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ON THE CONVOLUTION WITH A WEIGHT-FUNCTION FOR THE COSINE-FOURIER INTEGRAL TRANSFORM

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Abstract. The convolution with a weight-function for the cosine-Fourier transform is formulated and its properties are studied. A Titchmarch type theorem, non-existence of the unit element of the convolution are proved. The relationship between the convolution with and without a weight-function is estabilished and the application to solving integral equations is outlined.

1. Introduction

The convolutions of many integral transforms have interesting applications to numerous problems such as evaluating integrals [9], summing a series, solving equations of mathematical physics [3], [5], [6], [1], [9], [11], [12], [14]. Initially in 1941 Churchill [11] proposed the convolution of two functions f and g for the Fourier integral transform F [11]:

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-y)g(y)dy \tag{1}$$

for which the factorization property holds

$$F(f * g)(y) = (Ff)(y) \cdot (Fg)(y), \quad \forall y \in R.$$

Next, the convolutions of other integral transforms such as the Mellin, Laplace, cosine-Fourier, Hilbert... transforms were studied [2]. For example, the convolution of two functions f and g for the Laplace integral transform L:

$$(f * g)(x) = \int_0^x f(x-t)g(t)dt$$

for which the factorization property holds

$$L(f * g)(y) = (Lf)(y) \cdot (Lg)(y), \quad \forall y > 0.$$

The convolution with a weight-function for the Mehler Fox integral transform was studied by I. Ya. Vilenkin in [13].

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Afterwards, in 1967, V. A. Kakichev [7] proposed a constructive method for defining the convolution with a weight-function which is more general than the convolution (1). And as by-products, convolutions of many integral transforms such as the Meijer, Hankel, sine-Fourier, Sommerfeld were found [8]. For instance, the convolution with the weight-function $\gamma(y) = \sin y$ of the functions f and g for the sine-Fourier integral transform F_s was studied in [7], [10]

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(t)\big[g(x+1+t) + g(|x+1-t|)\mathrm{sign}(x+1-t) + \\ + g(|x-1+t|)\mathrm{sign}(x-1+t) + g(|x-1-t|)\mathrm{sign}(x-1-t)\big]dt$$

for which the factorization property holds

$$F_s(f \overset{\gamma}{*} g)(y) = \sin y \cdot (F_s f)(y) \cdot (F_s g)(y), \quad \forall y > 0.$$

Meanwhile, the convolution with the weight-function $\gamma(y) = y^{-\nu}$ of two functions f, g for the integral transform Hankel $\mathcal{H}$ is defined by

$$(f \overset{\gamma}{*} g)(x) = \frac{x^\nu}{2^\nu \sqrt{\pi}\Gamma\big(\nu + \frac{1}{2}\big)} \int_0^{\pi} \sin^{2\nu} t dt \\ \times \int_0^{+\infty} \frac{u^{\nu+1} f(u) \cdot g\big(\sqrt{x^2 + u^2 - 2xu\cos t}\,\big)}{\big(x^2 + u^2 - 2xu\cos t\big)^{\frac{\nu}{2}}} du$$

for which we have the factorization property

$$\mathcal{H}_\nu(f * g)(y) = y^\nu (\mathcal{H}_\nu f)(y)(\mathcal{H}_\nu g)(y), \quad \forall\, y > 0,\ \nu > -\frac{1}{2}\cdot$$

Since the convolution is in fact an integral transform, it is a research object of [9]. In the theory of norm rings, the convolution was introduced as the operation of multiplication.

The integral cosine-Fourier transform is studied in [11]

$$\tilde{f}(y) = (F_c f)(y) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x)\cos(yx)dx.$$

Here, $f(x)$ is continuous and belongs to

$$L(R_+) = \Big\{ f \text{ defined in the set of positive real numbers and} \int_0^{+\infty} |f(x)|dx < +\infty \Big\}.$$

We have then the inverse formula

$$f(x) = (F_c \tilde{f})(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \tilde{f}(y)\cos(xy)dy.$$

The convolution of $f(x)$ and $g(x)$ for this integral transform was also given in 1941 by Churchill

$$(2)\qquad (f \underset{F_c}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(y)\big[g(|x-y|) + g(x+y)\big]dy$$

with the factorization property

$$F_c(f \underset{F_c}{*} g)(y) = (F_c f)(y) \cdot (F_c g)(y), \quad \forall y \in R_+.$$

In this paper we study only the convolution with a weight-function for the cosine-Fourier transform and prove the relation between this convolution with that for the cosine-Fourier transform without a weight-function. Futhermore, we prove a Titchmarch type theorem for this new convolution as well as describe the normed rings for it. Finally, we apply the covolution with a weight-function for the cosine-Fourier transform to solving an integral equation.

2. The convolution with a weight-function for the cosine-Fourier integral transform

Definition 1. The convolution with the weight-function $\gamma(y) = \cos y$ of two function f, g for the cosine-Fourier integral transform is defined by

$$(3)\qquad (f \overset{\gamma}{*} g)(x) = \frac{1}{2\sqrt{2\pi}} \Big[\int_0^{+\infty} f(t)\big[g(x+1+t) + g(|x+1-t|) + g(|x-1+t|) + g(|x-1-t|)\big]dt\Big].$$

Theorem 1. *Let f and g be continuous in $L(R_+)$. Then the convolution with the weight-function $\gamma(y) = \cos y$ of them for the cosine-Fourier integral transform belongs to $L(R_+)$ and the factorization property holds*

$$(4)\qquad F_c(f \overset{\gamma}{*} g)(y) = \cos y\ (F_c f)(y) \cdot (F_c g)(y), \quad \forall y \in R_+.$$

Proof. We have

$$\int_0^{+\infty} \big|(f \overset{\gamma}{*} g)(x)\big|dx = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty}\int_0^{+\infty} |f(t)| \cdot \Big|\big[g(x+1+t) + g(|x+1-t|) + g(|x-1+t|) + g(|x-1-t|)\big]\Big| \cdot dtdx$$

$$\le \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} |f(t)|\Big[\int_0^{+\infty} |g(x+1+t)|dx + \int_0^{+\infty} \big|g(|x+1-t|)\big|dx$$

$$(5)\qquad + \int_0^{+\infty} \big|g(|x-1+t|)\big|dx + \int_0^{+\infty} \big|g(|x-1-t|)\big|dx\Big]dt.$$

On the other hand,

$$\int_0^{+\infty} \big|g(x+1+t)\big|dx + \int_0^{+\infty} \big|g(|x-1-t|)\big|dx$$

$$= \int_{t+1}^{+\infty} |g(u)|du + \int_{-1-t}^{+\infty} \big|g(|u|)\big|du$$

$$= \int_{t+1}^{+\infty} \big|g(u)\big|du + \int_0^{+\infty} |g(u)|du + \int_0^{1+t} |g(u)|du$$

$$= 2\int_0^{+\infty} |g(u)|du. \tag{6}$$

Similarly, without loss of generality we can assume $t > 1$,

$$\int_0^{+\infty} \big|g(|x+1-t|)\big|dx + \int_0^{+\infty} \big|g(|x-1+t|)\big|dx$$

$$= \int_{1-t}^{+\infty} \big|g(|u|)\big|du + \int_{t-1}^{+\infty} \big|g(u)\big|du$$

$$= \int_0^{+\infty} \big|g(u)\big|du + \int_0^{t-1} |g(u)|du + \int_{t-1}^{+\infty} |g(u)|du$$

$$= 2\int_0^{+\infty} |g(u)|du. \tag{7}$$

Hence, by virtrue of (6) and (7),

$$\int_0^{+\infty} \Big[\big|g(x+1+t)\big| + \big|g(|x+1-t|)\big| + \big|g(|x-1+t|)\big| + \big|g(|x-1-t|)\big|\Big] du = 4\int_0^{+\infty} |g(u)|du. \tag{8}$$

It follows from (5) and (8) that

$$\int_0^{+\infty} \big|(f \overset{\gamma}{*} g)(x)\big|dx \leq \sqrt{\frac{2}{\pi}} \int_0^{+\infty} |f(t)|dt \int_0^{+\infty} |g(t)|dt < +\infty.$$

So $(f \overset{\gamma}{*} g)(x) \in L(R_+)$.

Now we prove the factorization property (4). Since

$$\cos x(F_cf)(x)(F_cg)(x) = \frac{2}{\pi}\int_0^{+\infty}\int_0^{+\infty} \cos x \cos xu.\cos xv f(u)g(v)dudv$$

and

$$\cos x \cos xu \cos xv = \frac{1}{4}\big[\cos x(u+1+v) + \cos x(u+1-v) + \cos x(u-1+v) + \cos x(u-1-v)\big],$$

we obtain

$$\cos x(F_cf)(x)(F_cg)(x) = \frac{1}{2\pi}\int_0^{+\infty}\int_0^{+\infty}\big[\cos x(u+1+v) + \cos x(u+1-v) + \cos x(u-1+v) + \cos x(u-1-v)\big] f(u)g(v)dudv. \tag{9}$$

Substituting $u = y$ and $u+1+v = t$, we get

$$\begin{aligned}\frac{1}{2\pi}\int_0^{+\infty}\int_0^{+\infty}\cos x(u+1+v)f(u)g(v)dudv &= \frac{1}{2\pi}\int_0^{+\infty}\int_{y+1}^{\infty}\cos xtf(y)g(t-y-1)dtdy\\ &= \frac{1}{2\pi}\int_0^{+\infty}\int_0^{+\infty}\cos xtf(y)g(|t-y-1|)dtdy\\ &\quad -\frac{1}{2\pi}\int_0^{+\infty}\int_0^{y+1}\cos xtf(y)g(y+1-t)dtdy.\end{aligned} \tag{10}$$

Similarly, with the substitutions $u = y$, $u+1-v = -t$, we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{+\infty}\int_0^{+\infty}\cos x(u+1-v)f(u)g(v)dudv &= \frac{1}{2\pi}\int_0^{+\infty}\int_{-1-y}^{+\infty}\cos xtf(y)g(t+y+1)dtdy\\ &= \frac{1}{2\pi}\int_0^{+\infty}\int_0^{+\infty}\cos xtf(y)g(t+y+1)dtdy\\ &\quad +\frac{1}{2\pi}\int_0^{+\infty}\int_{-1-y}^{0}\cos xtf(y)g(t+y+1)dtdy.\end{aligned} \tag{11}$$

Further,

$$\int_0^{+\infty}\int_{-1-y}^{0}\cos xtf(y)g(t+y+1)dtdy = -\int_0^{+\infty}\int_{1+y}^{0}\cos xtf(y)g(y+1-t)dtdy$$

$$(12)\qquad = \int_0^{+\infty}\int_0^{1+y}\cos xtf(y)g(y+1-t)dtdy.$$

From (10), (11) and (12) we obtain

$$(13)\qquad \frac{1}{2\pi}\int_0^{+\infty}\int_0^{+\infty}\big[\cos x(u+1+v)+\cos x(u+1-v)\big]f(u)g(v)dudv$$

$$= \frac{1}{2\pi}\int_0^{+\infty}\int_0^{+\infty}\cos xt\big[g(|t-y-1|)+g(t+y+1)\big]f(y)dtdy.$$

Similarly,

$$\frac{1}{2\pi}\int_0^{+\infty}\int_0^{+\infty}\cos x(u-1+v)f(u)g(v)dudv = \frac{1}{2\pi}\int_0^{+\infty}\int_{y-1}^{+\infty}\cos xtf(y)g(t-y+1)dtdy$$

$$= \frac{1}{2\pi}\int_0^{1}\int_{y-1}^{+\infty}\cos xtf(y)g(t-y+1)dtdy$$

$$+ \frac{1}{2\pi}\int_1^{+\infty}\int_{y-1}^{+\infty}\cos xtf(y)g(t-y+1)dtdy$$

$$= \frac{1}{2\pi}\int_0^{+\infty}\int_0^{+\infty}\cos xtf(y)g(|t-y+1|)dtdy$$

$$+ \frac{1}{2\pi}\int_0^{1}\int_{y-1}^{0}\cos xtf(y)g(t-y+1)dtdy$$

$$(14)\qquad - \frac{1}{2\pi}\int_1^{+\infty}\int_0^{y-1}\cos xtf(y)g(y-1-t)dtdy.$$

With the substitutions $u=y$, $u-1-v=-t$ we get

$$\frac{1}{2\pi}\int_0^{+\infty}\int_0^{+\infty}\cos x(u-1-v)f(u)g(v)dudv = \frac{1}{2\pi}\int_0^{+\infty}\int_{1-y}^{+\infty}\cos xtf(y)g(t+y-1)dtdy$$

$$= \frac{1}{2\pi}\int_0^{1}\int_{1-y}^{+\infty}\cos xtf(y)g(t+y-1)dtdy$$

$$+ \frac{1}{2\pi}\int_1^{+\infty}\int_{1-y}^{+\infty}\cos xtf(y)g(t+y-1)dtdy$$

$$= \frac{1}{2\pi}\int_0^{+\infty}\int_0^{+\infty}\cos xtf(y)g(|t+y-1|)dtdy$$

$$- \frac{1}{2\pi}\int_0^{1}\int_0^{1-y}\cos xtf(y)g(1-y-t)dtdy$$

$$+ \frac{1}{2\pi}\int_1^{+\infty}\int_{1-y}^{0}\cos xtf(y)g(t+y-1)dtdy. \tag{15}$$

On the other hand,

$$\int_0^{1}\int_{y-1}^{0}\cos xtf(y)g(t-y+1)dtdy = \int_0^{1}\int_0^{1-y}\cos xtf(y)g(1-y-t)dtdy \tag{16}$$

and

$$\int_1^{+\infty}\int_{1-y}^{0}\cos xtf(y)g(t+y-1)dtdy = \int_1^{+\infty}\int_0^{y-1}\cos xtf(y)g(y-1-t)dtdy \tag{17}$$

By (14), (15), (16) and (17),

$$\frac{1}{2\pi}\int_0^{+\infty}\int_0^{+\infty}\big[\cos x(u-1+v)+\cos x(u-1-v)\big]f(u)g(v)dudv \tag{18}$$

$$= \frac{1}{2\pi}\int_0^{+\infty}\int_0^{+\infty}\cos xt\big[g(|t-y+1|)+g(|t+y-1|)\big]f(y)dtdy.$$

Finally, by (9), (13) and (18),

$$\cos x(F_c f)(x)(F_c g)(x) = \frac{1}{2\pi}\int\limits_0^{+\infty} \cos xt\Big\{\int\limits_0^{+\infty} f(y)\big[g(t+1+y)+g(|t+1-y|) + g(|t-1+y|)+g(|t-1-y|)\big]dy\Big\}dt.$$

The last equality and (3) yield

$$\cos x(F_c f)(x)(F_c g)(x) = F_c(f \overset{\gamma}{*} g)(x).$$

The proof is complete. □

Theorem 2. *In the space of continuous functions belonging to $L(R_+)$, the convolution with a weight-function for the cosine-Fourier integral transform is commutative, associative and distributive.*

Proof. We prove that the convolution with a weight-function for the cosine-Fourier integral transform is associative, i.e.,

$$(f \overset{\gamma}{*} g) \overset{\gamma}{*} h = f \overset{\gamma}{*} (g \overset{\gamma}{*} h).$$

Indeed,

$$\begin{aligned}\Big(F_c\big((f \overset{\gamma}{*} g) \overset{\gamma}{*} h\big)\Big)(y) &= \cos y\,\big(F_c(f \overset{\gamma}{*} g)\big)(y)\cdot(F_c h)(y)\\ &= \cos y\cos y(F_c f)(y)(F_c g)(y)(F_c h)(y)\\ &= \cos y(F_c f)(y)\big[\cos y(F_c g)(y)(F_c h)(y)\big]\\ &= \cos y(F_c f)(y)\big(F_c(g \overset{\gamma}{*} h)\big)(y)\\ &= F_c\big(f \overset{\gamma}{*} (g \overset{\gamma}{*} h)\big)(y) \quad (\forall\, y>0)\end{aligned}$$

implies that

$$(f \overset{\gamma}{*} g) \overset{\gamma}{*} h = f \overset{\gamma}{*} (g \overset{\gamma}{*} h).$$

The commutative, distributive properties are similarly proved. □

Definition 2. The norm in the space $L(R_+)$ is defined by

$$\|f\| = \sqrt{\frac{2}{\pi}}\int\limits_0^{+\infty} |f(x)|dx.$$

Theorem 3. *If f and g are continuous functions in to $L(\mathbb{R}_+)$, then the following inequality holds*

$$\|f \overset{\gamma}{*} g\| \le \|f\|\cdot\|g\|.$$

Proof. From the proof of Theorem 1 we get

$$\int_0^{+\infty} |(f \overset{\gamma}{*} g)(x)|dx \leq \sqrt{\frac{2}{\pi}} \int_0^{+\infty} |f(x)|dx \int_0^{+\infty} |g(x)|dx.$$

Hence

$$\sqrt{\frac{2}{\pi}} \int_0^{+\infty} |(f \overset{\gamma}{*} g)(x)|dx \leq \sqrt{\frac{2}{\pi}} \int_0^{+\infty} |f(x)|dx \cdot \sqrt{\frac{2}{\pi}} \int_0^{+\infty} |g(x)|dx.$$

Thus,

$$\|(f \overset{\gamma}{*} g)\| \leq \|f\| \cdot \|g\|.$$

□

Theorem 4. *If f and g are continuous functions in $L(R_+)$, then the following equality holds*

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2}\big[(f \underset{F_c}{*} g)(x+1) + (f \underset{F_c}{*} g)(|x-1|)\big], \quad \forall\, x > 0.$$

Here $(f \underset{F_c}{*} g)$ is defined in (2).

Proof. By definition

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2}\Big\{\frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(t)\big[g(x+1+t) + g(|x+1-t|)\big]dt + $$
$$+ \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(t)\big[g(|x-1+t|) + g(|x-1-t|)\big]dt\Big\}.$$

On the other hand, for any $x > 0$ and $t > 0$,

$$g(|x-1+t|) + g(|x-1-t|) = g(|x-1|+t) + g(||x-1|-t|).$$

Indeed, for $x \geq 1$,

$$\begin{aligned} g(|x-1|+t) + g(||x-1|-t|) &= g(x-1+t) + g(|x-1-t|) \\ &= g(|x-1+t|) + g(|x-1-t|). \end{aligned}$$

Similarly, for $0 < x \leq 1$,

$$\begin{aligned} g(|x-1|+t) + g(||x-1|-t|) &= g(|1-x+t|) + g(|1-x-t|) \\ &= g(|x-1-t|) + g(|x-1+t|). \end{aligned}$$

Hence

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2}\Big\{\frac{1}{\sqrt{2\pi}}\int\limits_0^{+\infty} f(t)\big[g(1+x+t) + g(|1+x-t|)\big]dt + $$
$$+\frac{1}{\sqrt{2\pi}}\int\limits_0^{+\infty} f(t)\big[g(|x-1|+t) + g(||x-1|-t|)\big]dt\Big\}$$
$$= \frac{1}{2}\big[(f \underset{F_c}{*} g)(x+1) + (f \underset{F_c}{*} g)(|x-1|)\big], \quad \forall\, x > 0.$$

The theorem is proved. □

Theorem 5. *In the space of continuous functions in $L(R_+)$ there does not exist the unit element for the operation of the convolution with a weight-function for the cosine-Fourier integral transform.*

Proof. Suppose that there exists e, the unit element of the operation of convolution in the space of continuous functions in $L(R_+)$: $e \overset{\gamma}{*} g = g \overset{\gamma}{*} e = g$, for any continuous function g belonging to $L(R_+)$. Then we have

$$F_c(e \overset{\gamma}{*} g)(y) = (F_c g)(y), \quad \forall y > 0.$$

Hence

$$\cos y (F_c e)(y) \cdot (F_c g)(y) = (F_c g)(y), \quad \forall y > 0.$$

The last is equivalent to the equality

$$(F_c g)(y)\big[\cos y F_c e)(y) - 1\big] = 0, \quad \forall y > 0,$$

for any continuous function $g(y)$ belonging to $L(R_+)$.

Choosing g so that $(F_c g)(y) \neq 0$, $\forall y > 0$, we see that $\cos(F_c e)(y) - 1 = 0$ or $(F_c e)(y) = \dfrac{1}{\cos y}$, $\forall y > 0$. Thus, $(F_c e)(y) \notin L(R_+)$, and so $e \notin L(R_+)$. This is a contradiction. So there does not exist the unit element for the operation of convolution with a weight-function for the cosine-Fourier integral transform in the space of continuous functions belonging to $L(R_+)$. The proof is complete. □

Set

$$L(e^{-x}, R_+) = \big\{e^{-x} \cdot h, \text{ for all } h \in L(R_+)\big\}$$

Theorem 6. (A Titchmarch type theorem). *Let $f,\ g \in L(e^{-x}, R_+)$. If $(f \overset{\gamma}{*} g)(x) \equiv 0$, $\forall x > 0$, then either $f(x) = 0$ or $g(x) = 0$, $\forall x > 0$.*

Proof. Under the hypothesis $(f \overset{\gamma}{*} g)(x) \equiv 0$, $\forall\, x > 0$, it follows that $F_c(f \overset{\gamma}{*} g)(y) = 0$, $\forall\, y > 0$. Due to Theorem 1 we have

$$\cos y \cdot (F_c f)(y) \cdot (F_c g)(y) = 0, \quad \forall y > 0. \tag{19}$$

Consider the cosine-Fourier integral transform

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) \cos(yx) dx, \quad y \in R_+.$$

Since

$$\Big| \frac{d^n}{dy^n} \big[\cos(yx) f(x) \big] \Big| = \Big| f(x) \cdot x^n \cdot \cos\big(yx + n\frac{\pi}{2} \big) \Big| \leq |f(x) \cdot x^n|$$
$$:= |e^{-x} \cdot x^n \cdot f_1(x)|$$
$$:= |e^{-x} \cdot x^n| \cdot |f_1(x)| \leq C|f_1(x)|$$

for x large enough, due to Weierstranss' criterion, the intergral

$$\int_0^{+\infty} \frac{d^n}{dy^n} \big[\cos(yx) f(x) \big] dx$$

uniformly converges on R_+. Therefore, based on the differentiability of integrals depending on parameter, we conclude that $(F_c f)(y)$ is analytic for $y > 0$.

Similarly, $(F_c g)(y)$ is analytic for $y > 0$. So from (19) we have $(F_c f)(y) = 0$, $\forall\, y > 0$, or $(F_c g)(y) = 0$, $\forall\, y > 0$. It follows that either $f(x) = 0$, $\forall\, x > 0$, or $g(x) = 0$, $\forall\, x > 0$.

The theorem is proved. □

3. Application to solving integral equations

Consider the integral equation

$$f(x) + \frac{\lambda}{2\sqrt{2\pi}} \int_0^{+\infty} f(t) \psi(g)(x,t) dt = h(x). \tag{20}$$

Here $\lambda \in R$, g and h are continuous functions of $L(R_+)$, f is the unknown function, and

$$\psi(g)(x,t) = g(x+1+t) + g(|x+1-t|) + g(|x-1+t|) + g(|x-1-t|). \tag{21}$$

Theorem 7. *With the condition* $1 + \lambda \cos y (F_c g)(y) \neq 0$, $\forall\, y \in R_+$, *there exists a unique solution in* $L(R_+)$ *of* (20) *which is defined by*

$$f = h - \lambda (h \overset{\gamma}{*} \varphi).$$

Here, $\varphi(x) \in L(R_+)$ *and it is defined by*

$$(F_c \varphi)(y) = \frac{(F_c g)(y)}{1 + \lambda \cos y (F_c g)(y)} \cdot$$

Proof. The equation (20) can be rewritten in the form

$$f + \lambda (f \overset{\gamma}{*} g) = h.$$

Due to Theorem 1,

$$(F_c f)(y) + \lambda \cos y (F_c f)(y)(F_c g)(y) = (F_c h)(y).$$

It follows that

$$(F_c f)(y)\big[1 + \lambda \cos y (F_c g)(y)\big] = (F_c h)(y).$$

Since $1 + \lambda \cos y (F_c g)(y) \neq 0$,

$$(F_c f)(y) = (F_c h)(y) \cdot \frac{1}{1 + \lambda \cos y (F_c g)(y)} \cdot$$

Therefore

$$(F_c f)(y) = (F_c h)(y)\Big[1 - \frac{\lambda \cos y (F_c g)(y)}{1 + \lambda \cos y (F_c g)(y)}\Big] \cdot$$

Due to Wiener-Levi's theorem, there exists a continuous function $\varphi \in L(R_+)$ such that

$$(F_c \varphi)(y) = \frac{(F_c g)(y)}{1 + \lambda \cos y (F_c g)(y)} \cdot$$

It follows that

$$(F_c f)(y) = (F_c h)(y)\big[1 - \lambda \cos y (F_c \varphi)(y)\big].$$

Hence

$$f(x) = h(x) - \lambda F_c\big[\cos y (F_c h)_(y)(F_c \varphi)(y)\big](x).$$

Thus

$$f = h - \lambda (h \overset{\gamma}{*} \varphi).$$

By Theorem 1, $f \in L(R_+)$. The theorem is proved. □

We see that the above theorem only asserts the existence and uniqueness of solution of equation (18), but does not show how to find the function φ. In general, to find an explicit form for φ is a difficult problem. In the next example, we describe a case where φ can be represented by an explicit formula.

Example. Consider the integral equation

$$f(x) + \frac{\lambda}{2\sqrt{2\pi}} \int_0^{+\infty} \frac{1 - \cos at}{t^2} \psi(g)(x,t) = h(x) \tag{22}$$

with ψ being defined by (21), λ a positive real number, $0 < a \leq \frac{\pi}{2}$, h continuous function belonging to $L(R_+)$, h already known, f the unknown function.

We can rewrite (22) as

$$f(x) + \lambda (f \overset{\gamma}{*} g)(x) = h(x)$$

with

$$g(t) = \frac{1-\cos(at)}{t^2}\cdot$$

Applying formula 3.786.3 at page 447 in [5]

$$\int_0^{+\infty} \frac{1-\cos(at)}{t^2}\cos(bt)dt = \begin{cases} \frac{\pi}{2}(a-b), & b \le a \\ 0, & b > a \end{cases}$$

we obtain

$$(F_c g)(t) = \begin{cases} \sqrt{\frac{\pi}{2}}(a-t), & 0 < t \le a \\ 0, & t > a. \end{cases}$$

Thus $1 + \lambda \cos t (F_c g)(t) \neq 0$. Now we find a function φ satisfying

$$(F_c \varphi)(t) = \frac{(F_c g)(t)}{1 + \lambda \cos t (F_c g)(t)}\cdot$$

We have

$$\begin{aligned} \varphi &= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \cos(xt) \cdot \frac{(F_c g)(t)}{1 + \lambda \cos t (F_c g)(t)} dt \\ &= \sqrt{\frac{2}{\pi}} \int_0^{a} \cos(xt) \cdot \frac{\sqrt{\frac{\pi}{2}}(a-t)}{1 + \lambda \cos t (a-t)\sqrt{\frac{\pi}{2}}} dt \\ &= \sqrt{2} \int_0^{a} \frac{(a-t)\cos(xt)}{\sqrt{2} + \lambda\sqrt{\pi}\ (a-t)\cos t} dt. \end{aligned}$$

And the solution of the integral equation (22) has the form

$$f = h - \lambda(h \overset{\gamma}{*} \varphi).$$

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