ACTA MATHEMATICA VIETNAMICA
Volume 29, Number 2, 2004, pp. 163-176


# ON SOME CLASSES OF CODES DEFINED BY BINARY RELATIONS

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Abstract. We consider some classes of codes defined by length-increasing transitive binary relations. The embedding problem for these classes of codes is solved for both the finite and regular case. Characterizations of these codes, especially of the maximal ones are established.

## 1. Preliminaries

Defining codes by binary relations was initiated by G. Thierrin and H. Shyr in the middle of 1970s [8]. It appeared that this is a good method in introducing new classes of codes. The idea of this comes from the notion of independent sets in universal algebra [2].

One of the interesting problems in the theory of codes is that of embedding a code in a given class $C$ of codes into a code maximal in the same class (not necessarily maximal as a code) which preserves some property (usually, the finiteness or the regularity) of the given code. This is called the *embedding problem* for the class $C$ of codes.

Until now the answer for the embedding problem is known only for several cases using different combinatorial techniques. In [9] (see also [10]) it is proposed a general embedding schema for the classes of codes, which can be defined by length-increasing transitive binary relations. This allows to solve positively, in a unified way, the embedding problem for many classes of codes well-known as well as new (see [4, 9, 10, 11]).

In this paper, we consider several classes of codes, called p-superinfix codes, s-superinfix codes, superinfix codes and sucypercodes, which can be defined by length-increasing transitive binary relations. Using the approach in [9] we show that the embedding problem for these classes of codes is solved positively for both finite and regular case. Characterizations of these codes, especially of maximal ones are established.

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Received October 31, 2003; in revised form April 4, 2004.

1991 *Mathematics Subject Classification.* 94A45, 68Q45.

*Key words and phrases.* Code, Parikh vector, embedding problem, independent set.

This work was supported by the National Basic Research Program in Natural Sciences, Vietnam.

We now recall some notions, notations and facts, which will be used in the sequel. Let $A$ throughout be a finite alphabet. We denote by $A^*$ the free monoid generated by $A$ whose elements are called *words* on $A$. The empty word is denoted by 1 and $A^+ = A^* - 1$. The number of all occurrences of letters in a word $u$ is the *length* of $u$, denoted by $|u|$.

A language over $A$ is a subset of $A^*$. A non-empty language $X$ is a *code* over $A$ if for all $n, m \geq 1$ and $x_1, \ldots, x_n, y_1, \ldots, y_m \in X$, the condition

$$x_1 x_2 \cdots x_n = y_1 y_2 \cdots y_m,$$

implies $n = m$ and $x_i = y_i$ for $i = 1, \ldots, n$. A code $X$ is *maximal* over $A$ if $X$ is not properly contained in any other code over $A$. Let $C$ be a class of codes over $A$ and $X \in C$. The code $X$ is *maximal in* $C$ (not necessarily maximal as a code) if $X$ is not properly contained in any other code in $C$. For further details of the theory of code we refer to [1, 6, 7].

Given a binary relation $\prec$ on $A^*$. A subset $X$ in $A^*$ is an *independent set* with respect to the relation $\prec$ if any two elements of $X$ are not in this relation. We say that a class $C$ of codes is *defined by* $\prec$ if these codes are exactly the independent sets w.r.t. $\prec$. Then, we denote the class $C$ by $C_\prec$. Very often, the relation $\prec$ characterizes some property $\alpha$ of words. In this case, we write $\prec_\alpha$, instead of $\prec$ and also $C_\alpha$ stands for $C_{\prec_\alpha}$. The relation $\prec$ is said to be *length-increasing* if for any $u, v \in A^*, u \prec v$ implies $|u| < |v|$. We denote by $\preceq$ the reflexive closure of $\prec$, i.e. for any $u, v \in A^*, u \preceq v$ iff $u = v$ or $u \prec v$.

It is easy to verify that the following binary relations on $A^*$ are transitive (except for $\prec_b$) and length-increasing.

$u \prec_p v \Leftrightarrow v = ux$, with $x \neq 1$;

$u \prec_s v \Leftrightarrow v = xu$, with $x \neq 1$;

$u \prec_b v \Leftrightarrow (u \prec_p v) \vee (u \prec_s v)$;

$u \prec_{p.i} v \Leftrightarrow v = xuy$, with $y \neq 1$;

$u \prec_{s.i} v \Leftrightarrow v = xuy$, with $x \neq 1$;

$u \prec_i v \Leftrightarrow v = xuy$, with $xy \neq 1$;

$u \prec_{p.h} v \Leftrightarrow \exists n \geq 1 : u = u_1 \ldots u_n \wedge v = x_0 u_1 \ldots u_n x_n$, with $x_1 \ldots x_n \neq 1$;

$u \prec_{s.h} v \Leftrightarrow \exists n \geq 1 : u = u_1 \ldots u_n \wedge v = x_0 u_1 \ldots u_n x_n$, with $x_0 \ldots x_{n-1} \neq 1$;

$u \prec_h v \Leftrightarrow \exists n \geq 1 : u = u_1 \ldots u_n \wedge v = x_0 u_1 x_1 \ldots u_n x_n$, with $x_0 \ldots x_n \neq 1$;

$u \prec_{p.si} v \Leftrightarrow \exists w \in A^* : w \prec_p v \wedge u \preceq_h w$;

$u \prec_{s.si} v \Leftrightarrow \exists w \in A^* : w \prec_s v \wedge u \preceq_h w$;

$u \prec_{si} v \Leftrightarrow \exists w \in A^* : w \prec_i v \wedge u \preceq_h w$;

$u \prec_{p.scpi} v \Leftrightarrow (\exists v' : v' \prec_p v)(\exists v'' \in \sigma(v')) : u \preceq_h v''$;

$u \prec_{s.scpi} v \Leftrightarrow (\exists v' : v' \prec_s v)(\exists v'' \in \sigma(v')) : u \preceq_h v''$;

$u \prec_{scpi} v \Leftrightarrow (\exists v' : v' \prec_i v)(\exists v'' \in \sigma(v')) : u \preceq_h v''$;

$u \prec_{p.spi} v \Leftrightarrow (\exists v' : v' \prec_p v)(\exists v'' \in \pi(v')) : u \preceq_h v''$;

$u \prec_{s.spi} v \Leftrightarrow (\exists v' : v' \prec_s v)(\exists v'' \in \pi(v')) : u \preceq_h v''$;

$u \prec_{spi} v \Leftrightarrow (\exists v' : v' \prec_i v)(\exists v'' \in \pi(v')) : u \preceq_h v''$;

$u \prec_{scp} v \Leftrightarrow \exists v' \in \sigma(v) : u \prec_h v'$;

$u \prec_{sp} v \Leftrightarrow \exists v' \in \pi(v) : u \prec_h v'$;

where $\pi(v)$ and $\sigma(v)$ are the sets of all permutations and cyclic permutations of $v$ respectively. In the sequel, for any $X \subseteq A^*$, we put $\pi(X) = \bigcup_{u \in X} \pi(u)$ and $\sigma(X) = \bigcup_{u \in X} \sigma(u)$.

The above mentioned relations define corresponding classes of codes which are denoted by and called respectively the classes $C_p$ of *prefix codes*, $C_s$ of *suffix codes*, $C_b$ of *bifix codes*, $C_{p.i}$ of *p-infix codes*, $C_{s.i}$ of *s-infix codes*, $C_i$ of *infix codes*, $C_{p.h}$ of *p-hypercodes*, $C_{s.h}$ of *s-hypercodes*, $C_h$ of *hypercodes*, $C_{p.si}$ of *p-subinfix codes*, $C_{s.si}$ of *s-subinfix codes*, $C_{si}$ of *subinfix codes*, $C_{p.scpi}$ of *p-sucyperinfix codes*, $C_{s.spci}$ of *s-sucyperinfix codes*, $C_{spci}$ of *sucyperinfix codes*, $C_{p.spi}$ of *p-superinfix codes*, $C_{s.spi}$ of *s-superinfix codes*, $C_{spi}$ of *superinfix codes*, $C_{scp}$ of *sucypercodes* and $C_{sp}$ of *supercodes*.

To facilitate understanding we give now intuitive meaning of some kinds of codes introduced above which are the main research subject in this paper.

A subset $X \subseteq A^+$ is a *superinfix* (*p-superinfix, s-superinfix*) *code*, $X \in C_{spi}$ ($X \in C_{p.spi}, X \in C_{s.spi}$, resp.), if no word in $X$ is a **su**bword of a **per**mutation of a proper **infix** (i.e. factor) (**p**refix, **s**uffix, resp.) of another word in $X$. This definition itself explains the terminologies superinfix, p-superinfix, s-superinfix. And a subset $X \subseteq A^+$ is a *sucypercode*, $X \in C_{scp}$, if no word in $X$ is a proper **su**bword of a **cy**clic **per**mutation of another word in $X$. Thus, every sucypercode is a hypercode and therefore any sucypercode over a finite alphabet is finite.

The following fact has been shown in [11].

**Lemma 1.1.** *For any $u, v \in A^*$, $\exists v' \in \sigma(v) : u \preceq_h v'$ iff $\exists u' \in \sigma(u) : u' \preceq_h v$.*

Let $\prec$ be a binary relation on $A^*$ and $u, v \in A^*$. We say that $u$ *depends* on $v$ if $u \prec v$ or $v \prec u$ holds. Otherwise, $u$ is *independent* of $v$. These notions can be extended to subsets of words in a standard way. Namely, a word $u$ is dependent on a subset $X$ if it depends on some word in $X$. Otherwise, $u$ is independent of $X$. For brevity, we shall adopt the following notations

$$u \prec X \rightleftharpoons \exists v \in X : u \prec v;\ X \prec u \rightleftharpoons \exists v \in X : v \prec u.$$

An element $u$ in $X$ is *minimal* in $X$ if there is no word $v$ in $X$ such that $v \prec u$. When $X$ is finite, by $\max X$ we denote the maximal wordlength of $X$.

For every subset $X$ in $A^*$ we denote by $D_X, I_X, L_X$ and $R_X$ the sets of words dependent on $X$, independent of $X$, non-minimal in $I_X$ and minimal in $I_X$, respectively. More precisely,

$$D_X = \{u \in A^* \mid u \prec X \vee X \prec u\};$$
$$I_X = A^* - D_X;$$

$$L_X = \{u \in I_X \mid I_X \prec u\};$$
$$R_X = I_X - L_X.$$

The following result has been proved in [9].

**Theorem 1.1.** *Let $\prec$ be a length-increasing transitive binary relation on $A^*$ which defines the class $C_\prec$ of codes. Then, for any code $X$ in $C_\prec$, we have*

(i) *$R_X$ is a maximal code in $C_\prec$ which contains $X$;*
(ii) *If moreover the relation $\prec$ satisfies the condition*

(*) $\exists k \geq 1 \forall u, v \in A^+ : (|v| \geq |u| + k) \wedge (u \not\prec v) \Rightarrow \exists w : (|w| \geq |u|) \wedge (w \prec v)$,

*then the finiteness of $X$ implies the finiteness of $R_X$, and* $\max R_X \leq \max X + k - 1$.

## 2. Embedding problem

For any set $X$ we denote by $\mathcal{P}(X)$ the family of all subsets of $X$. Recall that a *substitution* is a mapping $f$ from $B$ into $\mathcal{P}(C^*)$, where $B$ and $C$ are alphabets. If $f(b)$ is regular for all $b \in B$ the substitution $f$ is called a *regular substitution*. When $f(b)$ is a singleton for all $b \in B$ it induces a *homomorphism* from $B^*$ into $C^*$. Let # be a new letter not being in $A$. Put $A_\# = A \cup \{\#\}$. Let's consider the substitutions $S_1, S_2$ and the homomorphism $h$ defined as follows

$S_1 : A \to \mathcal{P}(A_\#^*)$, where $S_1(a) = \{a, \#\}$ for all $a \in A$;
$S_2 : A_\# \to \mathcal{P}(A^*)$, with $S_2(\#) = A^+$ and $S_2(a) = \{a\}$ for all $a \in A$;
$h : A_\#^* \to A^*$, with $h(\#) = 1$ and $h(a) = a$ for all $a \in A$.

Factually, the substitution $S_1$ is used to mark the occurrences of letters to be deleted from a word. The homomorphism $h$ realizes the deletion by replacing # by the empty word. The inverse homomorphism $h^{-1}$ "chooses" in a word the positions where the words of $A^+$ inserted, while $S_2$ realizes the insertions by replacing # by $A^+$. Notice that regular languages are closed under substitution, homomorphism and inverse homomorphism (see [3]). This class is also closed under the permutation $\pi$ and the cyclic permutation $\sigma$ [11].

**Theorem 2.1.** *The embedding problem has positive answer in the regular case for each class $C_\alpha$ of codes, $\alpha \in \{spi, p.spi, s.spi\}$. More precisely, every regular code $X$ in $C_\alpha$ is included in a code $Y$, which is maximal in $C_\alpha$ and remains regular.*

*Proof.* Notice that all the relations $\prec_\alpha$, $\alpha \in \{spi, p.spi, s.spi\}$, are transitive and length-increasing. Given a regular code $X$ in $C_\alpha$. By Theorem 1.1(i), $Y = R_X$ is a maximal code in $C_\alpha$ which contains $X$. The code $R_X$ is still regular because it can be obtained from $X$ by applications of some operations preserving regularity. Indeed, for the

• *Case of superinfix codes.* Without difficulty we can check the following computations

$$\{u \in A^* \mid u \prec_{spi} X\} = \pi(h(S_1(X) \cap (\{\#\}A^*_\# \cup A^*_\#\{\#\})));$$
$$\{u \in A^* \mid X \prec_{spi} u\} = S_2(h^{-1}(\pi(X)) \cap (\{\#\}A^*_\# \cup A^*_\#\{\#\}));$$
$$D_X = \{u \in A^* \mid u \prec_{spi} X \vee X \prec_{spi} u\}$$
$$= \pi(h(S_1(X) \cap (\{\#\}A^*_\# \cup A^*_\#\{\#\}))) \cup S_2(h^{-1}(\pi(X)) \cap (\{\#\}A^*_\# \cup A^*_\#\{\#\}));$$
$$I_X = A^* - D_X = A^* - \pi(h(S_1(X) \cap (\{\#\}A^*_\# \cup A^*_\#\{\#\})))$$
$$- S_2(h^{-1}(\pi(X)) \cap (\{\#\}A^*_\# \cup A^*_\#\{\#\}));$$
$$L_X = \{u \in I_X \mid I_X \prec_{spi} u\} = I_X \cap S_2(h^{-1}(\pi(I_X)) \cap (\{\#\}A^*_\# \cup A^*_\#\{\#\}));$$
$$R_X = I_X - L_X = I_X - S_2(h^{-1}(\pi(I_X)) \cap (\{\#\}A^*_\# \cup A^*_\#\{\#\})).$$

• *Case of p-superinfix codes.* Similarly, we get

$$R_X = I_X - S_2(h^{-1}(\pi(I_X)) \cap A^*_\#\{\#\}),$$

where $I_X = A^* - \pi(h(S_1(X) \cap A^*_\#\{\#\})) - S_2(h^{-1}(\pi(X)) \cap A^*_\#\{\#\})$.

• *Case of s-superinfix codes.* Dually, one obtains

$$R_X = I_X - S_2(h^{-1}(\pi(I_X)) \cap \{\#\}A^*_\#),$$

where $I_X = A^* - \pi(h(S_1(X) \cap \{\#\}A^*_\#)) - S_2(h^{-1}(\pi(X)) \cap \{\#\}A^*_\#)$. □

Denote by $A^{[n]}$ the set of all the words in $A^*$ whose length is less than or equal to $n$.

**Theorem 2.2.** *The embedding problem has positive answer in the finite case for every class $C_\alpha$ of codes, $\alpha \in \{spi, p.spi, s.spi, scp\}$. More precisely, every finite code $X$ in $C_\alpha$, with $\max X = n$, is included in a code $Y$ which is maximal in $C_\alpha$ and remains finite with $\max Y = \max X$. Namely, $Y$ can be computed by the following formulas according to the case.*

(i) *For superinfix codes*

$$Y = Z - S_2(h^{-1}(\pi(Z)) \cap (\{\#\}A^{[n-1]}_\# \cup A^{[n-1]}_\#\{\#\})) \cap A^{[n]},$$

*where* $Z = A^{[n]} - \pi(h(S_1(X) \cap (\{\#\}A^*_\# \cup A^*_\#\{\#\}))) - S_2(h^{-1}(\pi(X)) \cap (\{\#\}A^{[n-1]}_\# \cup A^{[n-1]}_\#\{\#\})) \cap A^{[n]}$.

(ii) *For p-superinfix codes*

$$Y = Z - S_2(h^{-1}(\pi(Z)) \cap A^{[n-1]}_\#\{\#\}) \cap A^{[n]},$$

*where* $Z = A^{[n]} - \pi(h(S_1(X) \cap A^*_\#\{\#\})) - S_2(h^{-1}(\pi(X)) \cap A^{[n-1]}_\#\{\#\}) \cap A^{[n]}$.

(iii) *For s-superinfix codes*

$$Y = Z - S_2(h^{-1}(\pi(Z)) \cap \{\#\}A^{[n-1]}_\#) \cap A^{[n]},$$

*where* $Z = A^{[n]} - \pi(h(S_1(X) \cap \{\#\}A^*_\#)) - S_2(h^{-1}(\pi(X)) \cap \{\#\}A^{[n-1]}_\#) \cap A^{[n]}$.

(iv) *For sucypercodes*

$$Y = Z - \sigma(S_2(h^{-1}(Z) \cap (A_\#^*\{\#\}A_\#^*) \cap A_\#^{[n]}) \cap A^{[n]}),$$

*where* $Z = A^{[n]} - h(S_1(\sigma(X)) \cap (A_\#^*\{\#\}A_\#^*)) - \sigma(S_2(h^{-1}(X) \cap (A_\#^*\{\#\}A_\#^*) \cap A_\#^{[n]}) \cap A^{[n]})$.

*Proof.* All the relations $\prec_\alpha, \alpha \in \{spi, p.spi, s.spi, scp\}$, are transitive, length-increasing and satisfy the condition (*) of Theorem 1.1 with $k = 1$ [11]. Given a code $X$ in $C_\alpha$ with $\max X = n$. Then, by Theorem 1.1, $Y = R_X$ is a maximal code in $C_\alpha$ which contains $X$ and $\max Y = \max X = n$. Here, we prove only the formulas for sucypercodes. For the remaining cases the argument is similar. For the sucypercode $X$, we can compute $R_X$ as follows

$$\begin{aligned}
&\{u \in A^* \mid u \prec_{scp} X\} \\
&\quad = \{u \in A^* \mid u \prec_h \sigma(X)\} = h(S_1(\sigma(X)) \cap (A_\#^*\{\#\}A_\#^*)); \\
&\{u \in A^* \mid X \prec_{scp} u\} \\
&\quad = \sigma(\{u \in A^* \mid X \prec_h u\}) = \sigma(S_2(h^{-1}(X) \cap (A_\#^*\{\#\}A_\#^*))); \\
&D_X = \{u \in A^* \mid u \prec_{scp} X \vee X \prec_{scp} u\} \\
&\quad = h(S_1(\sigma(X)) \cap (A_\#^*\{\#\}A_\#^*)) \cup \sigma(S_2(h^{-1}(X) \cap (A_\#^*\{\#\}A_\#^*))); \\
&I_X = A^* - D_X \\
&\quad = A^* - h(S_1(\sigma(X)) \cap (A_\#^*\{\#\}A_\#^*)) - \sigma(S_2(h^{-1}(X) \cap (A_\#^*\{\#\}A_\#^*))); \\
&L_X = \{u \in I_X \mid I_X \prec_{scp} u\} = I_X \cap \sigma(S_2(h^{-1}(I_X) \cap (A_\#^*\{\#\}A_\#^*))); \\
&R_X = I_X - L_X = I_X - \sigma(S_2(h^{-1}(I_X) \cap (A_\#^*\{\#\}A_\#^*))).
\end{aligned}$$

Since $\max R_X = n$, the expressions of $R_X$ and $I_X$ established above may be restricted to $A^{[n]}$ and $A_\#^{[n]}$ instead of $A^*$ and $A_\#^*$ respectively. Setting $Z = I_X$ and $Y = R_X$ we obtain the formulas required to prove. ☐

**Example 2.1.** Consider the superinfix code $X = \{a^2b, ab^2a, b^3a\}$ over the alphabet $A = \{a, b\}$. Since $\max X = 4$, we may compute $Y$ by the formulas in Theorem 2.2(i) with $n = 4$. We shall do it now step by step

$$\begin{aligned}
&S_1(X) \cap (\{\#\}A_\#^* \cup A_\#^*\{\#\}) = \{\#ab, a^2\#, \#\#b, \#a\#, a\#\#, \#\#\#, \\
&\quad \#b^2a, ab^2\#, \#\#ba, \#b\#a, \#b^2\#, a\#b\#, ab\#\#, \#\#\#a, \#\#b\#, \\
&\quad \#b\#\#, a\#\#\#, \#\#\#\#, b^3\#, b\#b\#, b^2\#\#, b\#\#\#\}; \\
&\pi(h(S_1(X) \cap (\{\#\}A_\#^* \cup A_\#^*\{\#\}))) = \{1, a, b, a^2, ab, ba, b^2, ab^2, b^2a, bab, b^3\}; \\
&h^{-1}(\pi(X)) \cap (\{\#\}A_\#^{[3]} \cup A_\#^{[3]}\{\#\}) = \{\#a^2b, a^2b\#, \#aba, aba\#, \#ba^2, ba^2\#\}; \\
&S_2(h^{-1}(\pi(X)) \cap (\{\#\}A_\#^{[3]} \cup A_\#^{[3]}\{\#\})) \\
&\quad = \{a^3b, ba^2b, a^2ba, a^2b^2, baba, aba^2, abab, b^2a^2, ba^3\}; \\
&Z = \{a^3, a^2b, aba, ba^2, a^4, ab^2a, ab^3, bab^2, b^2ab, b^3a, b^4\}; \\
&h^{-1}(\pi(Z)) \cap (\{\#\}A_\#^{[3]} \cup A_\#^{[3]}\{\#\}) \\
&\quad = \{\#a^3, a^3\#, \#a^2b, a^2b\#, \#aba, aba\#, \#ba^2, ba^2\#\};
\end{aligned}$$

$$S_2(h^{-1}(\pi(Z)) \cap (\{\#\}A_{\#}^{[3]} \cup A_{\#}^{[3]}\{\#\})) \cap A^{[4]}$$
$$= \{a^4, ba^3, a^3b, ba^2b, a^2ba, a^2b^2, baba, aba^2, abab, b^2a^2\};$$
$$Y = \{a^3, a^2b, aba, ba^2, ab^2a, ab^3, bab^2, b^2ab, b^3a, b^4\}.$$

**Example 2.2.** Consider the sucypercode $X = \{abab, a^2b^3\}$ over $A = \{a, b\}$. Since $\max X = 5$, by Theorem 2.2(iv), in a similar way, $Y$ can be computed as follows

$$h(S_1(\sigma(X)) \cap (A_{\#}^{*}\{\#\}A_{\#}^{*})) = \{1, a, b, a^2, ab, ba, b^2, a^2b, aba, ab^2, ba^2,$$
$$bab, b^2a, b^3, a^2b^2, ab^2a, ab^3, ba^2b, bab^2, b^2a^2, b^2ab, b^3a\};$$
$$\sigma(S_2(h^{-1}(X) \cap (A_{\#}^{*}\{\#\}A_{\#}^{*}) \cap A_{\#}^{[5]}) \cap A^{[5]})$$
$$= \{a^2bab, ba^2ba, aba^2b, baba^2, ababa, babab, b^2aba, ab^2ab, bab^2a, abab^2\};$$
$$Z = \{a^3, a^4, a^3b, a^2ba, aba^2, abab, ba^2, baba, b^4, a^5, a^4b, a^3ba, a^3b^2,$$
$$a^2ba^2, a^2b^2a, a^2b^3, aba^3, ab^2a^2, ab^3a, ab^4, ba^4, ba^3b, ba^2b^2,$$
$$bab^3, b^2a^3, b^2a^2b, b^2ab^2, b^3a^2, b^3ab, b^4a, b^5\};$$
$$\sigma(S_2(h^{-1}(Z) \cap (A_{\#}^{*}\{\#\}A_{\#}^{*}) \cap A_{\#}^{[5]}) \cap A^{[5]}) = \{a^4, ba^3, aba^2, a^2ba,$$
$$a^3b, a^5, aba^3, a^2ba^2, a^3ba, a^4b, ba^4, b^2a^3, ab^2a^2, a^2b^2a, a^3b^2, ba^3b,$$
$$ba^2ba, aba^2b, baba^2, ababa, a^2bab, babab, b^2aba, ab^2ab,$$
$$bab^2a, abab^2, ab^4, bab^3, b^2ab^2, b^3ab, b^4a, b^5\};$$
$$Y = \{a^3, abab, baba, b^4, a^2b^3, ab^3a, ba^2b^2, b^2a^2b, b^3a^2\}.$$

## 3. Characterizations

Let $A = \{a_1, a_2, \ldots, a_k\}$ and $K = \{1, 2, \ldots, k\}$. For every $u \in A^*$, we denote by $p(u)$ the Parikh vector of $u$, namely

$$p(u) = (|u|_{a_1}, |u|_{a_2}, \ldots, |u|_{a_k})$$

where $|u|_{a_i}$ denotes the number of occurrences of $a_i$ in $u$. Thus $p$ is a mapping from $A^*$ into the set $V^k$ of all the $k$-vectors of non-negative integers. Now, to every $u \in A^+$ we associate two elements of the cartesian product $V^k \times K$, denoted by $p_L(u)$ and $p_F(u)$, and one element of $V^k \times K^2$, denoted by $p_{LF}(u)$, which are defined as follows

$p_L(u) = (p(u), i)$, where $i$ is the index of the last letter in the word $u$;

$p_F(u) = (p(u), j)$, where $j$ is the index of the first letter in the word $u$;

$p_{LF}(u) = (p(u), i, j)$, where $i$ and $j$ are the indices of the last and the first letter in $u$, respectively.

Thus $p_L$ and $p_F$ are mappings from $A^+$ into $V^k \times K$, while $p_{LF}$ is a mapping from $A^+$ into $V^k \times K^2$. These mappings are then extended to languages in a standard way: $p_L(X) = \{p_L(u) \mid u \in X\}$, $p_F(X) = \{p_F(u) \mid u \in X\}$ and $p_{LF}(X) = \{p_{LF}(u) \mid u \in X\}$.

Put $U = \{(\xi, i) \in V^k \times K \mid p_i(\xi) \neq 0\}$ and $W = \{(\xi, i, j) \in V^k \times K^2 \mid p_i(\xi), p_j(\xi) \neq 0\}$. To each of the sets $U$ and $W$ we associate a binary relation, denoted both by $\prec$, which are defined by

$$(\xi, i) \prec (\eta, j) \Leftrightarrow (\xi \leq \eta) \wedge (p_j(\xi) < p_j(\eta)),$$
$$(\xi, m, n) \prec (\eta, i, j) \Leftrightarrow (\xi \leq \eta) \wedge (p_i(\xi) < p_i(\eta) \vee p_j(\xi) < p_j(\eta)),$$

where $p_i(\xi), 1 \leq i \leq k$, denotes the $i$-th component of $\xi$. These relations on $U$ and on $W$, as easily verified, are transitive. Notice that for all language $X$, $\emptyset \neq X \subseteq A^+$, $p_L(X)$ and $p_F(X)$ are subsets of $U$ while $p_{LF}(X)$ is a subset of $W$.

The following result, which will be useful in the sequel, is easily verified.

**Lemma 3.1.** *For any $u, v \in A^+$ we have*

(i) $u \prec_{p.spi} v$ *iff* $p_L(u) \prec p_L(v)$;
(ii) $u \prec_{s.spi} v$ *iff* $p_F(u) \prec p_F(v)$;
(iii) $u \prec_{spi} v$ *iff* $p_{LF}(u) \prec p_{LF}(v)$.

To every non-empty subset $X$ of $A^+$, we associate the sets

$$E_X = \{x \in X \mid \exists y \in X : p(y) < p(x)\} \text{ and } S_X = X - E_X.$$

Clearly, if $E_X = \emptyset$ then $X$ is a supercode.

Let $u$ be a word in $A^+$, we define the following operations

$\pi_L(u) = \pi(u')b$, with $u = u'b, b \in A$;

$\pi_F(u) = a\pi(u')$, with $u = au', a \in A$;

$$\pi_{LF}(u) = \begin{cases} a\pi(u')b, & \text{if } |u| \geq 2 \text{ and } u = au'b, \text{with } a, b \in A; \\ u, & \text{if } u \in A, \end{cases}$$

which are extended to languages in a normal way:

$$\pi_L(X) = \bigcup_{u \in X} \pi_L(u),\ \pi_F(X) = \bigcup_{u \in X} \pi_F(u) \text{ and } \pi_{LF}(X) = \bigcup_{u \in X} \pi_{LF}(u).$$

**Lemma 3.2.** *Let $X$ be a non-empty subset of $A^+$. If $p_L(X)$ ($p_F(X)$) is an independent set w.r.t. $\prec$ on $U$ then so is $p_L(\pi(S_X) \cup \pi_L(E_X))$ ($p_F(\pi(S_X) \cup \pi_F(E_X))$, resp.). If $p_{LF}(X)$ is an independent set w.r.t. $\prec$ on $W$ then so is $p_{LF}(\pi(S_X) \cup \pi_{LF}(E_X))$.*

*Proof.* We treat only the case of $p_L(X)$. The reasonements for the other cases are similar. Let $p_L(X)$ be an independent set w.r.t. $\prec$ on $U$. If $p_L(\pi(S_X) \cup \pi_L(E_X))$ were not an independent set w.r.t. $\prec$ on $U$ then there would exist $s, t \in p_L(\pi(S_X) \cup \pi_L(E_X))$ such that $s \prec t$. Since $s, t \in p_L(\pi(S_X) \cup \pi_L(E_X))$, we have $s = p_L(u), t = p_L(v)$ for some $u, v \in \pi(S_X) \cup \pi_L(E_X)$. Because $p_L(u) \prec p_L(v)$, we must have $v \in \pi_L(E_X)$. If $u \in \pi_L(E_X)$ then $p_L(u), p_L(v) \in p_L(\pi_L(E_X)) = p_L(E_X) \subseteq p_L(X)$, a contradiction. If $u \in \pi(S_X)$ then on one hand there exists $u' \in S_X$ such that $p(u') = p(u)$ with $p_L(u') \in p_L(S_X) \subseteq p_L(X)$, and on the other hand $p_L(v) \in p_L(E_X) \subseteq p_L(X)$. From $p_L(u) \prec p_L(v)$ it follows $p_L(u') \prec p_L(v)$, which contradicts the hypothesis that $p_L(X)$ is an independent set w.r.t. $\prec$. $\square$

We give now characterizations of p-superinfix codes, s-superinfix codes and superinfix codes.

**Theorem 3.1.** *For any non-empty subset $X$ of $A^+$, the following assertions are equivalent*

(i) *$X$ is a p-superinfix code (resp., a s-superinfix code, a superinfix code);*
(ii) *$\pi(S_X) \cup \pi_L(E_X)$ is a p-superinfix code (resp., $\pi(S_X) \cup \pi_F(E_X)$ is a s-superinfix code, $\pi(S_X) \cup \pi_{LF}(E_X)$ is a superinfix code);*
(iii) *$p_L(X)$ is an independent set w.r.t. $\prec$ on $U$ (resp., $p_F(X)$ is an independent set w.r.t. $\prec$ on $U$, $p_{LF}(X)$ is an independent set w.r.t. $\prec$ on $W$).*

*Proof.* We treat only the case of p-superinfix codes. For the other cases the argument is similar.

(i) $\Leftrightarrow$ (iii) By definition, $X$ is a p-superinfix code iff it is an independent set w.r.t. $\prec_{p.spi}$. By Lemma 3.1(i), the later is equivalent to the fact that $p_L(u) \not\prec p_L(v)$ for all $u, v \in X$, or equivalently $p_L(X)$ is an independent set w.r.t. $\prec$ on $U$.

(iii) $\Rightarrow$ (ii) Let $p_L(X)$ be an independent set w.r.t. $\prec$ on $U$. According to Lemma 3.2, $p_L(\pi(S_X) \cup \pi_L(E_X))$ is also an independent set w.r.t. $\prec$ on $U$. Hence, by the above, $\pi(S_X) \cup \pi_L(E_X)$ is a p-superinfix code.

(ii) $\Rightarrow$ (i) It is evident because any subset of a p-superinfix code is also a p-superinfix code. $\square$

**Example 3.1.** Consider the language $X = \{a^2ba,\ aba^2,\ ab^3,\ ba^3,\ bab^2,\ b^2ab,\ a^2b^2a,\ a^2b^3,\ ababa,\ abab^2,\ ab^2a^2,\ ab^2ab,\ ba^2ba,\ ba^2b^2,\ baba^2,\ babab,\ b^2a^3,\ b^2a^2b\}$ over $A = \{a, b\}$. It is easy to check that $p_L(X) = \{((3,1),1), ((3,2),1), ((2,3),2), ((1,3),2)\}$ and that it is an independent set w.r.t. $\prec$ on $U = \{(\xi, j) \in V^2 \times \{1,2\} \mid p_j(\xi) \neq 0\}$. By Theorem 3.1, $X$ is a p-superinfix code.

To end this section, we formulate a simple characterization of sucypercodes.

**Proposition 3.1.** *For any non-empty subset $X$ of $A^+$, $X$ is a sucypercode iff so is $\sigma(X)$.*

*Proof.* The sufficiency is trivial. Conversely, let $X$ be a sucypercode. If $\sigma(X)$ were not a sucypercode, there would exist $u, v \in \sigma(X)$ and $v' \in \sigma(v)$ such that $u \prec_h v'$. Then we have $u \in \sigma(x), v \in \sigma(y)$ for some $x, y \in X$. Thus $u \prec_h v'$ with $v' \in \sigma(y)$. By Lemma 1.1, there exists $u' \in \sigma(u)$ such that $u' \prec_h y$. Since $u' \in \sigma(x)$, again by Lemma 1.1, there exists $y' \in \sigma(y)$ such that $x \prec_h y'$. This means that $x \prec_{scp} y$, a contradiction. $\square$

## 4. Maximality

First we give characterizations of maximal p-superinfix, s-superinfix and superinfix codes by means of independent sets w.r.t. $\prec$ on $U$ and on $W$.

**Theorem 4.1.** *For any non-empty subset $X$ of $A^+$,*

(i) *$X$ is a maximal p-superinfix (s-superinfix) code iff $p_L(X)$ (resp., $p_F(X)$) is a maximal independent set w.r.t. $\prec$ on $U$ and $\pi(S_X) \cup \pi_L(E_X) = X$ (resp., $\pi(S_X) \cup \pi_F(E_X) = X$);*

(ii) *$X$ is a maximal superinfix code iff $p_{LF}(X)$ is a maximal independent set w.r.t. $\prec$ on $W$ and $\pi(S_X) \cup \pi_{LF}(E_X) = X$.*

*Proof.* (i) Let $X$ be a maximal p-superinfix code. If $\pi(S_X) \cup \pi_L(E_X) \neq X$ then, by Theorem 3.1, $\pi(S_X) \cup \pi_L(E_X)$ would be a p-superinfix code strictly containing $X$, a contradiction with the maximality of $X$. Hence, $\pi(S_X) \cup \pi_L(E_X) = X$. We next show that $p_L(X)$ is a maximal independent set w.r.t. $\prec$ on $U$. Indeed, by Theorem 3.1, $p_L(X)$ is an independent set w.r.t. $\prec$ on $U$. If $p_L(X)$ were not maximal then $\exists t \in U - p_L(X)$ such that $p_L(X) \cup \{t\}$ is still an independent set w.r.t. $\prec$. Let $t = (\xi, j)$. Since $p_j(\xi) \neq 0$, we can choose a word $u$ such that $p(u) = \xi$ and the last letter of $u$ has index $j$. Thus $p_L(u) = t$. Evidently $u \notin X$. We have $p_L(X \cup \{u\}) = p_L(X) \cup \{t\}$. Again by Theorem 3.1, $X \cup \{u\}$ is still a p-superinfix code, a contradiction with the maximality of $X$.

Conversely, let $p_L(X)$ be a maximal independent set w.r.t. $\prec$ on $U$ and $\pi(S_X) \cup \pi_L(E_X) = X$. By Theorem 3.1, $X$ is a p-superinfix code. Suppose $X$ is not maximal as a p-superinfix code. Then, there exists $u \notin X$ such that $X \cup \{u\}$ is still a p-superinfix code. If $p_L(u) \in p_L(X)$ then $p_L(u) = p_L(x)$ for some $x \in X$. This implies $p(u) = p(x)$ and the last letters of $u$ and $x$ are the same. Therefore $u \in \pi_L(x) \subseteq \pi(S_X) \cup \pi_L(E_X) = X$, a contradiction. Thus $t = p_L(u) \notin p_L(X)$. Again by Theorem 3.1, $p_L(X \cup \{u\}) = p_L(X) \cup \{t\}$ is still an independent set w.r.t. $\prec$, a contradiction with the maximality of $p_L(X)$. Thus $X$ must be maximal as a p-superinfix code. For the case of s-superinfix codes the argument is similar.

(ii) Let $X$ be a maximal superinfix code. If $\pi(S_X) \cup \pi_{LF}(E_X) \neq X$ then, by Theorem 3.1, $\pi(S_X) \cup \pi_{LF}(E_X)$ would be a superinfix code strictly containing $X$, a contradiction. So, $\pi(S_X) \cup \pi_{LF}(E_X) = X$. Now we show that $p_{LF}(X)$ is a maximal independent set w.r.t. $\prec$ on $W$. By Theorem 3.1, $p_{LF}(X)$ is an independent set w.r.t. $\prec$ on $W$. If $p_{LF}(X)$ were not maximal then $\exists t \in W - p_{LF}(X)$ such that $p_{LF}(X) \cup \{t\}$ is still an independent set. Let $t = (\xi, i, j)$. Since $p_i(\xi) \neq 0$ and $p_j(\xi) \neq 0$, we can choose a word $u$, whose the last and the first letter are $a_i$ and $a_j$ respectively, and such that $p(u) = \xi$. Thus $p_{LF}(u) = t$. Evidently $u \notin X$. We have $p_{LF}(X \cup \{u\}) = p_{LF}(X) \cup \{t\}$. Again by Theorem 3.1, $X \cup \{u\}$ is still a superinfix code, a contradiction with the maximality of $X$.

Conversely, let $p_{LF}(X)$ be a maximal independent set w.r.t. $\prec$ on $W$ and $\pi(S_X) \cup \pi_{LF}(E_X) = X$. By Theorem 3.1, $X$ is a superinfix code. Suppose $X$ is not maximal as a superinfix code. Then, there exists $u \notin X$ such that $X \cup \{u\}$ is still a superinfix code. If $p_{LF}(u) \in p_{LF}(X)$ then $p_{LF}(u) = p_{LF}(x)$ for some $x \in X$. This implies $p(u) = p(x)$, and that $u$ and $x$ have the same last and first letters. Therefore $u \in \pi_{LF}(x) \subseteq \pi(S_X) \cup \pi_{LF}(E_X) = X$, a contradiction. Thus $t = p_{LF}(u) \notin p_{LF}(X)$. Again by Theorem 3.1, $p_{LF}(X \cup \{u\}) = p_{LF}(X) \cup \{t\}$ is still an independent set w.r.t. $\prec$ on $W$, a contradiction with the maximality of $p_{LF}(X)$. Thus $X$ must be maximal as a superinfix code. □

**Example 4.1.** (i) Let $X = \{a^3, ab^2, bab, b^2a, b^3, a^2ba, a^2b^2, aba^2, abab, ba^3, ba^2b\}$. It is easy to see that $X = \pi(S_X) \cup \pi_L(E_X)$ and $p_L(X) = \{((3,0),1),\ ((3,1),1),$

$((2,2),2)$, $((1,2),1)$, $((1,2),2)$, $((0,3),2)\}$, which is easily verified to be a maximal independent set w.r.t. $\prec$ on $U = \{(\xi, i) \in V^2 \times \{1,2\} \mid p_i(\xi) \neq 0\}$. By Theorem 4.1(i), $X$ is a maximal $p$-superinfix code over $A = \{a, b\}$.

(ii) Consider the set $X = \{a^3,\ a^2ba,\ aba^2,\ b^4,\ a^2b^2a,\ ababa,\ ab^2a^2,\ bab^3,\ b^2ab^2,\ b^3ab,\ a^2b^3a,\ abab^2a,\ ab^2aba,\ ab^3a^2,\ ba^2b^3,\ babab^2,\ bab^2ab,\ b^2a^2b^2,\ b^2abab,\ b^3a^2b\}$ over $A = \{a, b\}$. A simple verification leads to $X = \pi(S_X) \cup \pi_{LF}(E_X)$ and also $p_{LF}(X) = \{((3,0),1,1),\ ((3,1),1,1),\ ((3,2),1,1),\ ((3,3),1,1),\ ((2,4),2,2),\ ((1,4),2,2)\ ((0,4),2,2)\}$. It is easy to see that the latter is a maximal independent set w.r.t. $\prec$ on $W = \{(\xi, i, j) \in V^2 \times \{1,2\}^2 \mid p_i(\xi), p_j(\xi) \neq 0\}$. By virtue of Theorem 4.1(ii), we may conclude that $X$ is a maximal superinfix code over $A$.

Recall that a subset $X$ of $A^+$ is an *infix* (*p-infix, s-infix*) *code* if no word in $X$ is an infix of a proper infix (prefix, suffix, resp.) of another word in $X$. The following result establishes relationship between maximal p-superinfix (s-superinfix) codes with p-infix (s-infix, resp.) codes.

**Theorem 4.2.** *Every maximal p-superinfix (s-superinfix) code is a maximal p-infix (s-infix, resp.) code.*

*Proof.* We treat only the case of p-superinfix codes. Let $X$ be a maximal p-superinfix code not being a maximal p-infix code. Then, there exists a word $y, 1 \neq y \notin X$, such that $Y = X \cup \{y\}$ is still a p-infix code. By Theorem 4.1(i), we have $\pi(S_X) \cup \pi_L(E_X) = X$ and $p_L(X)$ is a maximal independent set w.r.t. $\prec$ on $U$. If $p_L(y) \in p_L(X)$ then there is an $x \in X$ such that $p(y) = p(x)$ and the last letters of $y$ and $x$ are the same. Then, $y \in \pi_L(x) \subseteq \pi(S_X) \cup \pi_L(E_X) = X$, a contradiction with $y \notin X$. Thus we must have $p_L(y) \notin p_L(X)$ and therefore $p_L(X) \cup \{p_L(y)\}$ is not an independent set w.r.t. $\prec$ on $U$, i.e. either $p_L(y) \prec p_L(x)$ or $p_L(x) \prec p_L(y)$, for some $x \in X$. Suppose $p_L(y) \prec p_L(x)$, and let $a_j$ be the last letter of $x$. Since $p(y) \leq p(x)$ and $p_j(y) < p_j(x)$, there exists $x' \in \pi_L(x) \subseteq \pi(S_X) \cup \pi_L(E_X) = X$ such that $x'$ is of the form $x' = zya_j$ with $z \in A^*$. This is impossible because $Y$ is a p-infix code. Suppose now $p_L(x) \prec p_L(y)$. Without loss of generality we may assume $x \in S_X$. Let $a_j$ be the last letter of $y$. We have $p(x) \leq p(y)$ and $p_j(x) < p_j(y)$. Therefore there exists $x'' \in \pi(x) \subseteq \pi(S_X) \subseteq X$ such that $y$ has the form $y = zx''a_j$, a contradiction. Thus $X$ must be maximal as a p-infix code that required to prove. □

A subset $X$ in $A^+$ is a *subinfix* (*p-subinfix, s-subinfix*) *code* if no word in $X$ is a **sub**word of a proper **infix** (**p**refix, **s**uffix, resp.) of another word in $X$. The subset $X$ is called a *sucyperinfix* (*p-sucyperinfix, s-sucyperinfix*) *code* if no word in $X$ is a **su**bword of a **cy**clic **per**mutation of a proper **infix** (**p**refix, **s**uffix, resp.) of another word in $X$. We have $C_{spi} \subset C_{scpi} \subset C_{si} \subset C_i$, and the similar hierarchies for corresponding classes of codes.

As a direct consequence of Theorem 4.2 we have

**Corollary 4.1.** *We have the following assertions*

(i) *Every maximal p-superinfix (s-superinfix) code is a maximal p-subinfix (s-subinfix, resp.) code;*

(ii) *Every maximal p-superinfix (s-superinfix) code is a maximal p-sucyperinfix (s-sucyperinfix, resp.) code.*

**Remark 4.1.** It is easy to see that the inverses of Theorem 4.2, of assertions (i) and (ii) in Corollary 4.1 are false. For example, the maximal p-infix (p-subinfix) code $X = \{ab, b^2, a^3, a^2b, ba^2, bab\}$ over $A = \{a, b\}$ is not a p-superinfix code since $ab$ is a subword of a permutation of $ba$, a proper prefix of $ba^2$. The maximal p-sucyperinfix code $Y = \{a^3, b^3, a^2ba, aba^2,\ abab, ab^3, ba^3,\ baba, bab^2, b^2ab, a^2b^2a, a^2b^3,\ ab^2a^2,\ ab^2ab,\ ba^2ba,\ ba^2b^2, b^2a^3, b^2a^2b\}$ over $A = \{a, b\}$ is not also a p-superinfix code because $abab$ is a subword of a permutation of the proper prefix $a^2b^2$ of $a^2b^3$.

We have moreover

**Corollary 4.2.** *Every maximal p-superinfix (s-superinfix) code is a maximal code.*

*Proof.* Recall that a code $X$ is *thin* if there is a word $w$, which cannot be a factor of any word in $X$. Any p-infix (s-infix) code $X$ is thin because any word of the form $axa$ with $x \in X, a \in A$ cannot be a factor of any word in $X$. Every maximal p-infix (s-infix) code is a maximal prefix (suffix) code [5]. Thus, by Theorem 4.2, every maximal p-superinfix (s-superinfix) code is a maximal prefix (suffix) code which is thin. As well-known, for a thin code $X$, it is a maximal prefix (suffix) code if and only if it is a maximal code (see [1]). Hence, every maximal p-superinfix (s-superinfix) code is a maximal code. □

This corollary in combination with Theorems 2.1 and 2.2 give us immediately:

**Corollary 4.3.** *Every finite (regular) p-superinfix (s-superinfix) code is included in a finite (regular, resp.) p-superinfix (s-superinfix) code which is maximal as a code.*

The following assertion characterizes maximal p-superinfix (s-superinfix) codes among maximal p-infix (s-infix) codes.

**Theorem 4.3.** *A maximal p-infix (s-infix) code $X$ is a maximal p-superinfix (s-superinfix, resp.) code iff $\pi(S_X) \cup \pi_L(E_X) = X$ $(\pi(S_X) \cup \pi_F(E_X) = X$, resp.).*

*Proof.* The necessity is obvious by Theorem 4.1(i). Conversely, let $X$ be a maximal p-infix code with $\pi(S_X) \cup \pi_L(E_X) = X$. We first show that $p_L(X)$ is an independent set w.r.t. $\prec$ on $U$. Suppose the contrary that there exist $u, v \in X$ such that $p_L(u) \prec p_L(v)$, and let $a_j$ be the last letter of $v$. Then, we have $p(u) \leq p(v)$ and $p_j(u) < p_j(v)$. Therefore, there is $v' \in \pi_L(v) \subseteq X$ such that $v' = zua_j$, which contradicts the hypothesis that $X$ is a p-infix code. Thus $p_L(X)$ must be an independent set w.r.t. $\prec$ on $U$ and hence $X$ is a p-superinfix code. The maximality of $X$ as a p-superinfix code is then evident. For the remainning case the argument is similar. □

**Remark 4.2.** While, as seen above, a maximal p-superinfix (s-superinfix) code is always a maximal prefix (suffix) code, a maximal superinfix code is not necessarily a maximal subinfix code. Indeed, consider the code $X = ab^*a$ over $A = \{a, b\}$ which is easily verified to be a maximal superinfix code. But it is not a maximal subinfix code because $X \cup \{bab\}$ is still a subinfix code.

Now we consider some properties of maximal sucypercodes and their relationship with other kinds of codes, namely with supercodes and hypercodes. Recall that a subset $X$ of $A^+$ is a *hypercode*, $X \in C_h$, if no word in $X$ is a proper subword of another word in $X$. The subset $X$ is a *supercode*, $X \in C_{sp}$, if no word in $X$ is a proper **su**bword of a **per**mutation of another word in $X$. Note that $C_{sp} \subset C_{scp} \subset C_h$. Supercodes have been introduced and considered in [10].

**Theorem 4.4.** *For any subset $X$ of $A^+$, we have*

(i) *If $X$ is a maximal sucypercode then $\sigma(X) = X$;*
(ii) *Every maximal sucypercode is a maximal hypercode;*
(iii) *Every maximal supercode is a maximal sucypercode.*

*Proof.* (i) Let $X$ be a maximal sucypercode. If $\sigma(X) \neq X$ then, by Proposition 3.1, $\sigma(X)$ is a sucypercode strictly containing $X$, a contradiction with the maximality of $X$.

(ii) Let $X$ be a maximal sucypercode not being a maximal hypercode. Then, there exists $1 \neq y \notin X$ such that $X \cup \{y\}$ is still a hypercode. By (i), $\sigma(X) = X$. Thus $Y = \sigma(X) \cup \{y\}$ is a hypercode. We now prove that $Y$ is still a sucypercode. Suppose the contrary that it is not the case. Then either $y \prec_{scp} x$ or $x \prec_{scp} y$, for some $x \in \sigma(X)$. If $y \prec_{scp} x$ then there is $x' \in \sigma(x) \subseteq Y$ such that $y \prec_h x'$, which contradicts the fact that $Y$ is a hypercode. If $x \prec_{scp} y$ then there exists $y' \in \sigma(y)$ such that $x \prec_h y'$. By Lemma 1.1, there exists $x'' \in \sigma(x)$ such that $x'' \prec_h y$, again a contradiction. Thus $Y = X \cup \{y\}$ is a sucypercode, which contradicts the maximality of $X$ as a sucypercode. This contradiction shows that $X$ must be a maximal hypercode.

(iii) Let $X$ be a maximal supercode. By Proposition 3.1(i) in [10], $X$ is a maximal hypercode. Because $C_{sp} \subset C_{scp} \subset C_h$, this implies that $X$ is a maximal sucypercode. □

**Theorem 4.5.** *For any subset $X$ of $A^+$, we have*

(i) *A maximal hypercode $X$ is a maximal sucypercode iff $\sigma(X) = X$;*
(ii) *A maximal sucypercode $X$ is a maximal supercode iff $\pi(X) = \sigma(X)$.*

*Proof.* (i) The necessity is evident by Theorem 4.4(i). Conversely, let $X$ be a maximal hypercode with $\sigma(X) = X$. As $X$ is a hypercode, we have $u \not\prec_h v$ for all $u, v \in X$. Since $\sigma(X) = X$ this implies $u \not\prec_{scp} v$ for all $u, v \in X$. Thus $X$ is a sucypercode and hence a maximal sucypercode.

(ii) Let $X$ be a maximal sucypercode. By Theorem 4.4(i), $\sigma(X) = X$. By Theorem 4.4(ii), $X$ is a maximal hypercode. According to Proposition 3.4 in [10], $X$ is a maximal supercode iff $\pi(X) = X$, or equivalently $\pi(X) = \sigma(X)$. □

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