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MINIMAX THEOREMS REVISITED

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Abstract. Very general conditions are established that ensure the existence of a saddle-value for a function $F(x, y) : C \times D \to \mathbb{R}$, where C, D are subsets of two topological spaces X, Y, respectively. These conditions are much weaker than those generally required in the literature. As consequences, several minimax theorems are obtained that include as special cases refinements of various minimax theorems developed recently in nonlinear analysis and optimization for quasiconvex quasiconcave functions. Despite the generality of the results, the proof is very simple and is independent of separation or fixed point arguments on which most best known minimax theorems are based.

1. Introduction

Let X, Y be two topological spaces. Given two sets $C \subset X$, $D \subset Y$ and a function $F(x, y) : C \times D \to \mathbb{R}$, we define

$$\eta := \sup_{y \in D} \inf_{x \in C} F(x, y), \quad \gamma := \inf_{x \in C} \sup_{y \in D} F(x, y). \tag{1}$$

We say that the function $F(x, y)$ possesses a *saddle-value* on $C \times D$ if $\eta = \gamma$, i.e.

$$\sup_{y \in D} \inf_{x \in C} F(x, y) = \inf_{x \in C} \sup_{y \in D} F(x, y). \tag{2}$$

Investigations on the existence of a saddle-value for a given function $F(x, y)$ date back to von Neumann, in the context of game theory. A classical result of von Neumann, later improved by Kneser [7], states that a saddle-value exists if C, D are compact convex subsets of $X = \mathbb{R}^n$, and $Y = \mathbb{R}^m$, respectively, while the function $F(x, y)$ is continuous convex in x and continuous concave in y. Since minimax theorems have found important applications in different fields of mathematics, there has been afterwards a great deal of work on generalizing von Neumann's theorem. Especially, much effort has been spent on relaxing the assumptions on convexity-concavity of $F(x, y)$ and also compactness of both C, D. The best known result in this direction is due to Sion [14], who replaced the convexity-concavity of $F(x, y)$ by quasiconvexity-concavity and relaxed its continuity while dropping the compactness condition for one of the sets C, D. Subsequently, Wu [20] established a minimax theorem in topological spaces, using for the proof of

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his results a one-dimensional form of Helly's theorem rather than separation or fixed point arguments as usual. However, Wu's theorem did not contain some important results such as Sion's or Nikaido's [9] theorems. In 1974, following a different approach, a general minimax theorem in the same vein as Wu's theorem was established by the author of the present paper [15], [16] (see also [18], [11], [17]), without using any separation or fixed point argument. Although this theorem did contain Wu's as well as Sion's and Nikaido's results as special cases, the compactness condition required for at least one of the sets C, D turned out to be too restrictive for recent developments of mathematical programming and nonlinear analysis (see e.g. Rockafellar [12], [13], Golshtein [4], Ekeland-Temam [2], Aubin-Ekeland [1]). To cover the cases considered in the just mentioned works, weaker conditions than compactness of one of the sets C, D are required, while convexity-concavity of $F(x, y)$ is still imposed to ensure the existence of a saddle-value. Meanwhile, further topological minimax theorems have also been developed [8], [10], [5],..., in which convexity-concavity is relaxed to some kind of connectedness, but again compactness is required for at least one of the sets C, D as in Sion's theorem. To our knowledge, this compactness assumption is essential in almost all existing general topological minimax theorems. Furthermore, from a conceptual point of view, all the mentioned minimax theorems look somewhat disparate, and so far little work has been done on clarifying the relationship between different existence conditions formulated in these theorems.

The purpose of the present paper is to show that the earlier approach in [15] can be improved to produce stronger versions of minimax theorems which unify and include as special cases refinements of various recent minimax results by simultaneously weakening compactness assumption, convexity-concavity and continuity assumptions. Despite the generality of the new results, the proof is simple, making use only of elementary facts from analysis independent of separation or fixed point theories. As an application, we shall briefly indicate how general duality results and optimality conditions for modern mathematical programming can be derived in a simple way from the general minimax theorem.

2. The basic lemma

For every real number α and every point $x \in C$ we set

$$D_\alpha(x) = \{y \in D |\ F(x, y) \geq \alpha\}.$$

We say that a function $F(x, y)$ is *α-connected* on $C \times D$ if

1) For any nonempty finite set $M \subset C$ the set $D_\alpha^M = \bigcap_{x \in M} D_\alpha(x)$ is connected and closed.

2) For any pair $a, b \in C$ there exists a continuous mapping $u_{ab} : [0, 1] \to C$ such that $u_{ab}(0) = a, u_{ab}(1) = b$, while the function $\lambda \mapsto F(x_\lambda, y)$, with $x_\lambda = u_{ab}(\lambda)$, satisfies

$$F(x_\lambda, y) \leq \max\{F(x_\mu, y), F(x_\nu, y)\} \text{ whenever } 0 \leq \mu \leq \lambda \leq \nu \leq 1. \tag{3}$$

For example, it can easily be proved that if C, D are closed convex subsets of two topological vector spaces X, Y respectively, then any function $F(x, y)$: $C\times D \to \mathbb{R}$ which is quasiconvex in x and quasiconcave and upper semi-continuous in y is α-connected for every $\alpha \in \mathbb{R}$. Note that the above defined concept of α-connectedness is not quite symmetric with respect to x, y, as we would wish from a purely aesthetic viewpoint. This absence of symmetry is to reflect the "skew" symmetry underlying the minimax concept itself (see the equality (8) below).

The following lemma is fundamental for deriving all minimax theorems to be discussed in subsequent sections.

Lemma 1. *Assume that $F(x, y)$ is α-connected for every $\alpha \in (\alpha_0, \gamma)$, with $-\infty < \alpha_0 < \gamma := \inf_{x\in C} \sup_{y\in D} F(x, y)$, and that either of the following conditions holds:*

(A) *For every fixed $y \in D$ and $a, b \in C$, the function $\lambda \mapsto F(u_{ab}(\lambda), y)$ is lower semi-continuous (l.s.c.) in λ in the interval $0 \le \lambda \le 1$;*

(B) *D is a closed compact set while for every fixed $y \in D$ and $a, b \in C$, the function $\lambda \mapsto F(u_{ab}(\lambda), y)$ is upper semi-continuous (u.s.c.) in λ in the interval $0 \le \lambda \le 1$.*

Then for every nonempty finite set $M \subset C$ and every $\alpha \in (\alpha_0, \gamma)$ we have

$$\cap_{x\in M}\{y \in D|\ F(x, y) \ge \alpha\} \ne \emptyset.$$

Proof. This proposition was established some thirty years ago in [15] (Lemma 3-3') and [16] (Lemma 2.1), under somewhat stronger assumptions (lower and upper semi-continuity of $x \mapsto F(x, y)$ were assumed in condition (A), and (B), respectively). It turns out, however, that the proof given in [15] and [16] remains valid without any modification for the present Lemma 1. Below we present this proof.

We first prove the proposition for $|M| = 2$. Consider any $\alpha \in (\alpha_0, \gamma)$ and for every $x \in C$ let

$$D(x) := \{y \in D|\ F(x, y) \ge \alpha\}.$$

(For simplicity we omit the subscript α and write $D(x)$ instead of $D_\alpha(x)$). Since $\gamma > \alpha$, clearly $\sup_{y\in D} F(x, y) > \alpha\ \forall x \in C$ and it follows from the assumptions on $F(x, y)$, that every set $D(x)$, $x \in C$, is nonempty, and closed.

Arguing by contradiction, assume there are $a, b \in C$ such that

$$D(a) \cap D(b) = \emptyset. \tag{4}$$

Let $u_{ab}(\lambda) : [0, 1] \to C$ be the continuous mapping mentioned in the definition of α-connectedness, and let $x_\lambda = u_{ab}(\lambda)$. For every $\lambda \in [0, 1]$, since by (3) $D(x_\lambda) \subset D(a) \cup D(b)$, if the set $D(x_\lambda)$ meets simultaneously $D(a)$ and $D(b)$ we would have $D(x_\lambda) = E_a \cup E_b$ where $E_a = D(a) \cap D(x_\lambda)$ and $E_b = D(b) \cap D(x_\lambda)$ are two closed, nonempty and disjoint sets, contradicting the connectedness of $D(x_\lambda)$). Consequently, for every $\lambda \in [0, 1]$ one and only one of the following alternatives holds:

$$\text{(a) } D(x_\lambda) \subset D(a); \qquad \text{(b) } D(x_\lambda) \subset D(b).$$

Denote by $M_a(M_b$, respectively) the set of all $\lambda \in [0,1]$ satisfying (a) (satisfying (b), respectively). Clearly $0 \in M_a, 1 \in M_b, M_a \cup M_b = [0,1]$ and, analogously to (3):

$$D(x_\lambda) \subset D(x_{\lambda_1}) \cup D(x_{\lambda_2}) \quad \forall \lambda \in [\lambda_1, \lambda_2] \tag{5}$$

Therefore, $\lambda \in M_a$ implies $[0, \lambda] \subset M_a$, and $\lambda \in M_b$ implies $[\lambda, 1] \subset M_b$. Let $s = \sup M_a = \inf M_b$ and assume for instance that $s \in M_a$ (the argument is similar if $s \in M_b$). We show that (4) leads to a contradiction.

We cannot have $s = 1$, for this would imply $D(b) \subset D(a)$. Therefore, $0 \le s < 1$. Since $\alpha < \gamma \le \sup\limits_{y \in D} F(x_s, y)$, it follows that $F(x_s, \bar{y}) > \alpha$ for some $\bar{y} \in D$. If assumption (A) holds, so that $F(x_\lambda, y)$ is l.s.c. in λ, then there is $\varepsilon > 0$ such that $F(x_{s+\varepsilon}, \bar{y}) > \alpha$ and so $\bar{y} \in D(x_{s+\varepsilon})$. But $\bar{y} \in D(x_s) \subset D(a)$, hence $D(x_{s+\varepsilon}) \subset D(a)$, i.e. $s + \varepsilon \in M_a$, contradicting the definition of s. Thus (4) cannot occur if assumption (A) holds. On the other hand, if assumption (B) holds, so that $F(x_\lambda, y)$ is u.s.c. in λ while D is compact, then for every $y \in D(b)$, since $s \in M_a$, i.e. $D(x_s) \subset D(a)$, we have $y \notin D(x_s)$, hence $F(x_s, y) < \alpha$, and by the u.s.c. of $F(x_\lambda, y)$ in λ there exists an open interval $I_y = (s_1, s_2)$ containing s ($s_1 = s_1(y), s_2 = s_2(y)$) such that $F(x_\lambda, y) < \alpha$ for all $\lambda \in I_y$. Then $F(x_{s_i}, y) < \alpha$, i.e. $y \notin D(x_{s_i})$, $i = 1, 2$ and using the closedness of the sets $D(x_{s_i}), i = 1, 2$ we can find for each $i = 1, 2$ a neighbourhood $W_i(y)$ of y such that $F(x_{s_i}, z) < \alpha \ \forall z \in W_i(y)$. Clearly $W_y = W_1(y) \cap W_2(y)$ is a neighbourhood of y such that $F(x_{s_i}, z) < \alpha$ for all $z \in W_y$, i.e. $z \notin D(x_{s_i})$, $i = 1, 2$, and hence, $z \notin D(x_\lambda)$ for all $\lambda \in I_y$. Thus for every $y \in D(b)$ we have found a neighbourhood W_y and an interval I_y satisfying

$$F(x_\lambda, z) < \alpha \quad \forall \lambda \in I_y, \forall z \in W_y.$$

Since $D(b)$ is a closed subset of the compact set D it is itself compact and from the family $\{W_y, y \in D(b)\}$ one can extract a finite collection $\{W_y, y \in N\}$, $|N| < +\infty$, still covering $D(b)$. If $\lambda \in I := \bigcap\limits_{y \in N} I_y$ and $y \in D(b)$ then $y \in W_{y'}$ for some $y' \in N$, hence $F(x_\lambda, y) < \alpha$. Therefore, $D(x_\lambda) \subset D(a)$ for all $\lambda \in I$, i.e. $I \subset M_a$, again contradicting the definition of s. Thus in any case the situation (4) cannot occur.

We have thus proved that $D(a) \cap D(b) \neq \emptyset$ for any $a, b \in C$, i.e. that the proposition holds when $|M| = 2$. Let us now assume that the proposition holds for $|M| = k$ and prove it for $|M| = k + 1$.

Let $M = \{x^1, \ldots, x^k, x^{k+1}\} \subset C$ and $D' = D(x^{k+1})$. From part I of the proof, for any $\alpha' \in (\alpha, \gamma)$ and any $x \in C$ we have $\{y \in D| \ F(x^{k+1}, y) \ge \alpha', F(x, y) \ge \alpha'\} \neq \emptyset$, hence $\{y \in D'| \ F(x, y) \ge \alpha'\} \neq \emptyset$, i.e.

$$\forall x \in C \quad \exists y \in D' \qquad F(x, y) \ge \alpha',$$

which implies that $\inf_{x\in C}\sup_{y\in D'} F(x,y) \geq \alpha' > \alpha$. By the induction hypothesis, the proposition holds for k points, so by applying it, with D replaced by D', we have

$$\bigcap_{i=1}^{k} D'(x^i) \neq \emptyset,$$

hence $\bigcap_{i=1}^{k+1} D(x^i) \neq \emptyset$. □

Remark 1. The earliest proofs for minimax theorems used separation or fixed point arguments in one form or another. In [20] Helly's theorem was used instead. The above proof, given originally in [15], [16], was the first one using only purely set-theoretical arguments for establishing minimax theorems. The results in the mentioned papers with their proofs have been presented, partially or in full, in several books (see e.g. [11], [18]). Exactly the same results were rediscovered in [3], with only a difference of notation.

3. Minimax theorems

Recall that η and γ are defined from (1). Since it is obvious that $\eta \leq \gamma$, we have $\eta = \gamma$ if $\eta = +\infty$, or $\gamma = -\infty$. Therefore, throughout the sequel we will assume

$$\eta < +\infty, \qquad \gamma > -\infty.$$

Theorem 1. *Assume that D is a closed compact set, while the function $F(x,y)$ is α-connected for every $\alpha \in (\alpha_0, \gamma)$ with $-\infty < \alpha_0 < \gamma$ and satisfies either condition (A) or condition (B) in Lemma 1. Then $F(x,y)$ possesses a saddle-value on $C \times D$.*

Proof. It suffices to show that

$$\sup_{y\in D}\inf_{x\in C} F(x,y) \geq \gamma. \tag{6}$$

The assumptions of the Theorem imply that the conditions in Lemma 1 are fulfilled. According to this Lemma, for any $\alpha \in (\alpha_0, \gamma)$ the sets $D(x) := \{y \in D|\ F(x,y) \geq \alpha\}, x \in C$, have the finite intersection property and since these are closed subsets of the compact set D, their intersection must be nonempty: $\bigcap_{x\in C} D(x) \neq \emptyset$, i.e. $\sup_{y\in D}\inf_{x\in C} F(x,y) \geq \alpha$. Since this is true for every $\alpha \in (\alpha_0, \gamma)$, the inequality (6) follows. □

Theorem 2. *Assume $F(x,y)$ is α-connected for every $\alpha \in \mathbb{R}$ and satisfies (A) and the condition:*

(K) *There exists a nonempty finite set $M \subset C$ such that the set $D^M = \{y \in D|\min_{x\in M} F(x,y) \geq \eta\}$ is compact, while for every fixed $y \in D$ and $a, b \in C$ the function $\lambda \mapsto F(u_{ab}(\lambda), y)$ is u.s.c. in λ in the interval $0 \leq \lambda \leq 1$.*

Then the function $F(x,y)$ possesses a saddle-value on $C \times D$.

Proof. Suppose $\eta < \gamma$. For any fixed $\alpha \in (\eta, \gamma)$ the set $D_\alpha = \{y \in D | \min_{x \in M} F(x, y) \geq \alpha\}$ is closed (α-connectedness), and contained in the compact set D^M, hence D_α is also compact. By Lemma 1, $D_\alpha \neq \emptyset$ and the sets $D(x), x \in C$, defined by

$$D(x) := \{y \in D_\alpha | \ F(x, y) \geq \alpha\} = \{y \in D | \ F(x, y) \geq 0 \ \forall x' \in M \cup \{x\}\},$$

have the finite intersection property. Since they are closed subsets of the compact set D_α, they must have a nonempty intersection: $\bigcap_{x \in C} D(x) \neq \emptyset$. Therefore,

$$\sup_{y \in D} \inf_{x \in C} F(x, y) \geq \sup_{y \in D_\alpha} \inf_{x \in C} F(x, y) \geq \alpha,$$

i.e. $\eta \geq \alpha$. This contradiction shows that $\eta = \gamma$. □

We say that a function $F(x, y) : C \times D \to \mathbb{R}$ is *l.s.c.* (*u.s.c.*, respectively) *in x in every line segment* if for every fixed $y \in D$ and every line segment $[a, b] \subset C$ the function $\lambda \mapsto F((1 - \lambda)a + \lambda b, y)$ is l.s.c. (u.s.c., respectively) in the interval $0 \leq \lambda \leq 1$.

Theorem 3. *Let C, D be closed convex subsets of two topological vector spaces X, Y respectively, and let $F(x, y) : C \times D \to \mathbb{R}$ be quasiconcave and u.s.c. in y, quasiconvex in x, and: either l.s.c. in x in every line segment, or u.s.c. in x in every line segment. Assume, furthermore, that one of the following conditions holds:*

(S) *D is a closed compact set;*

(T) *Y is a reflexive Banach space and there exists a nonempty finite set $M \subset C$ such that $\min_{x \in M} F(x, y) \to -\infty$ as $y \in D, \|y\| \to +\infty$.*

Then the function $F(x, y)$ possesses a saddle-value on $C \times D$.

Proof. Clearly $F(x, y)$ is α-connected for every $\alpha \in \mathbb{R}$, with $u_{ab}(\lambda) = (1-\lambda)a+\lambda b$. If (S) holds, then either condition (A) or (B) in Lemma 1 is fulfilled and by this Lemma, for every $\alpha \geq \eta$, the sets $D(x) = \{y \in D | \ F(x, y) \geq \alpha\}, \ x \in C$, have the finite intersection property. Since $D(x)$ are closed subsets of the compact set D, they have a nonempty intersection, which implies, as we saw above, $\eta = \gamma$.

If (T) holds, for every $\alpha \geq \eta$ the convex closed set $D_\alpha := \{y \in D | \min_{x \in M} F(x, y) \geq \alpha\}$ is closed in the weak topology of Y. Since $\min_{x \in M} F(x, y) \to -\infty$ as $y \in D, \|y\| \to +\infty$, there exists $r > 0$ such that $\min_{x \in M} F(x, y) < \alpha$ whenever $y \in D, \|y\| > r$. Then $D_\alpha \subset \{y \in Y | \ \|y\| \leq r\}$ and since this bounded set is closed it is compact (in the weak topology of Y), in view of the reflexivity of Y. The proof can then be completed in the same way as with Theorem 2. □

Theorem 4. *Let C be a closed convex set in a topological vector space X, D a closed convex set in a reflexive Banach space Y, and let $F(x, y) : C \times D \to \mathbb{R}$ be a function quasiconvex and l.s.c. in x in every line segment, quasiconcave and u.s.c. in y. Assume, furthermore, that one of the following conditions holds:*

(H) *The set $D^* = \{y \in D | \inf_{x \in C} F(x, y) = \eta\}$ is nonempty and bounded;*

(Q) $\inf_{x\in C} F(x,y) \to -\infty$ *as* $y \in D$, $\|y\| \to +\infty$.

Then the function $F(x,y)$ *possesses a saddle-value on* $C \times D$.

Proof. It is plain to verify that under the assumptions specified in the first part of the Theorem the function $F(x,y)$ is α-connected for every $\alpha \in \mathbb{R}$ and satisfies condition (A) in Lemma 1. We now show that

$$(H) \Rightarrow (K), \qquad (Q) \Rightarrow (H).$$

(H) $\Rightarrow$ (K) : From the definition of η we have $D^* = \{y \in D|\ \inf_{x\in C} F(x,y) \geq \eta\} = \bigcap_{x\in C} D(x)$, with $D(x) := \{y \in D|\ F(x,y) \geq \eta\}$. Since $F(x,y)$ is quasiconcave and u.s.c. in y, each set $D(x)$ is convex and closed in the weak topology of Y. Let $A(x)$ denote the intersection of the recession cone of $D(x)$ with the unit sphere S in Y. Since $D^* = \bigcap_{x\in C} D(x)$ is bounded by assumption (H), one must have $\bigcap_{x\in C} A(x) = \emptyset$, and hence, since S is compact in the weak topology of Y, there exists a nonempty finite set $M \subset C$ such that $\bigcap_{x\in M} A(x) = \emptyset$. The latter implies that the set $D^M := \bigcap_{x\in M} D(x) = \{y \in D|\ \min_{x\in M} F(x,y) \geq \eta\}$ is bounded, and since it is a closed subset of the compact set D^*, it is compact (in the weak topology of Y), i.e. (K) holds.

(Q) $\Rightarrow$ (H) : In the weak topology of Y the set D is closed and the quasiconcave function $F(x,y)$ is u.s.c. in y, hence the function $y \mapsto \inf_{x\in C} F(x,y)$ is u.s.c. on the closed set D. Condition (Q) then implies that $\inf_{x\in C} F(x,y)$ has a maximum on D, i.e. $\eta = \max_{y\in D} \inf_{x\in C} F(x,y) \in \mathbb{R}$. Furthermore, there exists a number $r > 0$ such that $\inf_{x\in C} F(x,y) < \eta$ whenever $y \in D$, $\|y\| \geq r$. Consequently, $D^* := \{y \in D|\ \inf_{x\in C} F(x,y) \geq \eta\} \subset \{y \in Y|\ \|y\| \leq r\}$, i.e. the set D^* is bounded. Since D^* is closed and bounded, it is compact by reflexivity of the space Y. □

Remark 2. In each of the above Theorems 1-4, the saddle-value is equal to

$$\max_{y\in \overline{D}} \inf_{x\in C} F(x,y) = \inf_{x\in C} \sup_{y\in D} F(x,y) \tag{7}$$

where $\overline{D}$ is some compact subset of D. Specifically,

$$\overline{D} = \begin{cases} D & \text{if } D \text{ is compact} \\ D^M & \text{if (K) or (T) holds} \\ D^* & \text{if (H) or (Q) holds.} \end{cases}$$

Using the "skew" symmetry in the equality

$$\sup_{y\in D} \inf_{x\in C} F(x,y) = -\inf_{y\in D} \sup_{x\in C} [-F(x,y)], \tag{8}$$

we can also state alternative propositions to the above minimax theorems by interchanging the roles of x and y. In particular the following theorem is true.

Theorem 4*. *Let C, D be two closed convex sets in two topological vector spaces X, Y, respectively, and let $F(x,y) : C \times D \to \mathbb{R}$ be a function quasiconvex and l.s.c. in x, quasiconcave and u.s.c. in y in every line segment. Assume, in addition, that one of the following conditions holds:*

($\tilde{\mathrm{K}}$) *There exists a nonempty finite set $N \subset D$ such that the set $C^N = \{x \in C| \max_{y\in N} F(x,y) \le \gamma\}$ is compact;*

($\tilde{\mathrm{T}}$) *X is a reflexive Banach space and there exists a nonempty finite set $N \subset D$ such that $\max_{y\in N} F(x,y) \to +\infty$ as $x \in C, \|x\| \to +\infty$;*

($\tilde{\mathrm{H}}$) *X is a reflexive Banach space and the set $C^* = \{x \in C| \sup_{y\in D} F(x,y) = \gamma\}$ is nonempty and bounded;*

($\tilde{\mathrm{Q}}$) *X is a reflexive Banach space, and $\sup_{y\in D} F(x,y) \to +\infty$ as $x \in C$, $\|x\| \to +\infty$,*

Then the function $F(x,y)$ has a saddle-value on $C \times D$.

Also note that the saddle-value is equal to

$$\min_{x\in\overline{C}} \sup_{y\in D} F(x,y) = \sup_{y\in D} \inf_{x\in C} F(x,y) \tag{9}$$

where $\overline{C}$ is some compact subset of C.

4. Special cases

Many important minimax theorems among the best so far known are special cases of the above theorems.

Corollary 1. (Sion, refined) *Let C, D be be two closed convex sets in two topological vector spaces X, Y, respectively, and let $F(x,y) : C \times D \to \mathbb{R}$ be a function quasiconvex in x and quasiconcave in y.*

(i)*If $F(x,y)$ is l.s.c (or u.s.c.) in x in every line segment and u.s.c. in y, while D is compact, then* (7) *holds with $\overline{D} = D$;*

(ii) *If $F(x,y)$ is u.s.c. (or l.s.c.) in y in every line segment and l.s.c. in x, , while C is compact then* (9) *holds, with $\overline{C} = C$;*

Proof. Part (i) follows from Theorem 3. Part (ii) can be deduced from Part (i) and observation (8). □

Note that when D (C, resp.) is compact, the function $F(x,y)$ is not required to be l.s.c. in x (u.s.c. in y, resp.) as in Sion's theorem, but only to be *l.s.c. or u.s.c. in x* (u.s.c. or l.s.c. in y, resp.) *in every line segment*, which is a weaker condition. Noting that a function $F(x,y)$ convex in x is always u.s.c. in x in every line segment, while a function $F(x,y)$ concave in x is always l.s.c. in x in every line segment, we see that a special case of Corollary 1 is the following sharpening of the *Lop Sided Minimax Theorem* in Aubin-Ekeland [1] (Chapter 6, page 295):

Let C, D be two closed convex sets in two topological vector spaces X, Y respectively. Then a function $F(x,y) : C \times D \to \mathbb{R}$ has a saddle value on $C \times D$ if either of the following conditions holds:

(i) *D is compact, while $F(x,y)$ is convex in x and quasiconcave u.s.c. in y.*

(ii) *C is compact, while $F(x,y)$ is quasiconvex l.s.c. in x and concave in y.*

An even stronger proposition is true:

Corollary 2. (Aubin-Ekeland, refined) *Let C, D be two closed convex sets in two topological vector spaces X, Y respectively. Then a function $F(x,y) : C \times D \to \mathbb{R}$ has a saddle value on $C \times D$ if either of the following conditions holds:*

(i) *$F(x,y)$ is convex in x, quasiconcave u.s.c. in y, while for some nonempty finite set $M \subset C$ the set $D^M = \{y \in D | \min_{x \in M} F(x,y) \geq \eta\}$ is compact;*

(ii) *$F(x,y)$ is concave in y, quasiconvex l.s.c. in x, while for some nonempty finite set $N \subset D$ the set $C^N = \{x \in C | \max_{y \in N} F(x,y) \leq \gamma\}$ is compact.*

Proof. This follows from Theorem 2 and observation (8). $\square$

When quasiconvexity and quasiconcavity are replaced by convexity and concavity, while M, N are singletons, case (ii) reduces to Theorem 7 in Aubin-Ekeland [1], Chapter 6, page 319, while case (i) is the symmetric counterpart of the latter. Also a theorem of Hartung [6] is a special case of Corollary 2 when M is a singleton.

Corollary 3. (Ekeland-Temam, refined) *Let $C, D, F(x,y)$ be as in Corollary 1.*

(i) *If Y is a reflexive Banach space, $F(x,y)$ is l.s.c. in x in every line segment, u.s.c. in y, and there exists $\bar{x} \in C$ such that $F(\bar{x},y) \to -\infty$ as $y \in D$, $\|y\| \to +\infty$, then (7) holds with $\overline{D} = \{y \in D | F(\bar{x},y) \geq \eta > -\infty\}$.*

(ii) *If X is a reflexive Banach space, $F(x,y)$ is u.s.c. in y in every line segment, l.s.c. in x, and there exists $\bar{y} \in D$ such that $F(x,\bar{y}) \to +\infty$ as $x \in C$, $\|x\| \to +\infty$, then (9) holds with $\overline{C} = \{x \in C | F(x,\bar{y}) \leq \gamma < +\infty\}$.*

Proof. This follows from Theorems 3 and 4* because (i) implies (T), while (ii) implies ($\tilde{\mathrm{T}}$). $\square$

In the special case when $F(x,y)$ is convex in x, concave in y, this proposition was established in [2] (Proposition 2.3, page 175).

Corollary 4. (Golshtein, refined) *Let $C, D, F(x,y)$ be as in Corollary 1 and let Y be a reflexive Banach space.*

(i) *If the set $D^* = \{y \in D | \inf_{x \in C} F(x,y) = \eta\}$ is nonempty and bounded then (7) holds with $\overline{D} = D^*$.*

(ii) *If the set* $C^* = \{x \in C | \sup_{y \in D} F(x,y) = \gamma\}$ *is nonempty and bounded then* (9) *holds with* $\overline{C} = C^*$.

Proof. This is a consequence of Theorems 4 and 4*. □

In the finite-dimensional case, when $F(x,y)$ is convex in x, concave in y and continuous in (x,y), this proposition was established in [4]. Also, if $F(x,y)$ is concave in y then (i) is equivalent to saying that the set $\{y \in D | \inf_{x \in \mathrm{ri}C} F(x,y) \geq \eta\}$ has a recession cone consisting of the vector 0 alone, i.e., that *the concave functions* $F(x,.)$ *for* $x \in \mathrm{ri}C$ *have no common direction of recession*; similarly, if $F(x,y)$ is convex in x then (ii) is equivalent to saying that *the convex functions* $F(.,y)$ *for* $y \in \mathrm{ri}D$ *have no common direction of recession.* Therefore, in the finite dimensional case, Corollary 4 includes the minimax theorem proved in [12] (Theorem 37.3).

5. Duality by minimax

Many central results of nonlinear optimization theory can be established on the basis of the above minimax theorems rather than separation theorems as in the traditional approach. To give an example, we show how the basic duality theorem for modern convex optimization can be derived in a simple way from Theorem 4 or 4*.

A closed convex cone $K \subset \mathbb{R}^m$ defines in $\mathbb{R}^p$ a partial ordering $\preceq_K$ such that $a \preceq b$ whenever $a - b \in K$. When $\mathrm{int}K \neq \emptyset$ we also write $a \prec 0$ to mean $a \in \mathrm{int}K$. We say that a mapping $g : \mathbb{R}^n \to \mathbb{R}^p$ is K-convex if for any $x, x' \in \mathbb{R}^n$ and $0 \leq \alpha \leq 1$:

$$g(\alpha x + (1-\alpha)x') \preceq_K \alpha g(x) + (1-\alpha)g(x').$$

Consider the convex programming problem

$$\inf\{f(x) | \ g_0(x) = 0, \ g_i(x) \preceq_{K_i} 0 \ (i = 1, \dots, m), \ x \in C\}, \tag{10}$$

where $f : \mathbb{R}^n \to R$ is a convex function, $g_0 : \mathbb{R}^n \to \mathbb{R}^q$ is an affine mapping and $g_i : \mathbb{R}^n \to \mathbb{R}^{p_i}$ $(i = 1, \dots, m)$ are K_i-convex mappings, while C is a closed convex subset of $\mathbb{R}^n$. Letting $K_i^* = \{\lambda_i \in \mathbb{R}^{p_i} | \ \langle \lambda_i, y\rangle \geq 0 \ \forall y \in K_i\}$, the conjugate cone of K_i, it is easily verified that

$$\begin{aligned} &\sup\{f(x) + \sum_{i=0} \langle \lambda_i, g_i(x)\rangle | \ \lambda_0 \in \mathbb{R}^q, \ \lambda_i \succeq_{K_i^*} 0, \ i = 1, \dots, m\} \\ &= \begin{cases} f(x) & \text{if } \ g_0(x) = 0, \ g_i(x) \preceq_{K_i} 0 \ (i = 1, \dots, m) \\ +\infty & \text{otherwise} \end{cases} \end{aligned}$$

Therefore, the problem (10) can be written as

$$\inf_{x \in C}\{\sup\{f(x) + \sum_{i=0}^{m} \langle \lambda_i, g_i(x)\rangle | \ \lambda_0 \in \mathbb{R}^q, \ \lambda_i \succeq_{K_i^*} 0, \ i = 1, \dots, m\}\}$$

and its Lagrange dual is

$$\sup\Big\{\inf_{x\in C}[f(x)+\sum_{i=0}^{m}\langle\lambda_i,g_i(x)\rangle] \mid \lambda_0\in\mathbb{R}^q,\ \lambda_i\succeq_{K_i^*}0,\ i=1,\dots,m\}.$$

By setting $F(x,\lambda):=f(x)+\sum_{i=0}^{m}\langle\lambda_i,g_i(x)\rangle,\ D:=\{\lambda=(\lambda_0,\lambda_1,\dots,\lambda_m)\mid\ \lambda_0\in R^q,\ \lambda_i\succeq_{K_i^*}0,\ i=1,\dots,m\}$, the duality gap is

$$\inf_{x\in C}\sup_{\lambda\in D}F(x,\lambda)-\sup_{\lambda\in D}\inf_{x\in C}F(x,\lambda)$$

Corollary 5. *Assume that*

$$(\exists\bar{x}\in C)\quad g_0(\bar{x})=0\in \mathrm{ri}g_0(C),\ g_i(\bar{x})\prec_{K_i}0,\ i=1,\dots,m \tag{11}$$

Then there is no duality gap, i.e.

$$\inf_{x\in C}\sup_{\lambda\in D}F(x,\lambda)=\sup_{\lambda\in D}\inf_{x\in C}F(x,\lambda). \tag{12}$$

Proof. The sets C, D are closed convex in $X=\mathbb{R}^n$, $Y=\mathbb{R}^{q+p_1+\dots+p_m}$, respectively, and the function $F(x,\lambda)$ is convex in $x\in\mathbb{R}^n$, affine in $\lambda=(\lambda_0,\lambda_1,\dots,\lambda_m)\in\mathbb{R}^{q+p_1+\dots+p_m}$. By Theorem 4, to prove (12) it suffices to show that condition (Q) in Theorem 4 holds, i.e.

$$\inf_{x\in C}F(x,\lambda)\to-\infty \text{ as } \lambda\in D,\ \|\lambda\|\to+\infty. \tag{13}$$

Without loss of generality it can be assumed that $g_0(C)=\mathbb{R}^q$. In view of condition (11), for every $j=1,\dots,q$ there exist $a^j\in C$ such that $g_{0j}(a^j)>0$ and $g_{0i}(a^j)=0$ for $i\neq j$. Then for sufficiently small $\varepsilon>0$ the vectors $x^j=\bar{x}+\varepsilon(a^j-\bar{x})$ and $\hat{x}^j=\bar{x}-\varepsilon(a^j-\bar{x})$ satisfy $x^j,\ \hat{x}^j\in C$ while

$$g_{0j}(x^j)>0,\ g_{0i}(x^j)=0\ (i\neq j),\quad g_i(x^j)\prec_{K_i}0\ (i=1,\dots,m).$$

$$g_{0j}(\hat{x}^j)<0,\ g_{0i}(\hat{x}^j)=0\ (i\neq j),\quad g_i(x^j)\prec_{K_i}0\ (i=1,\dots,m).$$

Let $M=\{x^j,\ \hat{x}^j, j=1,\dots,q\}$. For every $\lambda\in D\setminus\{0\}$ we must have either $\lambda_0=0$, or $\lambda_{0j}\neq 0$ for some j. Therefore

$$\rho(\lambda):=\min_{x\in M}\sum_{i=0}^{m}\langle\lambda_i,g_i(x)\rangle<0,\quad \theta:=\max\{\rho(\lambda)\mid\ \lambda\in D,\ \|\lambda\|=1\}<0,$$

and hence, as $\|\lambda\|\to+\infty$,

$$\begin{aligned}\inf_{x\in C}F(x,\lambda)&\le\max_{x\in M}f(x)+\|\lambda\|\min_{x\in M}\Big\{\sum_{i=0}^{m}\langle\frac{\lambda_i}{\|\lambda\|},g_i(x)\rangle\Big\}\\&\le\max_{x\in M}f(x)+\|\lambda\|\theta\to-\infty,\end{aligned}$$

concluding the proof. □

Note that condition (11) is fulfilled when

$$(\exists \bar{x} \in \mathrm{int}C) \quad g_0(\bar{x}) = 0,\ g_i(\bar{x}) \prec_{K_i} 0,\ i = 1, \dots, m$$

because $\bar{x} \in \mathrm{int}C,\ g_0(\bar{x}) = 0$ implies that $g_0(\bar{x}) = 0 \in \mathrm{int}g_0(C)$.

For example, let $\mathcal{S}^p$ be the space of symmetric $p \times p$ real matrices, equipped with the inner product

$$A \bullet B = \mathrm{Tr}(AB) = \sum_{i,j} a_{ij} b_{ij}$$

In this space the set of positive semidefinite matrices forms a closed convex cone $\mathcal{S}^p_+$ inducing a partial ordering $\preceq_{\mathcal{S}^p_+}$. We denote this partial ordering simply by $\preceq$, so that $B \preceq A \Leftrightarrow A - B \in \mathcal{S}^p_+$. Consider the semidefinite program

$$\text{(SDP)} \qquad \min\Big\{\langle c, x\rangle |\ A_0 + \sum_{j=1}^n x_j A_j \preceq 0\Big\}$$

where $c, x \in \mathbb{R}^n$, and $A_j \in \mathcal{S}^p, j = 0, 1, \dots, n$. By noting that the cone $\mathcal{S}^p_+$ is self-conjugate, i.e. $\mathcal{S}^p_+ = \{Y \in \mathcal{S}^p |\ X \bullet Y \geq 0\ \forall X \in \mathcal{S}^p_+\}$, the Lagrangian of (SDP) is

$$L(x, Y) = c^T x + Y \bullet (A_0 + \sum_{j=1}^n x_j A_j),$$

and the Lagrange function is

$$\varphi(Y) = \begin{cases} (A_0 \bullet Y) & \text{if } A_j \bullet Y + c_j = 0,\ j = 1, \dots, n \\ -\infty & \text{otherwise} \end{cases}$$

Hence the dual problem to (SDP) is

$$\text{(SDD)} \qquad \max\{A_0 \bullet Y |\ A_j \bullet Y + c_j = 0,\ j = 1, \dots, n,\ Y \succeq 0\}.$$

From the above Corollary, strong duality holds if there exists $\bar{x} \in C$ satisfying $A_0 + \sum\limits_{j=1}^n \bar{x}_j A_j \prec 0$.

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