}$. Denote $\det_\beta(B) = \prod_{i=1}^{n} (\det_i B)^{\beta_i}$. Then $\det_\beta \in cf\big(H_\beta(GL_{n_1,\dots,n_r})\big)$.*

Proof of Theorem A. It follows from Lemma 2.1 and Lemma 2.3 that

$$\det{}^*_\beta \in cf\big(H^*_\beta(GL_{n_1,\dots,n_r})\big)$$

and

$$\det{}^J_{\beta^*} \in cf\big(H_{\beta^*}(GL_{n_1,\dots,n_r})\big),$$

for each $J \in GL_{n_1,\dots,n_r}$. By Lemma 2.2 and Proposition 1.2, the theorem will be proved if we can show that for a suitable choice of J, $\det^*_\beta = \det^J_{\beta^*}$, or equivalently, $\det_\beta(B^{-1}) = \det_{\beta^*}(J^{-1}BJ)$ for each $B \in GL_{n_1,\dots,n_r}$.

For each positive integer m, define the $m \times m$-matrix J_m as follows

$$J_m = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & 0 \end{pmatrix}_{m\times m}.$$

It is easily checked that $J_m^{-1} = J_m$ and

$$(J_m^{-1} A J_m)\begin{pmatrix} 1 & \dots & m-i \\ 1 & \dots & m-i \end{pmatrix} = A\begin{pmatrix} i+1 & \dots & m \\ i+1 & \dots & m \end{pmatrix}$$

for $A \in GL_m$ and $1 \le i \le m$. Exercise 972 of [5] shows that

$$A^{-1}\begin{pmatrix} 1 & \dots & i \\ 1 & \dots & i \end{pmatrix} = \frac{A\begin{pmatrix} i+1 & \dots & m \\ i+1 & \dots & m \end{pmatrix}}{|A|},$$

where $|A|$ is the determinant of A. We have then

$$A^{-1}\begin{pmatrix} 1 & \dots & i \\ 1 & \dots & i \end{pmatrix} = \frac{(J_m^{-1} A J_m)\begin{pmatrix} 1 & \dots & m-i \\ 1 & \dots & m-i \end{pmatrix}}{|A|},$$

or

$$\det_i(A^{-1}) = \frac{\det_{m-i}(J_m^{-1}AJ_m)}{\det_m(J_m^{-1}AJ_m)}\cdot \tag{2.5}$$

For each $\gamma \in \mathbb{F}_p^{(m)}$, we have

$$\gamma^* = (\gamma_{m-1}, \gamma_{m-2}, \dots, \gamma_1, \overline{-(\gamma_1 + \cdots + \gamma_m)}),$$

and therefore

$$\begin{aligned}
\det_\gamma(A^{-1}) &= \prod_{i=1}^{m} \det_i^{\gamma_i}(A^{-1}) \\
&= \prod_{i=1}^{m} \Big(\frac{\det_{m-i}(J_m^{-1}AJ_m)}{\det_m(J_m^{-1}AJ_m)}\Big)^{\gamma_i} \quad \text{(by (2.5))} \\
&= \det_{\gamma^*}(J_m^{-1}AJ_m). \qquad (2.6)
\end{aligned}$$

Let

$$J = \begin{pmatrix} J_{n_1} & & 0 \\ & \ddots & \\ 0 & & J_{n_r} \end{pmatrix} \in GL_{n_1,\dots,n_r}.$$

We prove that J is a matrix satisfying the equality $\det_\beta(B^{-1}) = \det_{\beta^*}(J^{-1}BJ)$ for each $B \in GL_{n_1,\dots,n_r}$.

In fact, for each $B = \begin{pmatrix} B_1 & & * \\ & \ddots & \\ 0 & & B_r \end{pmatrix} \in GL_{n_1,\dots,n_r}$, it is clear that

$$J^{-1}BJ = \begin{pmatrix} J_{n_1}^{-1}B_1J_{n_1} & & * \\ & \ddots & \\ 0 & & J_{n_r}^{-1}B_rJ_{n_r} \end{pmatrix},$$

and hence

$$\begin{aligned}
\det_\beta(B^{-1}) &= \prod_{i=1}^{r} \det_{\beta(i)}(B_i^{-1}) \\
&= \prod_{i=1}^{r} \det_{\beta(i)^*}(J_{n_i}^{-1}B_iJ_{n_i}) \quad \text{(by (2.6))} \\
&= \det_{\beta^*}(J^{-1}BJ).
\end{aligned}$$

The theorem follows. □

Rermark 2.4. (a) By using the same arguments as above, we can prove that *the contravariant module of* $H_\beta(GL_{n_1,\dots,n_r})$ *is isomorphic to* $H_\beta(GL_{n_1,\dots,n_r})$.

The contravariant $H_\beta^0(GL_{n_1,\dots,n_r})$ of $H_\beta(GL_{n_1,\dots,n_r})$ is the left $\mathbb{F}_p[GL_{n_1,\dots,n_r}]$-module in which the underlying vector space is the dual space $H_\beta^*(GL_{n_1,\dots,n_r})$ and with the module operation given by

$$(B\phi)(\ell) = \phi(B^t\ell)$$

for $B \in GL_{n_1,\dots,n_r}$, $\phi \in H^*_\beta(GL_{n_1,\dots,n_r})$, $\ell \in H_\beta(GL_{n_1,\dots,n_r})$ and B^t the transpose of B.

Since $\det_\beta \in cf(H_\beta(GL_{n_1,\dots,n_r}))$, it is similar to Lemma 2.1 that

$$\det{}^0_\beta \in cf\big(H^0_\beta(GL_{n_1,\dots,n_r})\big),$$

where $\det^0_\beta(B) = \det_\beta(B^t)$ for each $B \in GL_{n_1,\dots,n_r}$.

For each $1 \le i \le n$, it is clear that $\det_i(B) = \det_i(B^t)$, and hence $\det_\beta(B) = \det_\beta(B^t)$ for each $B \in GL_{n_1,\dots,n_r}$. This obviously implies that $H^0_\beta(GL_{n_1,\dots,n_r})$ is isomorphic to $H_\beta(GL_{n_1,\dots,n_r})$ as an $\mathbb{F}_p[GL_{n_1,\dots,n_r}]$-module.

(b) For the irreducible modules of the general linear group GL_n, we have

$$H^*_\beta(GL_n) \cong H_{\beta^*}(GL_n)$$

and

$$H^0_\beta(GL_n) \cong H_\beta(GL_n)$$

as $\mathbb{F}_p[GL_n]$-modules, where $\beta^* = \big(\beta_{n-1}, \beta_{n-2}, \dots, \beta_1, \overline{-(\beta_1 + \cdots + \beta_n)}\big)$.

3. Proof of Theorem B

Let G_i $(i = 1, 2)$ be finite groups and $G = G_1 \times G_2$ their direct product. Let M_i be a $\mathbb{K}[G_i]$-module $(i = 1, 2)$. We equip $M_1 \otimes_{\mathbb{K}} M_2$ with a $\mathbb{K}[G]$-module structure by setting

$$(g_1, g_2)(m_1 \otimes_{\mathbb{K}} m_2) = g_1 m_1 \otimes_{\mathbb{K}} g_2 m_2$$

for $g_i \in G_i$, $m_i \in M_i$, $i = 1, 2$. The operation is then extended to all $\mathbb{K}[G]$ by linearity.

Lemma 3.1 ([1, 27.15]). *Let G_i $(i = 1, 2)$ be finite groups and $G = G_1 \times G_2$ their direct product. Let $\{M_j : 1 \le j \le \nu_1\}$ and $\{N_k : 1 \le k \le \nu_2\}$ be respectively the complete sets of distinct irreducible modules for the algebras $\mathbb{K}[G_1]$ and $\mathbb{K}[G_2]$. Assume that $\mathbb{K}$ is a splitting field for $\mathbb{K}[G_i]$ $(i = 1, 2)$. Then*

$$\{M_j \otimes_{\mathbb{K}} N_k : 1 \le j \le \nu_1, 1 \le k \le \nu_2\}$$

is a complete set of $\nu_1\nu_2$ distinct irreducible modules for the algebra $\mathbb{K}[G]$.

Let $GL_{n_1\times\cdots\times n_r}$ be the subgroup of $GL_{n_1,\dots,n_r}$ defined as follows

$$GL_{n_1\times\cdots\times n_r} = \left\{B = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_r \end{pmatrix} \in GL_n : B_i \in GL_{n_i}, 1 \le i \le r\right\}.$$

We identify $GL_{n_1\times\cdots\times n_r}$ with $GL_{n_1} \times \cdots \times GL_{n_r}$ by the group isomorphism given by $B \mapsto (B_1, \dots, B_r)$.

Lemma 3.2. $\{H_\beta(GL_{n_1\times\cdots\times n_r}) : \beta \in \mathbb{F}_p^{(n_1,\dots,n_r)}\}$ *is a complete set of $(p-1)^r p^{n-r}$ distinct irreducible modules for the algebra $\mathbb{F}_p[GL_{n_1\times\cdots\times n_r}]$ and these modules are absolutely irreducible.*

Proof. By Corollary 1.3 and Lemma 3.1, there are exactly $(p-1)^r p^{n-r}$ distinct irreducible modules for the algebra $\mathbb{F}_p[GL_{n_1\times\cdots\times n_r}]$, which is identified with the algebra $\mathbb{F}_p[GL_{n_1}\times\cdots\times GL_{n_r}]$. It is sufficient to prove that the modules $H_\beta(GL_{n_1\times\cdots\times n_r})$, for $\beta\in\mathbb{F}_p^{(n_1,\dots,n_r)}$, are absolutely irreducible and distinct.

For each matrix

$$B=\begin{pmatrix} B_1 & & * \\ & \ddots & \\ 0 & & B_r \end{pmatrix}\in GL_{n_1,\dots,n_r},$$

the matrix $\overline{B}\in GL_{n_1\times\cdots\times n_r}$ is defined as follows

$$\overline{B}=\begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_r \end{pmatrix}.$$

The mapping $B\mapsto\overline{B}$ homomorphically maps $GL_{n_1,\dots,n_r}$ onto $GL_{n_1\times\cdots\times n_r}$. It is clear that $BL_i=\overline{B}L_i$ for $B\in GL_{n_1,\dots,n_r}$, $1\le i\le n$, and hence $BL^\beta=\overline{B}L^\beta$ for each $\beta\in\mathbb{F}_p^{(n_1,\dots,n_r)}$.

Fix an element $\beta\in\mathbb{F}_p^{(n_1,\dots,n_r)}$. We first prove that, as $\mathbb{F}_p$-spaces, $H_\beta(GL_{n_1,\dots,n_r})$ is the same as $H_\beta(GL_{n_1\times\cdots\times n_r})$. Indeed, the generators of the spaces $H_\beta(GL_{n_1,\dots,n_r})$ and $H_\beta(GL_{n_1\times\cdots\times n_r})$ are respectively

$$S=\{BL^\beta : B\in GL_{n_1,\dots,n_r}\}\text{ and }\overline{S}=\{\overline{B}L^\beta : \overline{B}\in GL_{n_1\times\cdots\times n_r}\}.$$

It is clear that $\overline{S}$ is a subset of S. Since $BL^\beta=\overline{B}L^\beta$ for each $B\in GL_{n_1,\dots,n_r}$, it follows that S is a subset of $\overline{S}$. We have then $S=\overline{S}$, which implies that the $\mathbb{F}_p$-spaces $H_\beta(GL_{n_1,\dots,n_r})$ and $H_\beta(GL_{n_1\times\cdots\times n_r})$ are the same. We denote this space by H_β for short. We have an immediate remark that $Bh=\overline{B}h$ for $B\in GL_{n_1,\dots,n_r}$, $h\in H_\beta$.

Let W be an $\mathbb{F}_p[GL_{n_1\times\cdots\times n_r}]$-submodule of H_β. Since $Bw=\overline{B}w$ for $B\in GL_{n_1,\dots,n_r}$, $w\in W$, it follows that $BW=\overline{B}W=W$. Thus W is an $\mathbb{F}_p[GL_{n_1,\dots,n_r}]$-submodule of H_β. Then W is trivial since H_β is irreducible as $\mathbb{F}_p[GL_{n_1,\dots,n_r}]$-module. This establishes the irreducibility of H_β as $\mathbb{F}_p[GL_{n_1\times\cdots\times n_r}]$-module.

Let M, N be irreducible $\mathbb{K}[G]$-modules of finite dimensions. We recall the following elementary facts:

(i) M is absolutely irreducible if and only if $\mathrm{Hom}_{\mathbb{K}[G]}(M,M)=\mathbb{K}$,

(ii) M and N are distinct if and only if $\mathrm{Hom}_{\mathbb{K}[G]}(M,N)=0$.

By the above facts and Proposition 1.2, in order to prove the modules H_β, for $\beta\in\mathbb{F}_p^{(n_1,\dots,n_r)}$, are absolutely irreducible and distinct, it is sufficient to show that

$$\mathrm{Hom}_{\mathbb{F}_p[GL_{n_1\times\cdots\times n_r}]}(H_\beta,H_{\beta'})=\mathrm{Hom}_{\mathbb{F}_p[GL_{n_1,\dots,n_r}]}(H_\beta,H_{\beta'})$$

for $\beta, \beta' \in \mathbb{F}_p^{(n_1,\dots,n_r)}$. However, this equality follows immediately from the fact that $Bh = \overline{B}h$ for each $B \in GL_{n_1,\dots,n_r}$, $h \in H_\beta$ and $\beta \in \mathbb{F}_p^{(n_1,\dots,n_r)}$. The lemma is proved. □

Proof of Theorem B. It follows from Lemma 2.3 that $\det_\beta \in cf\big(H_\beta(GL_{n_1,\dots,n_r})\big)$. Since the $\mathbb{F}_p$-spaces $H_\beta(GL_{n_1,\dots,n_r})$ and $H_\beta(GL_{n_1\times\cdots\times n_r})$ are the same and $\det_\beta(B) = \det_\beta(\overline{B})$ for each $B \in GL_{n_1,\dots,n_r}$, we have $\det_\beta \in cf(H_\beta(GL_{n_1\times\cdots\times n_r}))$.

From Lemma 2.3 it also follows that $\det_{\beta(i)} \in cf\big(H_{\beta(i)}(GL_{n_i})\big)$ for each $1 \le i \le r$. Therefore

$$\prod_{i=1}^{r} \det_{\beta(i)} \in cf\big(H_{\beta(1)}(GL_{n_1}) \otimes_{\mathbb{F}_p} \cdots \otimes_{\mathbb{F}_p} H_{\beta(r)}(GL_{n_r})\big),$$

where

$$(\prod_{i=1}^{r} \det_{\beta(i)})(B) = \prod_{i=1}^{r} \det_{\beta(i)}(B_i)$$

for $B = (B_1, \dots, B_r) \in GL_{n_1\times\cdots\times n_r}$. By the definitions of $\det_\beta$ and $\beta(i)$ we have

$$\det_\beta(B) = (\prod_{i=1}^{r} \det_{\beta(i)})(B).$$

This fact together with Lemmas 2.2, 3.1 and 3.2 imply that

$$H_\beta(GL_{n_1\times\cdots\times n_r}) \cong H_{\beta(1)}(GL_{n_1}) \otimes_{\mathbb{F}_p} \cdots \otimes_{\mathbb{F}_p} H_{\beta(r)}(GL_{n_r})$$

as $\mathbb{F}_p[GL_{n_1\times\cdots\times n_r}]$-modules. As a result,

$$\dim_{\mathbb{F}_p} H_\beta(GL_{n_1,\dots,n_r}) = \dim_{\mathbb{F}_p} H_\beta(GL_{n_1\times\cdots\times n_r}) = \prod_{i=1}^{r} \dim_{\mathbb{F}_p} H_{\beta(i)}(GL_{n_i}).$$

The theorem is proved. □

Remark 3.3. Denote by $R(G)$ the representation ring of a group G. Then it follows easily from the above proof that

$$R(GL_{n_1,\dots,n_r}) \cong R(GL_{n_1}) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} R(GL_{n_r}).$$

4. Proof of Proposition C

For each $\beta \in \mathbb{F}_2^{(n)}$, denote $H_\beta(GL_n)$ by H_β for brevity. We note that

- If $\beta = (0, \dots, 0, \underbrace{1}_{i}, 0, \dots, 0) \in \mathbb{F}_2^{(n)}$, then $\dim_{\mathbb{F}_2} H_\beta = \binom{n}{i}$ by [6, 1.4].
- If $\beta = (1, 1, \dots, 1, 0) \in \mathbb{F}_2^{(n)}$, then H_β has been known to be the Steinberg module for $\mathbb{F}_2[GL_n]$. The dimension of the Steinberg module for $\mathbb{F}_2[GL_n]$ is equal to the order of the Sylow 2-subgroup of GL_n, namely $2^{\frac{n(n-1)}{2}}$.

By the above facts, in order to determine the dimensions of all irreducible $\mathbb{F}_2[GL_4]$-modules, we only need to compute those of $H_{(1,1,0,0)}$, $H_{(1,0,1,0)}$ and $H_{(0,1,1,0)}$. However, Theorem A implies that $\dim_{\mathbb{F}_2} H_{(1,1,0,0)} = \dim_{\mathbb{F}_2} H_{(0,1,1,0)}$, and hence we only deal with $H_{(1,1,0,0)}$ and $H_{(1,0,1,0)}$.

For each $1 \leq k_1 < \cdots < k_i \leq n$, $\sigma \in GL_n$, let $L_{k_1,\ldots,k_i} = L_k(x_{k_1}, \ldots, x_{k_i})$ and $\sigma_{k_1,\ldots,k_i} = \sigma \begin{pmatrix} k_1 & \ldots & k_i \\ 1 & \ldots & i \end{pmatrix}$. The following formula is of basic importance

$$\sigma L_{1,\ldots,i} = \sum_{1 \leq k_1 < \cdots < k_i \leq n} \sigma_{k_1,\ldots,k_i} L_{k_1,\ldots,k_i}.$$

Dimension of $H_{(1,1,0,0)}$. We have $H_{(1,1,0,0)}$ is an $\mathbb{F}_2$-vector space generated by $\{\sigma(L_1 L_{1,2}) : \sigma \in GL_4\}$. For each $\sigma \in GL_4$,

$$\begin{aligned} \sigma(L_1 L_{1,2}) &= \Big(\sum_{1 \leq i \leq 4} \sigma_i L_i\Big)\Big(\sum_{1 \leq j < k \leq 4} \sigma_{j,k} L_{j,k}\Big) \\ &= \sum_{1 \leq i < j \leq 4} T_{i,j} + \sum_{1 \leq i < j < k \leq 4} T_{i,j,k}, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} T_{i,j} &= \sigma_i \sigma_{i,j} L_i L_{i,j} + \sigma_j \sigma_{i,j} L_j L_{i,j}, \\ T_{i,j,k} &= \sigma_i \sigma_{j,k} L_i L_{j,k} + \sigma_j \sigma_{i,k} L_j L_{i,k} + \sigma_k \sigma_{i,j} L_k L_{i,j}. \end{aligned}$$

It is clear that $\sigma_i \sigma_{j,k} + \sigma_j \sigma_{i,k} + \sigma_k \sigma_{i,j} = 0$ and $L_i L_{j,k} + L_j L_{i,k} + L_k L_{i,j} = 0$. We have then

$$\begin{aligned} T_{i,j,k} &= \sigma_i \sigma_{j,k}(L_i L_{j,k} + L_k L_{i,j}) + \sigma_j \sigma_{i,k}(L_k L_{i,j} + L_j L_{i,k}) \\ &= \sigma_i \sigma_{j,k} L_j L_{i,k} + \sigma_j \sigma_{i,k} L_i L_{j,k}. \end{aligned}$$

We also denote by $T_{i,j}$ and $T_{i,j,k}$ the $\mathbb{F}_2$-vector spaces generated by the sets $\{L_i L_{i,j}, L_j L_{i,j}\}$ and $\{L_j L_{i,k}, L_i L_{j,k}\}$, respectively. Let $T_{(1,1,0,0)}$ be the sum of these spaces. Note that if $f \in T_{i,j}$, $g \in T_{i,j,k}$, then

$$\begin{aligned} f &= x_i x_j f_1(x_i, x_j), \\ g &= x_i x_j x_k g_1(x_i, x_j, x_k). \end{aligned}$$

Therefore $T_{(1,1,0,0)}$ is the direct sum of all spaces $T_{i,j}$ and $T_{i,j,k}$,

$$T_{(1,1,0,0)} = \bigoplus_{1 \leq i < j \leq 4} T_{i,j} \oplus \bigoplus_{1 \leq i < j < k \leq 4} T_{i,j,k}. \tag{4.2}$$

It is easy to verify that the sets $\{L_i L_{i,j},\, L_j L_{i,j}\}$ and $\{L_j L_{i,k},\, L_i L_{j,k}\}$ are linearly independent over $\mathbb{F}_2$. Hence, from (4.2), the dimension of $T_{(1,1,0,0)}$ is

$$\dim_{\mathbb{F}_2} T_{(1,1,0,0)} = 2.\binom{4}{2} + 2.\binom{4}{3} = 20.$$

We prove that $H_{(1,1,0,0)} = T_{(1,1,0,0)}$, and therefore $\dim_{\mathbb{F}_2} H_{(1,1,0,0)} = 20$. From (4.1), it follows that $H_{(1,1,0,0)} \subset T_{(1,1,0,0)}$. In order to show $T_{(1,1,0,0)} \subset H_{(1,1,0,0)}$, it suffices to prove that $H_{(1,1,0,0)}$ contains the sets $\{L_i L_{i,j}, L_j L_{i,j}\}$ and $\{L_j L_{i,k},$

$L_iL_{j,k}\}$ for $1 \le i < j \le 4$ and $1 \le i < j < k \le 4$. We will prove the cases where $(i,j) = (1,2)$ and $(i,j,k) = (1,2,3)$.

Let

$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \tau_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \tau_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\tau_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \tau_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \tau_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have then

$$\begin{aligned} L_1L_{1,2} &= \tau_1(L_1L_{1,2}), \\ L_2L_{1,2} &= \tau_2(L_1L_{1,2}), \\ L_1L_{2,3} &= \tau_2(L_1L_{1,2}) + \tau_3(L_1L_{1,2}) + \tau_5(L_1L_{1,2}), \\ L_2L_{1,3} &= \tau_1(L_1L_{1,2}) + \tau_4(L_1L_{1,2}) + \tau_6(L_1L_{1,2}). \end{aligned}$$

Since $H_{(1,1,0,0)}$ is the $\mathbb{F}_2$-vector space generated by $\{\sigma(L_1L_{1,2}) : \sigma \in GL_4\}$, it follows from the above equations that $\{L_1L_{1,2}, L_2L_{1,2}\}$ and $\{L_1L_{2,3}, L_2L_{1,3}\}$ are contained in $H_{(1,1,0,0)}$. □

Dimension of $H_{(1,0,1,0)}$. $H_{(1,0,1,0)}$ is an $\mathbb{F}_2$-vector space generated by $\{\sigma(L_1L_{1,2,3}) : \sigma \in GL_4\}$. For each $\sigma \in GL_4$,

$$\begin{aligned} \sigma(L_1L_{1,2,3}) &= \Big(\sum_{1\le i\le 4} \sigma_i L_i\Big)\Big(\sum_{1\le j<k<l\le 4} \sigma_{j,k,l} L_{j,k,l}\Big) \\ &= \sum_{1\le i<j<k\le 4} T_{i,j,k} + T_{1,2,3,4}, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} T_{i,j,k} &= \sigma_i\sigma_{i,j,k}L_iL_{i,j,k} + \sigma_j\sigma_{i,j,k}L_iL_{i,j,k} + \sigma_k\sigma_{i,j,k}L_kL_{i,j,k}, \\ T_{1,2,3,4} &= \sigma_1\sigma_{2,3,4}L_1L_{2,3,4} + \sigma_2\sigma_{1,3,4}L_2L_{1,3,4} \\ &\quad + \sigma_3\sigma_{1,2,4}L_3L_{1,2,4} + \sigma_4\sigma_{1,2,3}L_4L_{1,2,3}. \end{aligned}$$

Since

$$\sigma_1\sigma_{2,3,4} + \sigma_2\sigma_{1,3,4} + \sigma_3\sigma_{1,2,4} + \sigma_4\sigma_{1,2,3} = 0$$

and

$$L_1L_{2,3,4} + L_2L_{1,3,4} + L_3L_{1,2,4} + L_4L_{1,2,3} = 0,$$

we have

$$\begin{aligned} T_{1,2,3,4} = \ &(\sigma_1\sigma_{2,3,4} + \sigma_3\sigma_{1,2,4})(L_1L_{2,3,4} + L_4L_{1,2,3}) \\ &+ (\sigma_2\sigma_{1,3,4} + \sigma_3\sigma_{1,2,4})(L_2L_{1,3,4} + L_4L_{1,2,3}). \end{aligned}$$

We also denote by $T_{i,j,k}$ and $T_{1,2,3,4}$ the $\mathbb{F}_2$-vector spaces generated by the sets $\{L_iL_{i,j,k},\ L_jL_{i,j,k},\ L_kL_{i,j,k}\}$ and $\{L_1L_{2,3,4}+L_4L_{1,2,3}, L_2L_{1,3,4}+L_4L_{1,2,3}\}$, respectively. Let $T_{(1,0,1,0)}$ be the sum of these spaces. It is clear that

$$T_{(1,0,1,0)} = \bigoplus_{1\leq i<j<k\leq 4} T_{i,j,k} \oplus T_{1,2,3,4}. \tag{4.4}$$

Since the sets $\{L_iL_{i,j,k},\ L_jL_{i,j,k},\ L_kL_{i,j,k}\}$ and $\{L_1L_{2,3,4} + L_4L_{1,2,3},\ L_2L_{1,3,4} + L_4L_{1,2,3}\}$ are linearly independent over $\mathbb{F}_2$, it follows from (4.4) that

$$\dim_{\mathbb{F}_2} T_{(1,1,0,0)} = 3.\binom{4}{3} + 2 = 14.$$

We finally prove that $H_{(1,0,1,0)} = T_{(1,0,1,0)}$, and hence $\dim_{\mathbb{F}_2} H_{(1,1,0,0)} = 14$. It is sufficient to show that the sets $\{L_iL_{i,j,k}, L_jL_{i,j,k}, L_kL_{i,j,k}\}$ and $\{L_1L_{2,3,4} + L_4L_{1,2,3}, L_2L_{1,3,4}+L_4L_{1,2,3}\}$ are contained in $H_{(1,0,1,0)}$. We only consider the case $(i,j,k) = (1,2,3)$.

Let

$$\tau_1 = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ \tau_2 = \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ \tau_3 = \begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix},$$

$$\tau_4 = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\ \tau_5 = \begin{pmatrix}1&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\ \tau_6 = \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\1&0&0&1\end{pmatrix}.$$

We have

$$\begin{aligned}
L_1L_{1,2,3} &= \tau_1(L_1L_{1,2,3}),\\
L_2L_{1,2,3} &= \tau_2(L_1L_{1,2,3}),\\
L_3L_{1,2,3} &= \tau_3(L_1L_{1,2,3}),\\
L_1L_{2,3,4}+L_4L_{1,2,3} &= \tau_1(L_1L_{1,2,3}) + \tau_4(L_1L_{1,2,3}) + \tau_5(L_1L_{1,2,3}),\\
L_2L_{1,3,4}+L_4L_{1,2,3} &= \tau_2(L_1L_{1,2,3}) + \tau_4(L_1L_{1,2,3}) + \tau_6(L_1L_{1,2,3}).
\end{aligned}$$

Since $H_{(1,0,1,0)}$ is the $\mathbb{F}_2$-vector space generated by $\{\sigma(L_1L_{1,2,3}) : \sigma \in GL_4\}$, it follows from the above equations that $H_{(1,0,1,0)}$ contains the sets $\{L_1L_{1,2,3}, L_2L_{1,2,3}, L_3L_{1,2,3}\}$ and $\{L_1L_{2,3,4} + L_4L_{1,2,3}, L_2L_{2,3,4} + L_4L_{1,2,3}\}$.

The proposition is proved. □

Acknowledgement

We would like to thank Professor Nguyen Huu Viet Hung and Professor Pham Anh Minh for useful references and helpful advices in completing this paper.

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Department of Mathematics
Hue College of Sciences
Hue, Vietnam

E-mail address: hohai.kh@hueuni.edu.vn