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ON THE WEAK LAW OF LARGE NUMBERS FOR ADAPTED SEQUENCES IN VON NEUMANN ALGEBRA

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Abstract. We investigate the weak law of large numbers for adapted sequences of measurable operators. As corollaries of the main theorem we obtain non-commutative versions of some related results presented in the monograph "Martingale limit theory and its application" of P. Hall and C. C. Heyde.

1. Introduction and notations

The weak law of large numbers in von Neumann algebra was considered by some authors. This law was proved for martingale difference and quadratic forms in [7]. Some other results for independent sequences of measurable operators would be found in [1] and [4].

The aim of this note is to give the weak law of large numbers for adapted sequence of measurable operators. As corollaries we get non-commutative versions of some known results for real-valued martingale difference.

Let us begin with some definitions and notations. Throughout this note, let $\mathcal{A}$ denote a von Neumann algebra with faithfull normal tracial state τ and $\tilde{\mathcal{A}}$ denote the algebra of measurable operators in Segal-Nelson's sense (see [6]). For every fixed real number $r \geq 1$ one can define the Banach space $L^r(\mathcal{A}, \tau)$ of (possibly unbounded) operators as the non-commutative analogue of the Lebesgue spaces of r^{th}-intergrable random variables (see [6]). If $\mathcal{B}$ is a von Neumann subalgebra of $\mathcal{A}$ then $L^r(\mathcal{B}, \tau) \subset L^r(\mathcal{A}, \tau)$ for all $r \geq 1$. Umegaki ([9]) defined the conditional expectation $E^{\mathcal{B}} : L^1(\mathcal{A}, \tau) \to L^1(\mathcal{B}, \tau)$ by

$$\tau(XY) = \tau((E^{\mathcal{B}}X)Y), \quad X \in \mathcal{A}, Y \in \mathcal{B}. \tag{1.1}$$

Then $E^{\mathcal{B}}$ is a positive linear mapping of norm one and uniquely defined by (1.1). Moreover, the restriction of $E^{\mathcal{B}}$ to the Hilbert space $L^2(\mathcal{A}, \tau)$ is an orthogonal projection from $L^2(\mathcal{A}, \tau)$ onto $L^2(\mathcal{B}, \tau)$.

Now let $(\mathcal{A}_n)$ be an increasing sequence of von Neumann subalgebras of $\mathcal{A}$. A sequence (X_n) of measurable operators is said to be adapted to $(\mathcal{A}_n)$ if for all $n \in N$, $X_n \in \tilde{\mathcal{A}}_n$,. Note that if (X_n) is an arbitrary sequence of measurable operators in $\tilde{\mathcal{A}}$ and $\mathcal{A}_n = W(X_1, X_2, \cdots, X_n)$ (the von Neumann subalgebra generated by $X_1, X_2, \cdots, X_n$) then (X_n) is the sequence adapted to the sequence $(\mathcal{A}_n)$.

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A sequence $(X_n, \mathcal{A}_n)$ is said to be a martingale if for all $n \in N$ we have (i) $X_n \in L^1(\mathcal{A}_n, \tau)$ and (ii) $E^{\mathcal{A}n} X_{n+1} = X_n$.

If a sequence $(X_n, \mathcal{A}_n)$ satisfies the conditions (i) $X_n \in L^1(\mathcal{A}_n, \tau)$ and (ii') $E^{\mathcal{A}_n} X_{n+1} = 0$, then it is said to be a martingale difference.

For further information about the theory of probability in von Neumann algebras we refer to [1], [5] and [8].

2. Results

The main aim of this note is to prove the following result.

Theorem 2.1. *Let $(\mathcal{A}_n)$ be an increasing sequence of von Neumann subalgebras of $\mathcal{A}$; $(S_n = \sum_{i=1}^{n} X_i)$ a sequence of measurable operators adapted to $(\mathcal{A}_n)$ and (b_n) a sequence of positive numbers with $b_n \uparrow \infty$ as $n \to \infty$. Then, writing $X_{n_i} = X_i e_{[0,b_n]}(|X_i|)$ $(1 \le i \le n)$, we have*

$$\frac{1}{b_n} S_n \xrightarrow{\tau} 0 \ \ \text{as} \ \ n \to \infty; \tag{2.1}$$

if

$$\sum_{i=1}^{n} \tau(e_{(b_n,\infty)}(|X_i|)) \to 0 \ \ \text{as} \ \ n \to \infty; \tag{2.2}$$

$$\frac{1}{b_n} \sum_{i=1}^{n} E^{\mathcal{A}_{i-1}} X_{n_i} \xrightarrow{\tau} 0 \ \ \text{as} \ \ n \to \infty \tag{2.3}$$

and

$$\frac{1}{b_n^2} \sum_{i=1}^{n} \left\{ \tau |X_{n_i}|^2 - \tau \left| E^{\mathcal{A}_{i-1}} X_{n_i} \right|^2 \right\} \to 0 \ \ \text{as} \ \ n \to \infty. \tag{2.4}$$

Proof. Put

$$S_{nn} = \sum_{i=1}^{n} X_{n_i}, \qquad m_{nn} = \sum_{i=1}^{n} E^{\mathcal{A}_{i-1}} X_{n_i}.$$

Suppose that $\mathcal{A}$ acts in a Hilbert space H. For an arbitrary $\gamma > 0$, we have

$$p = e_{[2\gamma,\infty)}(|S_n - m_{nn}|) \wedge_{[0,\gamma)} (|S_{nn} - m_{nn}|) \wedge (\bigwedge_{i=0}^{n} e_{[0,b_n]}(|X_n|)) = 0.$$

Indeed, if, for some h of norm one, $h \in p(H)$, then $h \in e_{[0,b_n]}(|X_n|)(H)$ and, consequently

$$X_{ni}(h) = X_i(h) \quad i = 1, 2 \cdots n.$$

which yields $S_n(h) = S_{nn}(h)$ and

$$\begin{aligned}2\gamma = 2\gamma\|h\| &\le \||S_n - m_{nn}|e_{[2\gamma,\infty)}(|S_n - m_{nn}|)(h)\| \\ &= \||S_n - m_{nn}|(h)\| = \|(S_n - m_{nn})(h)\| \\ &\le \|(S_n - S_{nn})(h)\| + \|(S_{nn} - m_{nn})(h)\| \\ &= \||S_{nn} - m_{nn}|e_{[0,\gamma)}(|S_{nn} - m_{nn}|)(h)\| \le \gamma\|h\| = \gamma.\end{aligned}$$

It is impossible; so $p = 0$ and this implies

$$e_{[2\gamma,\infty)}(|S_n - m_{nn}|) \prec e_{[\gamma,\infty)}(|S_{nn} - m_{nn}|) \vee (\bigvee_{i=1}^{n} e_{(b_n,\infty)}(|X_i|)).$$

From the positivity of trace τ, we obtain

$$\tau\big(e_{[2\gamma,\infty)}(|S_n - m_{nn}|)\big) \le \tau\big(e_{[\gamma,\infty)}(|S_{nn} - m_{nn}|)\big) + \sum_{i=1}^{n} \tau\big(e_{(b_n,\infty)}(|X_i|)\big).$$

By the Tchebyshev inequality we get

$$\tau(e_{[\gamma,\infty)}(|S_{nn} - m_{nn}|)) \le \gamma^{-2}\tau(|S_{nn} - m_{nn}|^2). \tag{2.5}$$

Since the sequence $(S_n = \sum_{i=1}^{n} X_i)$ is adapted to the sequence $(\mathcal{A}_n)$, the sequence (X_n) is also adapted to the sequence $(\mathcal{A}_n)$. From the definition of conditional expectation, we have that the elements $X_{n_i} - EX_{n_i}^{\mathcal{A}_{i-1}}$ $(i = \overline{1,n})$ are the pairwise orthogonal elements in the Hilbert space $L^2(\mathcal{A},\tau)$ and $E_{(.)}^{\mathcal{A}_{i-1}}$ $(i = \overline{1,n})$ are the orthogonal projections on $L^2(\mathcal{A},\tau)$. It follows that

$$\begin{aligned}\tau\big(|S_{nn} - m_{nn}|^2\big) &= \tau\big(|\sum_{i=1}^{n}(X_{n_i} - E^{\mathcal{A}_{i-1}}X_{n_i})|^2\big) = \\ &= \sum_{i=1}^{n} \tau\big(|X_{n_i} - E^{\mathcal{A}_{i-1}}X_{n_i}|^2\big) \\ &= \sum_{i=1}^{n}(\tau|X_{n_i}|^2 - \tau|E^{\mathcal{A}_{i-1}}X_{n_i}|^2).\end{aligned} \tag{2.6}$$

Now, for given $\varepsilon > 0$, we put $\gamma = \dfrac{b_n\varepsilon}{2}$, then from (2.5), (2.6), (2.2) and (2.4) we get

$$\begin{aligned}\tau\Big(e_{[\varepsilon,\infty)}\big(|\frac{S_n - m_{nn}}{b_n}|\big)\Big) &= \tau\Big(e_{[b_n\epsilon,\infty)}\big(|S_n - m_{nn}|\big)\Big) \\ &\le \frac{4}{b_n^2\varepsilon}\sum_{i=1}^{n}(\tau|X_{ni}|^2 - \tau|E^{\mathcal{A}_{i-1}}X_{n_i}|^2) \\ &\quad + \sum_{i=1}^{n} \tau\big(e_{(b_n,\infty)}(|X_i|)\big) \to 0\end{aligned}$$

as $n \to \infty$.

It folows that

$$\frac{S_n - m_{nn}}{b_n} \xrightarrow{\tau} 0 \text{ as } n \to \infty$$

which together with (2.3) yields (2.1) and the proof of the theorem is complete. $\square$

The following example shows that, in the particular case, when (X_i) is a sequence of random variables, then the above theorem is stronger than Theorem 2.13 in [2] which considered the same problem under the assumption: $(S_n = \sum\limits_{i=1}^{n} X_i, \ ; \ \mathcal{A}_n)$ is a martingale. Let (Y_i) be a sequence of independent and identically distributed random variables such that

$$P(Y_i = -1) = P(Y_i = 1) = \frac{1}{2}.$$

Then $EY_i = 0$ ($\forall i = 1, 2, \cdots$) and

$$\frac{1}{n}\sum_{i=1}^{n} Y_i \xrightarrow{P} 0 \text{ as } n \to \infty.$$

Put

$$X_i = Y_i + \frac{1}{i} \ .$$

Then $EX_i = \dfrac{1}{i}$ $(\forall i = 1, 2, \cdots)$ and

$$\frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n}\sum_{i=1}^{n} Y_i + \frac{1}{n}\sum_{i=1}^{n}\frac{1}{i} \xrightarrow{P} 0 + 0 = 0 \text{ as } n \to \infty.$$

Thus, $(S_n = \sum\limits_{i=1}^{n} X_i)$ satisfies the condition (2.1) and it also satisfies the conditions (2.2), (2.3), (2.4) (with $b_n = n$). (Because (X_i) is the sequence of independent random variables and in this case, the conditions (2.2), (2.3), (2.4) are necessary as well as sufficient for the condition (2.1) (see [3], pp. 290)). But $(S_n = \sum\limits_{i=1}^{n} X_i, \ ; \ \mathcal{A}_n)$ is not a martingale. ($\mathcal{A}_n$ denotes the σ- algebra generated by $(X_i; \ 1 \leq i \leq n)$). This shows that the martingale condition of $(S_n = \sum\limits_{i=1}^{n} X_i, \ ; \ \mathcal{A}_n)$ in Theorem 2.13 of [2] is too strong.

Let $(X_n) \subset \tilde{\mathcal{A}}, \ X \in \tilde{\mathcal{A}}$. If there exists a constant $C > 0$ such that for all $\lambda > 0$ and all $n \in N$

$$\tau\big(e_{[\lambda,\infty)}(|X_n|)\big) \leq C\tau(e_{[\lambda,\infty}(|X|))$$

then we write $(X_n) \prec X$. With some additional condition we get the following corollaries which can be considered as non-commutative versions of the related results, given in [2].

Corollary 2.1. *If $(X_n, \mathcal{A}_n)$ is a martingale difference such that $(X_n) \prec X$ and $\tau(|X|) < \infty$, then*

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\tau} 0. \tag{2.7}$$

Proof. Put $X_{n_i} = X_i e_{[0,n]}(|X_i|),\ 1 \le i \le n$. From the assumption we have

$$\sum_{i=1}^{n} \tau\big(e_{[n,\infty)}(|X_i|)\big) \le Cn\tau(e_{[n,\infty)}(|X|)) \to 0, \tag{2.8}$$

$$\begin{aligned}
\tau\Big|\frac{1}{n}\sum_{i=1}^{n} E^{\mathcal{A}_{i-1}} X_{ni}\Big| &= \tau\Big|\frac{1}{n}\sum_{i=1}^{n} E^{\mathcal{A}_{i-1}} X_i - X_i e_{(n,\infty)}(|X_i|))\Big| \\
&= \tau\Big|\frac{1}{n}\sum_{i=1}^{n} E^{\mathcal{A}_{i-1}} X_n e_{(n,\infty)}(|X_i|))\Big| \\
&\le \frac{1}{n}\sum_{i=1}^{n} \tau|X_i e_{[n,\infty)}| \\
&= \frac{1}{n}\sum_{i=1}^{n} \int\limits_{\lambda\in(n,\infty)} \lambda d\tau(e_{[0,\lambda)}(|X_i|)) \ \to 0
\end{aligned}$$

as $n \to \infty$. Because

$$\tau(|X_i|) = \int\limits_R \tau(e_{[t,\infty)}(|X_i|))dt \le \int\limits_R C\tau(e_{[t,\infty)}(|X_i|))dt \le C\tau(|X|) < \infty.$$

This implies

$$\frac{1}{n}\sum_{i=1}^{n} E^{\mathcal{A}_{i-1}} X_{n_i} \ \to 0 \tag{2.9}$$

in $L^1(\mathcal{A}, \tau)$ and so in measure as $n \to \infty$.

At the end we have

$$\begin{aligned}
\frac{1}{n^2}\sum_{i=1}^{n}(\tau|X_{n_i}|^2 - \tau|E^{\mathcal{A}_{i-1}}|^2) &\le \frac{1}{n^2}\sum_{i=1}^{n}\tau|X_{n_i}|^2 \\
&= \frac{1}{n^2}\sum_{i=1}^{n} 2 \int\limits_{\lambda\in(0,n]} \lambda\tau(e_{[\lambda,\infty)}(|X_i|))d\lambda \\
&\le \frac{2C}{n} \int\limits_{\lambda\in(0,n]} \lambda\tau(e_{[\lambda,\infty)}(|X|))d\lambda \to 0
\end{aligned} \tag{2.10}$$

as $n \to \infty$

Combining (2.8), (2.9) and (2.10) we get (2.7). $\square$

Corollary 2.2. *Let* $(\mathcal{A}_n)$ *be an increasing sequence of von Neumann subalgebras of* $\mathcal{A}$, $(S_n = \sum\limits_{i=1}^{n} X_i)$ *a sequence of measurable operators adapted to* $(\mathcal{A}_n)$ *such that* $(X_n) \prec X$ *and* $\tau(|X|) < \infty$. *Then*

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - E^{\mathcal{A}_{i-1}}X_i) \xrightarrow{\tau} 0.$$

Proof. Put

$$Y_i = X_i - E^{\mathcal{A}_{i-1}}X_i\ ;\ \ Y = X.$$

It is easy to see that $(Y_n, \mathcal{A}_n)$ and Y satisfy all the assumptions of Corollary 2.1. So the desired conclution follows from Corollary 2.1. □

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