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THE STRUCTURE OF IDEALS IN THE POLYNOMIAL RING OVER A PID

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Introduction

The polynomial ring over a PID is not necessary to be a PID, but it is a commutative Noetherian UFD. In the article, we study the structure of ideals in the polynomial ring $D[x]$ over a PID D. Beside the properties known for an arbitrary commutative Noetherian UFD, ideals in $D[x]$ also have special structures due to the properties of ideals in D. We first examine the structure of prime, maximal, primary, and irreducible ideals. Then, using the irredundant primary decomposition in commutative Noetherian rings, we completely describe the structure of arbitrary ideals in $D[x]$.

1. The structure of prime and maximal ideals in $D[x]$

In this section, we will describe the structure of prime and maximal ideals in $D[x]$, where D is a PID.

Note that, in an arbitrary integral domain, every nontrivial prime element is irreducible. The converse is not necessarily true. However, it holds in an UFD. In this section, p is a nontrivial prime element of D. For an ideal I of $D[x]$, if there are no additional remarks, then I is nontrivial (i.e., $I \neq 0$ and $I \neq D[x]$).

For a nontrivial prime element p of D, D/pD is a field, so that $D[x]/pD[x] \cong (D/pD)[x]$ is a PID.

Lemma 1.1. *If J is an ideal of $D[x]$ and $p \in J$, then J is prime if and only if it can be represented as follows:*

1) $J = pD[x]$, or

2) $J = \langle \varphi(x), p \rangle$, where $\varphi(x) \in D[x]$ such that $\varphi(x) + pD[x]$ is irreducible in $D[x]/pD[x]$.

Proof. It is a fact that J is prime in $D[x]$ if and only if $J/pD[x]$ is prime in $D[x]/pD[x]$. This is equivalent to

$$J/pD[x] = \{0 + pD[x]\} \quad \text{or} \quad J/pD[x] = \langle \varphi(x) + pD[x] \rangle,$$

where $\varphi(x) + pD[x]$ is a nontrivial prime element of D[x]/pD[x]. Thus $J = pD[x]$ or $J = \langle \varphi(x), p \rangle$. □

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Theorem 1.1. (Description of prime ideals) *An ideal J of $D[x]$ is prime if and only if J can be represented as follows:*

1) $J = \langle \psi(x) \rangle$, where $\psi(x)$ is irreducible in $D[x]$, or

2) $J = \langle \varphi(x), p \rangle$, where $\varphi(x) \in D[x]$ such that $\varphi(x) + pD[x]$ is irreducible in $D[x]/pD[x]$.

In the first case (resp., second case), J is called a *prime ideal of type* 1 (resp., *prime ideal of type* 2).

Proof. ($\Longleftarrow$) If $J = \langle \psi(x) \rangle$, then J is obviously prime in $D[x]$.

If $J = \langle \varphi(x), p \rangle$, then J is prime by Lemma 1.1.

($\Longrightarrow$) Let J be prime in $D[x]$, then $J \cap D$ is prime in D. Since D is a PID, either $J \cap D = \{0\}$ or $J \cap D = pD$.

If $J \cap D = pD$, then $p \in J$. By Lemma 1.1, $J = pD[x]$ or $J = \langle \varphi(x), p \rangle$, where $\varphi(x) \in D[x]$ such that $\varphi(x) + pD[x]$ is irreducible in $D[x]/pD[x]$, and hence the theorem is proved.

Suppose that $J \cap D = \{0\}$. Let $f(x)$ be the polynomial of the least degree in $J \setminus \{0\}$. Since $J \cap D = \{0\}$, $\deg f(x) \geq 1$. We have $f(x) = \mathrm{Vol}(f(x)) f_1(x)$, where $\mathrm{Vol}(f(x))$ is the volume of $f(x)$ and $f_1(x)$ is primitive in $D[x]$. Since J is prime, either $\mathrm{Vol}(f(x)) \in J$ or $f_1(x) \in J$. Because $J \cap D = \{0\}$, we have $\mathrm{Vol}\, f(x) \notin J$, whence $f_1(x) \in J$. Thus, $f(x)$ can be considered as a primitive polynomial. We claim that $J = \langle f(x) \rangle$. Since $f(x) \in J$, $\langle f(x) \rangle \subset J$. Conversely, let $g(x) \in J$, we will prove $g(x) \in \langle f(x) \rangle$. Let F be the quotient field of D. By the Euclidean algorithm, we have $q(x)$, $r(x) \in F[x]$ such that

$$g(x) = f(x)q(x) + r(x),$$

where $\deg r(x) < \deg f(x)$ if $r(x) \neq 0$. Since $q(x)$, $r(x)$ can be written as

$$q(x) = \frac{a}{b} q_1(x),$$
$$r(x) = \frac{c}{d} r_1(x),$$

where $a, c \in D$; $b, d \in D \setminus \{0\}$; $q_1(x)$, $r_1(x)$ are primitive in $D[x]$,

$$g(x) = \frac{a}{b} f(x) q_1(x) + \frac{c}{d} r_1(x),$$

or

$$(bd)g(x) = (ad)f(x)q_1(x) + (bc)r_1(x).$$

Because both $(bd)g(x)$ and $(ad)f(x)q_1$ are in J, we have $(bc)r_1(x) = b(cr_1(x)) \in J$. Furthermore, since $b \neq 0$, $cr_1(x) \in J$. If $r(x) \neq 0$, then $\deg r(x) = \deg(cr_1(x))$, and hence $\deg(cr_1(x)) < \deg f(x)$, which contradicts the definition of $f(x)$. Thus, $r(x) = 0$, whence

$$g(x) = \frac{a}{b} f(x) q_1(x),$$

or

$$bg(x) = af(x)q_1(x).$$

Because both $f(x)$ and $q_1(x)$ are primitive, $f(x)q_1(x)$ is primitive, whence $\mathrm{Vol}(f(x)q_1(x)) \sim 1$. Therefore $b\,\mathrm{Vol}(g(x)) \sim a$, so b divides a, whence $q(x) \in D[x]$. Hence $g(x) = f(x)q(x) \in < f(x) >$. Consequently, $J = \langle f(x) \rangle$ where $f(x)$ is irreducible in $D[x]$ for J is prime. □

Remarks. 1) The first and second type of prime ideals do not coincide because $\langle \varphi(x), p \rangle$ is not principal. In fact, suppose, on the contrary, that $I = \langle \varphi(x), p \rangle$ is principal, i.e., $I = \langle \psi(x) \rangle$ for some irreducible $\psi(x) \in D[x]$. Since $p \in I$, $p = \psi(x)\phi(x)$, where $\phi(x) \in D[x]$. Since p is prime (and then irreducible), p and $\psi(x)$ are conjugate, whence $\langle \psi(x) \rangle = pD[x]$, or $\langle \varphi(x), p \rangle = pD[x]$. It follows that $\varphi(x) \in pD[x]$ and hence $\varphi(x) + pD[x] = 0 + pD[x]$, a contradiction. Thus, $I = \langle \varphi(x), p \rangle$ is not principal and we have the conclusion.

2) The polynomial $\varphi(x)$ in the prime ideal of type 2 has $\deg \varphi(x) \geq 1$, and we can choose $\varphi(x)$ monic and irreducible in $D[x]$.

Lemma 1.2. *$J = \langle \varphi(x), p \rangle$ is a maximal ideal of $D[x]$, where $\varphi(x) \in D[x]$ such that $\varphi(x) + pD[x]$ is irreducible in $D[x]/pD[x]$.*

Proof. Since $J/pD[x] = \langle \varphi(x) + pD[x] \rangle$, $J/pD[x]$ is maximal in $D[x]/pD[x]$. Thus J is maximal in $D[x]$. □

Lemma 1.3. *Let D be a PID but not a field and I a principal ideal of $D[x]$. Then I is maximal in $D[x]$ if and only if the following conditions are satisfied:*

1) *D has a finite number of nontrivial prime ideals, which are $\langle p_1 \rangle, \langle p_2 \rangle, \ldots, \langle p_n \rangle$ $(n \geq 1)$.*

2) *$I = \langle axg(x)+1 \rangle$, where $a = \prod_{i=1}^{n} p_i$, $g(x) \in D[x]$, and $axg(x)+1$ is irreducible in $D[x]$.*

Proof. ($\Longrightarrow$) Suppose $I = \langle f(x) \rangle$ is maximal in $D[x]$. Then $f(x)$ is irreducible in $D[x]$. First, we claim that $\deg f(x) \geq 1$. Suppose, on the contrary, that $\deg f(x) = 0$, or equivalently, $f(x) = q$ is irreducible in D. We have $I = \langle q \rangle \subset \langle x, q \rangle$ and $I \neq \langle x, q \rangle$. By Lemma 1.2, $\langle x, q \rangle$ is maximal in $D[x]$, so I is not maximal, a contradiction. Thus $\deg f(x) \geq 1$.

Next, we are going to show that D has finite nontrivial prime ideals. Suppose, on the contrary, that D has infinite nontrivial prime ideals, then D has infinite nontrivial prime elements. Let b be the leading coefficient of $f(x)$, then there exists a nontrivial prime element p of D such that p does not divide b. Therefore $f(x) + pD[x] \neq 0 + pD[x]$ and $\deg(f(x) + pD[x]) \geq 1$, whence $f(x) + pD[x]$ is nontrivial in $D[x]/pD[x]$. We have $I = \langle f(x) \rangle \subset \langle f(x), p \rangle$ and $\langle f(x), p \rangle \neq D[x]$, for $f(x) + pD[x]$ is non-unit. Since $\deg f(x) \geq 1$, we have $p \notin I$. Thus, I is properly contained in $\langle f(x), p \rangle$, which contradicts the definition of I . On the other hand, since D is a PID, but not a field, D has at least a nontrivial prime ideal. Thus, D has finite nontrivial prime ideals, which are $\langle p_1\rangle, \langle p_2\rangle, \dots, \langle p_n\rangle$ $(n \geq 1)$.

Set $a = \prod_{i=1}^{n} p_i$. We claim that $I = \langle axg(x)+1\rangle$ for some $g(x) \in D[x]$. We write $f(x)$ in the form

$$f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 \quad (m \geq 1, a_m \neq 0).$$

If $a_0 = 0$, then

$$f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x,$$

whence $f(x) \subset \langle x\rangle$, which implies I is properly contained in $\langle x, p\rangle$ for some nontrivial prime element of D, a contradiction. Thus, $a_0 \neq 0$. If a_0 is non unit, then it has a prime factor q, whence $a_0 = qa_0'$ for some $a_0' \in D \setminus \{0\}$. We have

$$f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + qa_0',$$

whence $I = \langle f(x)\rangle$ is properly contained in $\langle x, q\rangle$, which contradicts the fact that I is maximal in $D[x]$. Thus, a_0 is a unit. Set $d = \text{g.c.d}\{a_m, \dots, a_1\}$. We claim that a divides d, or equivalently, that p_i divides d for every i $(1 \leq i \leq n)$. Suppose, on the contrary, that there exists i_0 $(1 \leq i_0 \leq n)$ such that p_{i_0} does not divide d, whence $\deg(f(x) + p_{i_0}D[x]) \geq 1$ and hence $f(x) + p_{i_0}D[x]$ is neither zero nor unit in $D[x]/p_{i_0}D[x]$. We have $p_{i_0} \notin I$, for $\deg f(x) \geq 1$ and $\langle f(x), p_{i_0}\rangle \neq D[x]$, for $f(x) + p_{i_0}D[x]$ is non-unit. Therefore $I = \langle f(x)\rangle$ is properly contained in $\langle f(x), p_{i_0}\rangle$, which contradicts the maximum of I. Thus a divides d.

Summing up, $f(x)$ can be written in the form

$$f(x) = axh(x) + a_0,$$

or

$$f(x) = a_0(a_0^{-1}axh(x) + 1).$$

Therefore,

$$I = \langle axg(x) + 1\rangle,$$

where $g(x) = a_0^{-1}h(x)$. Since I is maximal, $axg(x) + 1$ is irreducible in $D[x]$.

$(\Longleftarrow)$ Let D have finite nontrivial prime ideals, namely $\langle p_1\rangle, \langle p_2\rangle, \dots, \langle p_n\rangle$ $(n \geq 1)$, and $I = \langle axg(x) + 1\rangle$, where $a = \prod_{i=1}^{n} p_i$, $g(x) \in D[x]$ such that $axg(x) + 1$ is irreducible in $D[x]$. Set $f(x) = axg(x) + 1$, whence $f(x)$ is primitive. We claim that $I = \langle f(x)\rangle$ is maximal. Suppose, on the contrary, that I is not maximal. It follows that there exists $J \neq D[x]$ such that I is properly contained in J. Choose $h(x) \in J \setminus I$ and let F be the quotient field of D. First, we claim that $f(x)F[x]$ is properly contained in $f(x)F[x] + h(x)F[x]$. Clearly, $f(x)F[x] \subset f(x)F[x] + h(x)F[x]$. In addition, $h(x) \notin f(x)F[x]$. Suppose, on the contrary, that $h(x) \in f(x)F[x]$, and hence $h(x) = f(x)k_F(x)$, where $k_F(x) \in F[x]$. Since $k_F(x) \in F[x]$,

$$k_F(x) = \frac{c}{d}k_D(x),$$

where $c, d \in D \setminus \{0\}$ and $k_D(x)$ is primitive in $D[x]$. Thus

$$h(x) = f(x)(\frac{c}{d}k_D(x)),$$

or

$$dh(x) = c(f(x)k_D(x)).$$

Since $f(x)$ and $k_D(x)$ are primitive, so is $f(x)k_D(x)$. It follows from the last equality that

$$d\,\mathrm{Vol}(h(x)) \sim c.$$

Then d divides c, and hence $k_F(x) \in D[x]$, whence $h(x) \in \langle f(x) \rangle = I$, a contradiction. Thus $f(x)F[x]$ is properly contained in $f(x)F[x] + h(x)F[x]$. Next, since $f(x)$ is irreducible in $D[x]$, in $F[x]$, whence $f(x)F[x]$ is maximal in $F[x]$. Combining with the previous paragraph, we have $f(x)F[x] + h(x)F[x] = F[x]$, whence there exists $m \in D \setminus \{0\}$ and $u(x), v(x) \in D[x]$ such that

$$1 = f(x)\frac{u(x)}{m} + h(x)\frac{v(x)}{m},$$

or

$$m = f(x)u(x) + h(x)v(x).$$

Therefore, $m \in f(x)D[x] + h(x)D[x]$. Since $m \neq 0$, m can be factorized as

$$m = \varepsilon \prod_{i=1}^{n} p_i^{\alpha_i},$$

where ε is unit, $\alpha_i \geq 0$, and p_i are prime. Choose $r = 1 + 2\max\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, so r is odd, and whence

$$(axg(x))^r + 1 = (axg(x) + 1)\zeta(x),$$

where $\zeta(x) \in D[x]$. Thus

$$(axg(x))^r + 1 \in I$$

and hence

$$(axg(x))^r + 1 \in f(x)D[x] + h(x)D[x]. \tag{1}$$

On the other hand

$$a^r = \prod_{i=1}^{n} p_i^r.$$

Because $r \geq \alpha_i$ for all $1 \leq i \leq n$, $a^r \in \big(\prod_{i=1}^{n} p_i^{\alpha_i}\big)D[x] = mD[x]$. Since $mD[x] \subset f(x)D[x] + h(x)D[x]$, $a^r \in f(x)D[x] + h(x)D[x]$, whence

$$(axg(x))^r = a^r(xg(x))^r \in f(x)D[x] + h(x)D[x]. \tag{2}$$

From (1) and (2), we have $1 \in f(x)D[x] + h(x)D[x]$. It follows that $1 \in J$ (since $f(x)D[x] + h(x)D[x] \subset J$), whence $J = D[x]$, which contradicts the fact that J is maximal in $D[x]$. Thus, $I = \langle f(x) \rangle$ is maximal. $\square$

Theorem 1.2. (Description of maximal ideals) *Let D be a PID and I an ideal of $D[x]$. Then*

(i) *If D has no nontrivial prime ideal, or equivalently, D is a field, then I is maximal if and only if $I = \langle \psi(x) \rangle$, where $\psi(x)$ is irreducible in $D[x]$.*

(ii) *If D has a finite number of nontrivial prime ideals, namely $\langle p_1 \rangle, \langle p_2 \rangle, \dots, \langle p_n \rangle$ $(n \geq 1)$, then I is maximal if and only if it can be written in one of the following forms:*

1) *$I = \langle axg(x) + 1 \rangle$, where $a = \prod_{i=1}^{n} p_i$, $g(x) \in D[x]$ such that $axg(x) + 1$ is irreducible in $D[x]$.*

2) *$I = \langle \varphi(x), p \rangle$, where p is nonzero prime element of D, $\varphi(x) \in D[x]$ such that $\varphi(x) + pD[x]$ is irreducible in $D[x]/pD[x]$.*

(iii) *If D has an infinite number of prime ideals, then I is maximal if and only if $I = \langle \varphi(x), p \rangle$, where p is a nonzero prime element of D, $\varphi(x) \in D[x]$ such that $\varphi(x) + pD[x]$ is irreducible in $D[x]/pD[x]$.*

Proof. (i) It is easy to see that D is a field and hence $D[x]$ is a PID, so conclusion follows.

(ii) Let D have a finite number of nontrivial prime ideals, namely $\langle p_1 \rangle, \langle p_2 \rangle, \dots,$ $\langle p_n \rangle$ $(n \geq 1)$.

$(\Longrightarrow)$ Let I be maximal and hence prime. By Theorem 1.1, either $I = \langle \psi(x) \rangle$, where $\psi(x)$ is irreducible in $D[x]$, or $I = \langle \varphi(x), p \rangle$, where $\varphi(x) \in D[x]$ such that $\varphi(x) + pD[x]$ is irreducible in $D[x]/pD[x]$. If I is not of the second form, then $I = \langle \psi(x) \rangle$, whence by Lemma 1.3, $I = \langle axg(x) + 1 \rangle$, where $a = \prod_{i=1}^{n} p_i$, $g(x) \in D[x]$ such that $axg(x) + 1$ is irreducible.

$(\Longleftarrow)$ By Lemma 1.3, the ideal in 1) is maximal. By Lemma 1.2, the ideal in 2) is maximal.

(iii) Let D have an infinite number of prime ideals.

$(\Longrightarrow)$ Let I be maximal and hence I is prime. By Theorem 1.1, either $I = \langle \psi(x) \rangle$, where $\psi(x)$ is irreducible, or $I = \langle \varphi(x), p \rangle$, where $\varphi(x) \in D[x]$ such that $\varphi(x) + pD[x]$ is irreducible in $D[x]/pD[x]$. Since D has infinite prime ideals, $\langle \psi(x) \rangle$ is not maximal (Lemma 1.3). By Remark 1) after Theorem 1.1, $\langle \varphi(x), p \rangle$ is not principal, whence $I = \langle \varphi(x), p \rangle$.

$(\Longleftarrow)$ It follows from Lemma 1.2. □

Remark. We can always establish a PID having only a finite number of nontrivial prime ideals. In fact, consider the ring of integers $\mathbb{Z}$, and distinct nonzero prime elements $p_1, p_2, \dots, p_n$ of $\mathbb{Z}$. Set $M = \mathbb{Z} \setminus (\langle p_1, p_2, \dots, p_n \rangle)$, then M is a multiplicative set of $\mathbb{Z}$. Consider the localization $\mathbb{Z}_M$ of $\mathbb{Z}$ at M, then it is a PID having exactly n nontrivial prime ideals $\left\langle \frac{p_1}{1} \right\rangle, \left\langle \frac{p_2}{1} \right\rangle, \dots, \left\langle \frac{p_n}{1} \right\rangle$.

2. The structure of primary and irreducible ideals in $D[x]$

An arbitrary ideal of $D[x]$ can be represented by means of primary ideals or irreducible ideals. In order to describe the structure of an ideal of $D[x]$, we examine the structure of primary and irreducible ideals of $D[x]$ first.

In this section, p is a nontrival prime elememt in D, $\varphi(x) \in D[x]$ such that $\varphi(x) + pD[x]$ is irreducible in $D[x]/pD[x]$, and $\psi(x)$ is irreducible in $D[x]$.

If Q is a primary ideal of $D[x]$, then $\mathrm{Rad}(Q)$ is a prime one. Thus, by Theorem 1.1, $\mathrm{Rad}(Q) = \langle \psi(x) \rangle$ or $\mathrm{Rad}(Q) = \langle \varphi(x), p \rangle$. A primary ideal Q is called *primary of type 1* (resp., *primary of type 2*) if $\mathrm{Rad}(Q)$ is prime of type 1 (resp., prime of type 2).

Theorem 2.1. (Description of primary ideals of type 1) *An ideal Q of $D[x]$ is primary of type 1 if and only if $Q = \langle \psi^n(x) \rangle$, where n is a positive integer.*

Proof. ($\Longrightarrow$) Suppose that Q is a primary ideal of type 1. According to the definition above, $\mathrm{Rad}(Q) = \langle \psi(x) \rangle$. Hence we have $\psi(x) \in \mathrm{Rad}(Q)$, so there exists a positive integer n such that $\psi^n(x) \in Q$. We can consider n as the least positive integer satisfying that condition. Clearly, $\langle \psi^n(x) \rangle \subset Q$. Conversely, if $f(x) \in Q \subset \mathrm{Rad}(Q) = \langle \psi(x) \rangle$, then $f(x) = \psi(x) f_1(x)$, where $f_1(x) \in D[x]$. By induction, $f(x)$ can be written in the form

$$f(x) = \psi^i(x) f_i(x),$$

where $1 \leq i \leq n$ and $f_i(x) \in D[x]$.

When $i = n$, we have $f(x) = \psi^n(x) f_n \in \langle \psi^n(x) \rangle$. Hence $Q \subset \langle \psi^n(x) \rangle$, whence $Q = \langle \psi^n(x) \rangle$.

($\Longleftarrow$) Suppose that $Q = \langle \psi^n(x) \rangle$, where $n > 0$. It is easy to prove Q is primary and $\mathrm{Rad}(Q) = \langle \psi(x) \rangle$. $\square$

Theorem 2.2. (Description of primary ideals of type 2) *An ideal Q of $D[x]$ is primary of type 2 if and only if Q can be written in the form*

$$Q = \langle \varphi^m(x), p^n, \varphi(x) h_1(x) + p k_1(x), \ldots, \varphi(x) h_t(x) + p k_t(x) \rangle,$$

where m, n are positive integers, t is a non-negative integer, $h_i(x), k_i(x) \in D[x]$ $(1 \leq i \leq t)$. In addition, $\mathrm{Rad}(Q) = \langle \varphi(x), p \rangle$ and we can choose $h_i(x), k_i(x)$ such that

$$\deg h_i(x) \leq \deg \varphi^{m-1}(x),$$
$$\deg k_i(x) \leq \deg \varphi^m(x).$$

Proof. ($\Longrightarrow$) Since Q is a primary ideal of type 2, $\mathrm{Rad}(Q) = \langle \varphi(x), p \rangle$, whence there are positive integers m, n such that $\varphi^(x), p^n \in Q$. Because $D[x]$ is a Noetherian ring, Q is finitely generated

$$Q = \langle f_1(x), f_2(x), \ldots, f_r(x) \rangle,$$

where $r \geq 1$. Therefore,

$$Q = \langle \varphi^m(x), p^n, f_1(x), f_2(x), \ldots, f_r(x) \rangle.$$

Since $Q \subset \mathrm{Rad}(Q)$, $f_i(x) \in \mathrm{Rad}(Q) = \langle \varphi(x), p \rangle$. Hence $f_i(x) = \varphi(x)h_i(x) + pk_i(x)$, where $h_i(x), k_i(x) \in D[x]$.

($\Longleftarrow$) Since $1 \notin Q$, $Q \neq D[x]$. Because $\varphi^m(x), p^n \in Q$, $\langle \varphi(x), p \rangle \subset \mathrm{Rad}(Q)$, whence $\mathrm{Rad}(Q) = \langle \varphi(x), p \rangle$. Therefore, Q is primary of type 2. Note that we can choose $\varphi(x)$ such that it is a monic polynomial, so $\varphi^m(x), \varphi^{m-1}(x)$ are monic. The last statement in the theorem comes from dividing $h_i(x)$ by $\varphi^{m-1}(x)$ and $k_i(x)$ by $\varphi^m(x)$. □

In view of Theorem 2.2, we have the following corollary

Corollary 2.1. *If an ideal Q of $D[x]$ satisfies the condition* $\mathrm{Rad}(Q) = \langle \varphi(x), p \rangle$, *then Q is primary of type 2, and furthermore*

(i) *if $p \in Q$, then $Q = \langle \varphi^n(x), p \rangle$;*

(ii) *if $\varphi(x) \in Q$, then $Q = \langle \varphi(x), p^n \rangle$,*

where n is a positive integer.

Because $D[x]$ is a commutative Noetherian ring with identity, every irreducible ideal is primary. Therefore, in order to describe all irreducible ideals of D[x], we only need to consider ideals in the set of primary ideals. An irreducible ideal Q of $D[x]$ is called *irreducible of type 1* (resp., *irreducible of type 2*) if Q is primary of type 1 (resp., primary of type 2).

Theorem 2.3. (Description of irreducible ideals of type 1) *Let Q be an ideal of $D[x]$. Then, Q is irreducible of type 1 if only if Q is primary of type 1.*

Proof. It is sufficient to prove the converse. Let Q be a primary ideal of type 1. Suppose that Q is reducible. Since Q is a primary ideal of type 1, $Q = \langle [\psi^k(x) \rangle$ and $\mathrm{Rad}(Q) = \langle \psi(x) \rangle$. Because Q is reducible in the Noetherian ring $D[x]$, there are primary ideals $Q_1, Q_2, Q_1 \neq Q \neq Q_2$ such that $Q = Q_1 \cap Q_2$ and $\mathrm{Rad}(Q_1) = \mathrm{Rad}(Q_2) = \langle \psi(x) \rangle$. By Theorem 2.1, there are positive integers r, s such that $Q_1 = \langle \psi^r(x) \rangle$, $Q_2 = \langle \psi^s(x) \rangle$. Without loss of generality, we can assume that $r \geq s$, so $Q_1 \subset Q_2$. Thus, $Q = Q_1$, a contradiction. Therefore Q is irreducible. Furthermore, since Q is a primary ideal of type 1, Q is irreducible of type 1. □

Thus, by Theorem 2.3, every primary ideal of type 1 is irreducible. However, a primary ideal of type 2 is not necessarily irreducible. For instance, $Q = \langle x^2, px, p^2 >$ is a primary ideal of type 2 in $\mathbb{Z}[x]$. Since it can be written in the form

$$Q = \langle x, p^2 \rangle \cap \langle x^2, p \rangle,$$

it is reducible.

In order to describe the structure of irreducible ideals of type 2, we need the following lemmas

Lemma 2.1. *If $Q = \langle \varphi^r(x), p^s \rangle$, with $r, s > 0$, then r, s are respectively the least positive integers such that $\varphi^r(x)$, $p^s \in Q$.*

We omit the proof, which is trivial.

Lemma 2.2. *Let Q be a primary ideal of type 2 with* $\mathrm{Rad}(Q) = \langle \varphi(x), p\rangle$. *Let m, n be the least positive integers such that $\varphi^m(x), p^n \in Q$. Then there are positive integer k, l with $k \le m$, $l \le n$ satisfying*

$$Q : \langle \varphi^{m-1}(x)\rangle = \langle \varphi(x), p^k\rangle,$$
$$Q : \langle p^{n-1}\rangle = \langle \varphi^l(x), p\rangle.$$

Proof. We prove the first equality. If $m = 1$, then

$$\begin{aligned} Q : \langle [\varphi(x)]^{m-1}\rangle &= Q : R \\ &= Q. \end{aligned}$$

Since $\varphi^1(x) \in Q$, $Q = \langle \varphi(x), p^r\rangle$, with $r \ge 0$, by Corollary 2.1. Moreover, it is easy to see that r is the least positive integer such that $\varphi^r(x) \in Q$, whence $r = n$.

If $m \ge 1$, then $\varphi(x) \in Q : \langle \varphi^{m-1}(x)\rangle$. Since $\varphi^{m-1}(x) \notin Q$, $Q : \varphi^{m-1}(x)$ is $\langle \varphi(x), p\rangle$-primary. By Corollary 2.1, we have

$$Q : \varphi^{m-1}(x) = \langle \varphi(x), p^k\rangle,$$

where k is a positive integer. Obviously, k is the least positive integer such that $p^k \in \langle \varphi(x), p^k\rangle$. Besides, $p^n \in Q \subset Q : \varphi^{m-1}(x) = \langle \varphi(x), p^k\rangle$, so $k \le n$. The first equality is completely proved.

The second equality is proved similarly. □

Lemma 2.3. *Let Q be an ideal of $D[x]$ satisfying $\varphi^r(x), p^s \in Q$, where r, s are positive integers; moreover, $Q : \langle \varphi^{r-1}(x)\rangle = \langle \varphi(x), p^s\rangle$ or $Q : \langle p^{s-1}\rangle = \langle \varphi^r(x), p\rangle$. Then $Q = \langle \varphi^r(x), p^s\rangle$.*

Proof. It suffices to prove that $Q \subset \langle \varphi^r(x), p^s\rangle$.

We consider the case of $Q : \langle \varphi^{r-1}(x)\rangle = \langle \varphi(x), p^s\rangle$. If $f(x) \in Q$, then $f(x)Q : \langle \varphi^{r-1}(x)\rangle = \langle \varphi(x), p^s\rangle$, whence $f(x) = \varphi(x)u_1(x) + p^s v_1(x)$, where $u_1(x), v_1(x) \in D[x]$. By induction on i ($1 \le i \le r$), we have the equality

$$f(x) = \varphi^i(x)u_i(x) + p^s v_i(x),$$

where $u_i(x), v_i(x) \in D[x]$.

With $i = r$, we have

$$f(x) = \varphi^r(x)u_r(x) + p^s v_r(x) \in \langle \varphi^r(x), p^s\rangle.$$

Therefore $Q \subset \langle \varphi^r(x), p^s\rangle$.

The case $Q : \langle p^{s-1}\rangle = \langle \varphi^r(x), p\rangle$ is proved similarly.

Summing up, $Q = \langle \varphi^r(x), p^s\rangle$ and the proof is completed. □

Lemma 2.4. *The ideal $Q = \langle \varphi(x), p^n\rangle$ is irreducible in $D[x]$ for every positive integer n.*

Proof. Suppose that Q is reducible, or equivalently, there are two ideals Q_1, Q_2 such that

$$Q = Q_1 \cap Q_2,$$

where $Q_1 \neq Q \neq Q_2$.

Since $\varphi(x)$, $p^n \in Q$, $\varphi(x)$, $p^n \in Q_1$, Q_2, whence Q_1, Q_2 are primary and $\mathrm{Rad}(Q_1) = \langle \varphi(x), p\rangle = \mathrm{Rad}(Q_2)$. By Corollary 2.1, there are positive integer n_1, n_2 such that

$$\begin{aligned} Q_1 &= \langle \varphi(x), p^{n_1}\rangle, \\ Q_2 &= \langle \varphi(x), p^{n_2}\rangle. \end{aligned}$$

Thus,

$$\langle \varphi(x), p^n\rangle = \langle \varphi(x), p^{n_1}\rangle \cap \langle \varphi(x), p^{n_2}\rangle.$$

Without loss of generality, we may assume $n_1 \geq n_2$, whence $Q_1 \subset Q_2$. Therefore $Q = Q_1$, a contradiction. The proof is completed. □

Lemma 2.5. *The ideal $Q = \langle \varphi^m(x), p^n\rangle$ is irreducible in $D[x]$ for every positive integers m, n.*

Proof. The case $m = 1$ results from Lemma 2.4.

Now let $m > 1$. By Lemma 2.1 and Lemma 2.2, we have

$$Q : \langle \varphi^{m-1}(x)\rangle = \langle \varphi(x), p^k\rangle,$$

where $k \leq n$. We are going to prove $k = n$. Suppose that $k < n$. Since $p^k \in \langle \varphi(x), p^k\rangle = Q : \langle \varphi^{m-1}(x)]\rangle$, $p^k\varphi^{m-1}(x)] \in Q$, whence there are $f(x), g(x) \in D[x]$ such that

$$p^k\varphi^{m-1}(x) = \varphi^m(x)f(x) + p^n g(x),$$

or

$$p^k(\varphi^{m-1}(x) - p^{n-k}g(x)) = \varphi^m(x)f(x).$$

Hence p^k divides $\varphi^m(x)f(x)$. Since p^k and $\varphi^m(x)$ are relatively prime, p^k divides $f(x)$, whence there exists $h(x) \in D[x]$ such that

$$f(x) = p^k h(x).$$

Therefore

$$p^k(\varphi^{m-1}(x) - p^{n-k}g(x)) = p^k(\varphi^m(x)h(x)),$$

or equivalently,

$$\varphi^{m-1}(x) - p^{n-k}g(x) = \varphi^m(x)h(x),$$

so

$$\varphi^{m-1}(x) + pD[x] = \varphi^m(x)h(x) + pD[x],$$

or

$$(\varphi(x) + pD[x])^{m-1} = (\varphi(x) + pD[x])^m(h(x) + pD[x]),$$

whence

$$1 + pD[x](\varphi(x) + pD[x])(h(x) + pD[x]).$$

The last equality means $\varphi(x) + pD[x]$ is unit, a contradiction. Thus $k = n$.

We will now prove Q is irreducible. Suppose, on the contrary, that Q is reducible. Then there exist Q_1, Q_2, ideals of $D[x]$, such that

$$Q = Q_1 \cap Q_2,$$

where $Q_1 \neq Q \neq Q_2$. Thus

$$Q : \langle(\varphi(x))^{m-1}\rangle = (Q_1 : \langle(\varphi(x))^{m-1}\rangle) \cap (Q_2 : \langle(\varphi(x))^{m-1}\rangle),$$

or

$$\langle\varphi(x), p^n\rangle = (Q_1 : \langle(\varphi(x))^{m-1}\rangle) \cap (Q_2 : \langle(\varphi(x))^{m-1}\rangle).$$

Since $\langle\varphi(x), p^n\rangle$ is irreducible (Lemma 2.4), either $Q_1 : \langle(\varphi(x))^{m-1}\rangle = \langle\varphi(x), p^n\rangle$ or $Q_2 : \langle(\varphi(x))^{m-1}\rangle = \langle\varphi(x), p^n\rangle$. Furthermore, since $\varphi^m(x)$, $p^n \in Q_1$, Q_2, $Q_1 = \langle\varphi^m(x), p^n\rangle$ or $Q_2 = \langle\varphi^m(x), p^n\rangle$ (Lemma 2.3). Therefore $Q_1 = Q$ or $Q_2 = Q$, a contradiction. Thus, Q is irreducible in $D[x]$. □

Does every irreducible ideal Q of type 2 with $\mathrm{Rad}(Q) = \langle\varphi(x), p\rangle$ have the form of $\langle\varphi^m(x), p^n\rangle$, in sense of Lemma 2.5 ? The answer is affirmative and we need some more lemmas.

Lemma 2.6. *Consider the local ring R_P, where $R = D[x]$ and $P = \langle\varphi(x), p\rangle$. Then*

$$\langle\varphi^k(x), p^l\rangle_P = \langle\varphi^{k'}(x), p^{l'}\rangle_P \text{ if and only if } k = k' \text{ and } l = l'.$$

Lemma 2.7. *If $Q = \langle\varphi^r(x), p^s\rangle$, where r, s are two positive integers, then*

$$Q : \langle p\rangle = \langle\varphi^r(x), p^{s-1}\rangle,$$
$$Q : \langle\varphi(x)\rangle = \langle\varphi^{r-1}(x), p^s\rangle.$$

Moreover, if M is a multiplicative set of $D[x]$, then

$$Q_M : \langle p\rangle_M = \langle\varphi^r(x), p^{s-1}\rangle_M,$$
$$Q_M : \langle\varphi(x)\rangle_M = \langle\varphi^{r-1}(x), p^s\rangle_M.$$

We omit the proofs of the above lemmas, which are easy.

The two following lemmas are in [1]

Lemma 2.8. *If R is a commutative local Noetherian ring with identity and Q is an irreducible ideal of R such that $\mathrm{Rad}(Q)$ is maximal, then*

$$Q : (Q : A) = A$$

for every ideal A containing Q.

Lemma 2.9. *Let R be a commutative local Noetherian ring with identity, Q be an irreducible ideal of R, and A be an ideal of R containing Q. Then A is irreducible if and only if $Q : A$ is principal mod Q.*

Lemma 2.10. *If Q is an irreducible ideal of $D[x]$ with $P = \mathrm{Rad}(Q) = \langle \varphi(x), p \rangle$, then there are positive integers m, n such that*

$$Q = \langle \varphi^m(x), p^n \rangle.$$

Proof. We put

$$\mathcal{S} = \big\{ \ Q : Q \text{ is irreducible in } D[x], \ \mathrm{Rad}(Q) = \langle \varphi(x), p \rangle, \ Q \neq \langle \varphi^a(x), p^b \rangle \\ \text{for any positive integers } a, b \big\}.$$

We are going to show $\mathcal{S} = \emptyset$. Suppose, on the contrary, that $\mathcal{S} \neq \emptyset$. Since $D[x]$ is Noetherian ring, $\mathcal{S}$ has a maximal element, namely Q_0. Because $\mathrm{Rad}(Q_0) = \langle \varphi(x), p \rangle$, there are positive integers m, n such that $\varphi^m(x)$, $p^n \in Q_0$, and we consider m, n as the least positive integers satisfying these properties.

If $m = 1$, then $Q_0 = \langle \varphi(x), p^n \rangle$ by Corollary 2.1, a contradiction.

If $n = 1$, then $Q_0 = \langle \varphi^m(x), p \rangle$ by Corollary 2.1, a contradiction.

If $m > 1$ and $n > 1$ then

$$Q_0 : \langle \varphi^{m-1}(x) \rangle = \langle \varphi(x), p^k \rangle, \tag{3}$$

$$Q_0 : \langle p^{n-1} \rangle = \langle \varphi^l(x), p \rangle, \tag{4}$$

where $1 \leq k \leq n$ and $1 \leq l \leq m$ (Lemma 2.2).

If $k = n$ or $l = m$, $Q_0 = \langle \varphi^m(x), p^n \rangle$ by Lemma 2.3, a contradiction.

We consider the case $k < n$ and $l < m$.

We put

$$\mathcal{S}' = \big\{ Q_P : Q_P \text{ is irreducible in } (D[x])_P, \ \mathrm{Rad}(Q_P) = (\langle \varphi(x), p \rangle)_P, \text{ and} \\ Q_P \neq (\langle \varphi^a(x), p^b \rangle)_P \text{ for any positive integer } a, b \big\}.$$

We have $(Q_0)_P$ is a maximal element of $\mathcal{S}'$. On the contrary, suppose there exists $(Q_1)_P$ belonging to $\mathcal{S}'$ such that $(Q_0)_P \subset (Q_1)_P$ and $(Q_0)_P \neq (Q_1)_P$. Firstly, we prove $Q_1 \in \mathcal{S}$. Because $(Q_1)_P$ is irreducible, Q_1 is irreducible and $Q_1 \subset P$, whence $\mathrm{Rad}(Q_1) \subset P$. Furthermore, since $\mathrm{Rad}((Q_1)_P) = (\mathrm{Rad}(Q_1))_P = (\langle \varphi(x), p \rangle)_P$ and $\mathrm{Rad}(Q_1)$ is prime, we have $\mathrm{Rad}(Q_1) = \langle \varphi(x) \rangle, p)$. Moreover, $Q_1 \neq \langle \varphi^a(x), p^b \rangle$ for any positive integers a, b. Otherwise, $(Q_1)_P = (\langle \varphi^{a_0}(x), p^{b_0} \rangle)_P$, a contradiction. Thus $Q_1 \in \mathcal{S}$.

We prove $Q_0 \subset Q_1$. Suppose that $Q_0 \not\subset Q_1$. Then there exists $u \in Q_0 \setminus Q_1$, whence $\dfrac{u}{1} \in (Q_0)_P$, so $\dfrac{u}{1} \in (Q_1)_P$. Hence there are $g \in Q_1$, $h \notin P$ such that $\dfrac{u}{1} = \dfrac{g}{h}$. Thus $uh = g \in Q_1$. Since Q_1 is primary and $h \notin P = \mathrm{Rad}(Q_1)$, $u \in Q_1$, a contradiction. Thus $Q_0 \subset Q_1$.

Since Q_0 is a maximal element of $\mathcal{S}$, $Q_0 = Q_1$, whence $(Q_0)_P = (Q_1)_P$. Thus $(Q_0)_P$ is a maximal element of $\mathcal{S}'$.

Next, we prove that there are positive integers r, s, u, v such that

$$(Q_0 : \langle p\rangle)_P = \langle \varphi^r(x), p^s\rangle_P, \tag{5}$$

$$(Q_0 : \langle \varphi(x)\rangle)_P = \langle \varphi^u(x), p^v\rangle_P. \tag{6}$$

Since m, n are the least positive integers such that $\varphi^m(x), p^n \in Q$ and $m, n > 1$,

$$\begin{array}{lll} (Q_0)_P & \subseteq & (Q_0 : \langle p\rangle)_P, \qquad (Q_0)_P \neq (Q_0 : \langle p\rangle)_P, \\ (Q_0)_P & \subseteq & (Q_0 : \langle \varphi(x)\rangle)_P, \quad (Q_0)_P \neq (Q_0 : \langle \varphi(x)\rangle)_P. \end{array}$$

Since $(Q_0)_P$ is irreducible, by Lemma 2.8,

$$(Q_0)_P : [(Q_0)_P : (Q_0 + \langle p\rangle)_P] = (Q_0 + \langle p\rangle)_P,$$

or

$$(Q_0)_P : [(Q_0)_P : (Q_0 + \langle p\rangle)_P] = (Q_0)_P + \langle p\rangle_P.$$

By the last equality and Lemma 2.9, $(Q_0)_P : (Q_0 + \langle p\rangle)_P$ is irreducible. Besides

$$\begin{aligned} (Q_0)_P : (Q_0 + \langle p\rangle)_P &= (Q_0)_P : [(Q_0)_P + \langle p\rangle_P] \\ &= [(Q_0)_P : (Q_0)_P] \cap [(Q_0)_P : \langle p\rangle_P] \\ &= (Q_0)_P : \langle p\rangle_P, \end{aligned}$$

so $(Q_0)_P : \langle p\rangle_P$ is irreducible. Because $\mathrm{Rad}(Q_0)_P = \langle \varphi(x), p\rangle_P$, $\mathrm{Rad}((Q_0)_P :< p >_P) = \langle \varphi(x), p\rangle_P$. Thus, $(Q_0 : \langle p\rangle)_P$ is irreducible in $(D[x])_P$ and $\mathrm{Rad}(Q_0 : \langle p\rangle)_P = \langle \varphi(x), p\rangle$. Furthermore, since $(Q_0)_P$, a maximal element of $\mathcal{S}'$, is properly contained in $(Q_0 : \langle p\rangle)_P$, there are positive integers r, s such that

$$(Q_0 : \langle p\rangle)_P = \langle \varphi^r(x), p^s\rangle_P.$$

The last equality is (5). Similarly, we have (6).

Next, we will prove

$$(Q_0 : \langle p^i\rangle)_P = \langle \varphi^r(x), p^{s-i+1}\rangle_P$$

by induction on i ($1 \leq i \leq n-1$).

With $i = 1$, by (5), we have the equality. Suppose that the equality is true with $i = j$ ($1 \leq j \leq n-2$), or equivalently

$$(Q_0 : \langle p^j\rangle)_P = \langle \varphi^r(x), p^{s-j+1}\rangle_P.$$

Hence $\dfrac{p^{s-j+1}}{1}\dfrac{p^j}{1} = \dfrac{p^{s+1}}{1} \in (Q_0)_P$. Therefore, $\dfrac{p^{s+1}}{1} = \dfrac{f(x)}{w(x)}$, where $f(x) \in Q_0$, $w(x) \in D[x] \setminus P$, or we have $p^{s+1}w(x) = f(x) \in Q_0$. Since Q_0 is primary and $w(x) \notin \mathrm{Rad}(Q_0) = P$, $p^{s+1} \in Q_0$. Since n is the least positive integer such that $p^n \in Q_0$, we have $s + 1 \geq n$. For $1 \leq j \leq n-2$, $(s+1) - j = s - j + 1 \geq 2$, and

$$\begin{aligned}(Q_0 : \langle p^{j+1}\rangle)_P &= [(Q_0 : \langle p^j\rangle) : \langle p\rangle]_P \\ &= (Q_0 : \langle p^j\rangle)_P : \langle p\rangle_P \\ &= \langle \varphi^r(x), p^{s-j+1}\rangle_P : \langle p\rangle_P \\ &= \langle \varphi^r(x), p^{s-j}\rangle_P.\end{aligned}$$

Thus, the equality occurs in the case of $i = j+1$, whence it is proved. With $i = n-1$, we have

$$(Q_0 : \langle p^{n-1}\rangle)_P = \langle \varphi^r(x), p^{s-n+2}\rangle_P. \tag{7}$$

Similarly, by (6) and induction,

$$(Q_0 : \langle \varphi^{m-1}(x)\rangle)_P = \langle \varphi^{u-m+2}(x), p^v\rangle_P. \tag{8}$$

By (3), (4), (7), and (8), we have

$$\langle \varphi^\ell(x), p\rangle_P = \langle \varphi^r(x), p^{s-n+2}\rangle_P$$
$$\langle \varphi(x), p^k\rangle_P = \langle \varphi^{u-m+2}(x), p^v\rangle_P$$

By Lemma 2.6, we have

$$r = l, \quad s-n+2 = 1, \quad v = k, \quad u-m+2 = 1.$$

Or

$$r = l, \quad s = n-1, \quad v = k, \quad u = m-1. \tag{9}$$

By (5), (6), and (9), we have

$$\begin{aligned}(Q_0 : \langle p\rangle)_P &= \langle \varphi^l(x), p^{n-1}\rangle_P, \\ (Q_0 : \langle \varphi(x)\rangle)_P &= \langle \varphi^{m-1}(x), p^k\rangle_P.\end{aligned}$$

By the equalities above and Lemma 2.7,

$$\begin{aligned}(Q_0 : \langle p\rangle)_P : \langle \varphi(x)\rangle_P &= \langle \varphi^{l-1}(x), p^{n-1}\rangle_P, \\ (Q_0 : \langle \varphi(x)\rangle)_P : \langle p\rangle_P &= \langle \varphi^{m-1}(x), p^{k-1}\rangle_P.\end{aligned}$$

Since $D[x]$ is commutative,

$$(Q_0 : \langle p\rangle)_P : \langle \varphi(x)\rangle_P = (Q_0 : \langle \varphi(x)\rangle)_P : \langle p\rangle_P,$$

whence

$$\langle \varphi^{l-1}(x), p^{n-1}\rangle_P = \langle \varphi^{m-1}(x), p^{k-1}\rangle_P.$$

By the last equality and Lemma 2.6, we have

$$l-1 = m-1, \quad n-1 = k-1.$$

Or $l = m$, $k = n$, a contradiction with our case.

Summing up, $\mathcal{S} = \emptyset$ and the proof is completed. □

By Lemma 2.1, Lemma 2.5, and Lemma 2.10, we have

Theorem 2.4. (Description of irreducible ideals of type 2) *An ideal Q of $D[x]$ is irreducible of type 2 if and only if*

$$Q = \langle \varphi^m(x), p^n \rangle,$$

where m, n are positive integers. Furthermore, $\mathrm{Rad}(Q) = \langle \varphi(x), p \rangle$*, and m, n are the least positive integers such that $\varphi^m(x), p^n \in Q$*

Thus, the structure of irreducible ideals in $D[x]$ is described clearly. By this description, each primary ideal of type 1 is irreducible of type 1. A primary ideal of type 2, however, does not have to be an irreducible ideal of type 2. The following consequence of Theorem 2.4 shows another description of primary ideals of type 2, beside the description of Theorem 2.2.

Corollary 2.2. *An ideal Q of $D[x]$ is primary of type 2 if and only if*

$$Q = \bigcap_i \langle \varphi^{m_i}(x), p^{n_i} \rangle,$$

where i runs on a finite set of positive integer numbers. Moreover, $\mathrm{Rad}(Q) = \langle \varphi(x), p \rangle$.

3. The structure of ideals in $D[x]$

In this section, we will describe the structure of an arbitrary ideal of $D[x]$. Since $D[x]$ is commutative and Noetherian, each ideal of $D[x]$ is an intersection of a finite number of irreducible ideals. Therefore, combining with the theorem describing the structure of irreducible ideals in $D[x]$, we have following theorem about the structure of an arbitrary ideal of $D[x]$.

Theorem 3.1. *An ideal I of $D[x]$ can be represented by the form*

$$I = \Big(\bigcap_{i=1}^{r} \langle [\psi_i(x)]^{k_i} \rangle \Big) \bigcap \Big(\bigcap_{j=1}^{s} \langle [\varphi_j(x)]^{m_j}, p_j^{n_j} \rangle \Big),$$

where r, s, k_i $(1 \le i \le r)$, m_j, n_j $(1 \le j \le s)$ are positive integers, $\psi_i(x)$ is irreducible in $D[x]$, p_j is nonzero prime in D, $\varphi_j(x) \in D[x]$ such that $\varphi_j(x) + p_j D[x]$ is irreducible in $D[x]/p_j D[x]$.

Remark. The representation of ideal I in Theorem 3.1 is not unique. For example, with $D = \mathbb{Z}$, in $\mathbb{Z}[x]$, ideal I can be written in two distinct forms

$$I = \langle p \rangle \cap \langle x - p, p^2 \rangle$$

and

$$I = \langle p \rangle \cap \langle x - 2p, p^2 \rangle.$$

We are going to consider a unique representation of an arbitrary ideal of $D[x]$ via primary ideals. First, we have some lemmas.

Lemma 3.1. *Let $I_1, I_2, \ldots, I_n$ be primary ideals of the same type (either type 1 or type 2) such that* $\mathrm{Rad}(I_i)$ $(1 \le i \le n)$ *are distinct. Then*

$$I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n.$$

Proof. Clearly, we have

$$I_1 I_2 \cdots I_n \subset I_1 \cap I_2 \cap \cdots \cap I_n.$$

If I_i $(1 \le i \le n)$ are primary ideals of type 1, then $I_i = \langle [\psi_i(x)]^{k_i} \rangle$, where $\psi_i(x)$ are irreducible and k_i are positive. Since $\mathrm{Rad}(I_i)$ $(1 \le i \le n)$ are distinct, $\psi_i(x)$ $(1 \le i \le n)$ are pairwise prime. If $f(x) \in I_1 \cap I_2 \cap \cdots \cap I_n$, then $f(x) \in I_i$, whence $[\psi_i(x)]^{k_i}$ divides $f(x)$ for every $1 \le i \le n$. Thus, $\prod\limits_{i=1}^{n} [\psi_i(x)]^{k_i}$ divides $f(x)$ and hence

$$f(x) \in \langle \prod_{i=1}^{n} [\psi_i(x)]^{k_i} \rangle = \prod_{i=1}^{n} \langle \psi_i(x) \rangle = \prod_{i=1}^{n} I_i.$$

Therefore

$$I_1 \cap I_2 \cap \cdots \cap I_n \subset I_1 I_2 \ldots I_n$$

and hence

$$I_1 \cap I_2 \cap \cdots \cap I_n = I_1 I_2 \ldots I_n.$$

If I_i $(1 \le i \le n)$ are primary ideals of type 2, then $\mathrm{Rad}(I_i)$ $(1 \le i \le n)$ are maximal in $D[x]$. Furthermore, since $\mathrm{Rad}(I_i)$ $(1 \le i \le n)$ are distinct, they are pairwise prime, whence

$$I_1 \cap I_2 \cap \cdots \cap I_n = I_1 I_2 \ldots I_n.$$

□

Lemma 3.2. *If I, J are P-primary ideals of $D[x]$, then IJ is a P-primary one.*

Lemma 3.3. *If I is a principal ideal of $D[x]$, then I can be uniquely written as*

$$I = I_1 I_2 \ldots I_n,$$

where n is a positive integer, and I_i $(1 \le i \le n)$ are primary ideals of type 1 such that $\mathrm{Rad}(I_i)$ $(1 \le i \le n)$ *are distinct.*

We omit the proofs, which are easy.

Lemma 3.4. *Let J be an ideal of $D[x]$. If J is not contained in any nontrivial principal ideal, then J can be uniquely written in the form*

$$J = J_1 J_2 \ldots J_k,$$

where k is positive and J_i $(1 \le i \le k)$ are primary ideals of type 2 such that $\mathrm{Rad}(J_i)$ $(1 \le i \le k)$ *are distinct.*

Proof. Consider the irredundant primary representation of J

$$J = J_1 \cap J_2 \cap \cdots \cap J_k.$$

Since J is not contained in any nontrivial principal ideal, also J_i $(1 \leq i \leq k)$. Therefore J_i $(1 \leq i \leq k)$ are primary ideals of type 2. By Lemma 3.1,

$$J = J_1 J_2 \dots J_k.$$

We claim that this representation of J is unique. In fact, suppose that there are $J'_1, J'_2, \dots, J'_l$ such that

$$J = J'_1 J'_2 \cdots J'_l,$$

where J'_j $(1 \leq j \leq l)$ are primary ideals of type 2 and $\mathrm{Rad}(J'_j)$ $(1 \leq j \leq l)$ are distinct. By Lemma 3.1,

$$J'_1 J'_2 \cdots J'_l = J'_1 \cap J'_2 \cap \cdots \cap J'_l,$$

whence

$$J_1 \cap J_2 \cap \cdots \cap J_k = J = J'_1 \cap J'_2 \cap \cdots \cap J'_l.$$

Since $\mathrm{Rad}(J'_j)$ $(1 \leq j \leq l)$ are distinct, $J = J'_1 \cap J'_2 \cap \cdots \cap J'_l$ is an irredundant primary representation of J. Indeed, it is necessary to check that $\bigcap\limits_{1 \leq j \leq l, j \neq i} J'_j \not\subset J'_i$. Suppose, on the contrary, that $\bigcap\limits_{1 \leq j \leq l, j \neq i} J'_j \subset J'_i$, then $\bigcap\limits_{1 \leq j \leq l, j \neq i} \mathrm{Rad}(J'_j) \subset \mathrm{Rad}(J'_i)$. Since $\mathrm{Rad}(J'_i)$ is maximal (and then, prime), there exists j_0 such that $\mathrm{Rad}(J'_{j_0}) = \mathrm{Rad}(J'_i)$, a contradiction. Thus, J has two irredundant primary representations. Because $\mathrm{Rad}(J_j)$ $(1 \leq j \leq k)$ are maximal, they are isolated prime ideals, whence $k = l$ and $J_i = J'_i$ (after reordering if necessary) for every $1 \leq i \leq k$. □

Lemma 3.5. *Let I be an ideal of $D[x]$. Then, there exist uniquely ideals I', I'' such that*

$$I = I' I'',$$

where I' is principal and I'' is not contained in any nontrivial principle ideal.

Proof. Since $D[x]$ is a Noetherian ring, I is finitely generated

$$I = \langle f_1(x), f_2(x), \dots, f_n(x) \rangle.$$

Put $f(x) = \mathrm{g.c.d}(f_1(x), f_2(x), \dots, f_n(x))$. We claim that

$$I = \langle f(x) \rangle (I : \langle f(x) \rangle).$$

Clearly, $I \subset \langle f(x) \rangle$, so if $g(x) \in I$, then $g(x) \in \langle f(x) \rangle$, whence $g(x) = f(x)h(x)$, where $h(x) \in D[x]$, and hence $h(x) \in I : \langle f(x) \rangle$. Thus $g(x) \in \langle f(x) \rangle (I : \langle f(x) \rangle)$ and then $I \subset \langle f(x) \rangle (I : \langle f(x) \rangle)$. Conversely, if $g(x) \in \langle f(x) \rangle (I : \langle f(x) \rangle)$ then $g(x) = g_1(x) g_2(x)$, where $g_1(x) \in \langle f(x) \rangle$, $g_2(x) \in I : \langle f(x) \rangle$. Therefore $g_1(x) = h(x) f(x)$ and then $g(x) = h(x)(f(x) g_2(x)) \in I$. Thus, $f(x)(I : \langle f(x) \rangle) \subset I$ and hence $I = f(x)(I : \langle f(x) \rangle)$.

We claim that $I : \langle f(x)\rangle$ is not contained in any nontrivial principal ideal. Suppose, on the contrary, that $I : \langle f(x)\rangle \subset \langle g(x)\rangle$, where $g(x)$ is nonzero and nonunit in $D[x]$. Since $f(x)\dfrac{f_i(x)}{f(x)} = f_i(x) \in I$, $\dfrac{f_i(x)}{f(x)} \in I : \langle f(x)\rangle$ for every $1 \le i \le n$. Therefore $\dfrac{f_i(x)}{f(x)} \in \langle g(x)\rangle$, or equivalently, $g(x)$ divides $\dfrac{f_i(x)}{f(x)}$ for every $1 \le i \le n$, whence $g(x)$ divides g.c.d$\Big\{\dfrac{f_1(x)}{f(x)}, \dfrac{f_1(x)}{f(x)}, \cdots, \dfrac{f_n(x)}{f(x)}\Big\} = 1$, a contradiction. Put $I' = \langle f(x)\rangle$, $I'' = I : \langle f(x)\rangle$, then $I = I'I''$.

Next, we claim that the representation $I = I'I''$ is unique. Suppose that $I = J'J''$, where J' is principal, J'' is not contained in any nontrivial principal ideal. Set $J' = \langle g(x)\rangle$. Let $\psi(x)$ is an irreducible factor of $g(x)$ and α is the greatest positive integer such that $\psi^\alpha(x)$ divides $g(x)$. Since $I'' \not\subset \langle \psi(x)\rangle$, there exists $u(x) \in I''$ and $u(x) \notin \langle \psi(x)\rangle$. Since $f(x)u(x) \in f(x)I'' = g(x)J''$ and $g(x)J'' \subset \langle \psi^\alpha(x)\rangle$, $f(x)u(x) \in \langle \psi^\alpha(x)\rangle$, or equivalently, $\psi^\alpha(x)$ divides $f(x)u(x)$. Because $u(x)$ and $\psi^\alpha(x)$ are pairwise prime, $\psi^\alpha(x)$ divides $f(x)$. Thus, $\psi^\alpha(x)$ divides $f(x)$ if $\psi^\alpha(x)$ divides $g(x)$ for every irreducible factor $\psi(x)$ of $g(x)$ and hence $g(x)$ divides $f(x)$. Similarly, $f(x)$ divides $g(x)$, so $f(x)$ and $g(x)$ are conjugate. Therefore $I' = J'$. Hence, $I'' = J''$, and whence the representation $I = I'I''$ is unique. □

Remark. I'' in the above lemma can equal to $D[x]$, which happens when I is principal.

Theorem 3.2. *Any ideal I of $D[x]$ can be uniquely written in the form*

$$I = I_1 I_2 \dots I_n,$$

where n is a positive integer and I_i $(1 \le i \le n)$ are primary ideals such that $\mathrm{Rad}(I_i)$ $(1 \le i \le n)$ *are distinct.*

Proof. By Lemma 3.5, I can be uniquely written as

$$I = I'I'',$$

where I' is principal and I'' is not contained in any nontrivial principal ideal. By Lemma 3.3, I' can be uniquely written as

$$I' = I_1 I_2 \cdots I_k,$$

where I_i $(1 \le i \le k)$ are primary ideals of type 1 such that $\mathrm{Rad}(I_i)$ $(1 \le i \le k)$ are distinct. By Lemma 3.4, I'' can be uniquely written as

$$I'' = I_{k+1} I_{k+2} \cdots I_n,$$

where I_j $(k+1 \le j \le n)$ are primary ideals of type 2 such that $\mathrm{Rad}(I_j)$ $(k+1 \le j \le n)$ are distinct. Therefore I can be written by

$$I = I_1 I_2 \cdots I_k I_{k+1} \cdots I_n,$$

where I_i $(1 \le i \le n)$ are primary ideals with distinct radicals.

Next, we prove that the representation of x is unique. In fact, suppose that there are $J_1, J_2, \ldots, J_m$ such that

$$I = J_1 J_2 \cdots J_m,$$

where J_j $(1 \leq j \leq m)$ are primary ideals with distinct radials. By reordering if necessary, we may assume that $J_1, J_2, \ldots, J_k$ are primary of type 1, and $J_{k+1}, J_{k+2}, \ldots, J_m$ primary of type 2. Put

$$J' = J_1 J_2 \cdots J_k,$$
$$J'' = J_{k+1} J_{k+2} \cdots J_m.$$

Since J_j $(1 \leq j \leq k)$ are all principal, so is J'. On the other hand, J'' is not contained in any nontrivial principal ideal. In fact, on the contrary, by Lemma 3.3, J'' is contained in some prime ideal of type 1 K. Then

$$\mathrm{Rad}(J'') = \bigcap_{j=k+1}^{m} \mathrm{Rad}(J_j) \subset \mathrm{Rad}(K).$$

Since $\mathrm{Rad}(K)$ is prime, we have $\mathrm{Rad}(J_j) \subset \mathrm{Rad}(K)$ for some $k+1 \leq j \leq m$, and hence, by the maximality of $\mathrm{Rad}(J'')$, $\mathrm{Rad}(I_j) = \mathrm{Rad}(K)$, contrary to the fact that $\mathrm{Rad}(I_j)$ and $\mathrm{Rad}(K)$, are prime of different types. Thus, $I = J'J''$, where J' is principal and J'' is not contained in any nontrivial principal ideal. By the uniqueness of the representation in Lemma 3.5, we have $I' = J'$ and $I'' = J''$. From this, by the uniqueness of the representations in Lemma 3.3 and Lemma 3.4, we conclude that $n = m$ and $I_j = J_j$ for all $1 \leq j \leq n$, hence the representation of I is unique. This completes the proof. □

Corollary 3.1. *Let I be an ideal of $D[x]$ and n be the least positive integer such that*

$$I = I_1 I_2 \cdots I_n,$$

where I_i $(1 \leq i \leq n)$ are primary. Then this representation is unique.

Proof. By Theorem 3.2, we need only to prove $\mathrm{Rad}(I_i)$ $(1 \leq i \leq n)$ are distinct. Suppose, on the contrary, that $\mathrm{Rad}(I_i)$ $(1 \leq i \leq n)$ are not distinct. That means there are integers i_0, j_0 with $1 \leq i_0 \neq j_0 \leq n$ such that $\mathrm{Rad}(I_{i_0}) = \mathrm{Rad}(I_{j_0})$. By Lemma 3.2, $I_{i_0} I_{j_0}$ is primary. Therefore

$$I = (I_{i_0} I_{j_0}) \prod_{1 \leq i \leq n, i \neq i_0, j_0} I_i,$$

which is product of $n-1$ primary ideals, a contradiction. □

Remark. The authors have been informed by the referee that Theorem 3.2 can be deduced from the main results of [2], [3], [4]. In fact, Anderson characterized Noetherian rings in which every ideal is a produced of primary ideals as rings in which every non-maximal ideal is locally pricipal [2] (see [3] for the non-Noetherian case). Later, Anderson and Mahaney gave sufficient conditions for such a product to be unique [3]. For the polynomial ring over a PID, our description of primary and irreducible ideals provide more precise information on the decomposition of ideals.

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