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CASTELNUOVO-MUMFORD REGULARITY AND INITIAL IDEALS WITH NO EMBEDDED PRIME IDEAL

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Abstract. Let $A = K[x_1, \dots, x_n]$ be the polynomial ring over a field K and $P \subset A$ a homogeneous prime ideal with no linear form. Let $\text{reg}(P)$ denote the Castelnuovo–Mumford regularity of P and $e(A/P)$ the multiplicity of A/P. It will be shown that if P possesses an initial ideal with no embedded prime ideal, then the regularity of P satisfies the inequality $\text{reg}(P) \leq e(A/P) - \text{codim}(A/P) + 1$, where $\text{codim}(A/P)$ is the codimension of A/P.

Introduction

Let K be a field and $A = K[x_1, \dots, x_n]$ the polynomial ring in n variables over K. Let $P \subset A$ be a homogeneous prime ideal with no linear form. Write $\text{reg}(P)$ for the Castelnuovo–Mumford regularity of P and $e(A/P)$ for the multiplicity of A/P. In [2, p. 93] it is conjectured that the regularity of P satisfies the inequality

$$\text{reg}(P) \leq e(A/P) - \text{codim}(A/P) + 1, \tag{1}$$

where $\text{codim}(A/P)$ is the codimension of A/P.

Many papers in algebraic geometry and commutative algebra discuss the topics related with the inequality (1). However, the conjecture itself is widely open in general, except for limited special cases considered in algebraic geometry (e.g., [4], [8]) and in commutative algebra (e.g., [5]).

The main purpose of the present paper is to show that if P possesses an initial ideal with no embedded prime ideal, then $\text{reg}(P)$ satisfies the inequality (1). See Theorem 1.1. It then turns out to be an exciting research problem to find a reasonable class which consists of those prime ideals $P \subset A$ such that P possesses an initial ideal with no embedded prime ideal. Note, however, that the inequality (1) is automatically satisfied when A/P is Cohen–Macaulay. Thus we are interested in those prime ideals $P \subset A$ having an initial ideal with no embedded prime ideal such that A/P is not Cohen–Macaulay. Such a prime ideal will be presented in Examples 2.1 and 2.2. We refer the reader to, e.g., [9] for fundamental materials on initial ideals.

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1. Monomial ideals with no embedded prime ideal

Let $A = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K. An ideal $I \subset A$ is called *pure* if all minimal prime ideals of I have the same height.

For example, the ideal $I = (x_1^2, x_2) \cap (x_1)$ is pure, but has an embedded prime ideal. The ideal $J = (x_1, x_2) \cap (x_3)$ has no embedded prime ideal, but J is not pure.

We will associate each monomial $u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ belonging to A with the squarefree monomial

$$u^{pol} = x_{11}x_{12} \cdots x_{1a_1} x_{21}x_{22} \cdots x_{2a_2} \cdots x_{n1}x_{n2} \cdots x_{na_n}$$

in the variables x_{ij} with $1 \le i \le n$ and $1 \le j \le a_i$.

Let $I = (u_1, \ldots, u_s)$ be a monomial ideal of A, where $\{u_1, \ldots, u_s\}$ is the minimal set of monomial generators of I. Then fix a polynomial ring B over K such that all monomials $u_1^{pol}, \ldots, u_s^{pol}$ belong to B and consider the squarefree ideal $I^{pol} = (u_1^{pol}, \ldots, u_s^{pol})$ of B.

It is known [10, p. 107] that, for monomial ideals I and J, one has

$$(I \cap J)^{pol} = I^{pol} \cap J^{pol}.$$

Even though the following Lemma 1.1 is [10, Theorem 1.2 (ii)], we give its proof which will be required in the proof of Lemma 1.2.

Lemma 1.1. *If $I \subset A$ is a monomial ideal which is pure and has no embedded prime ideal, then I^{pol} is pure.*

Proof. Let $I = \cap_\lambda Q_\lambda$ denote the irreducible primary decomposition of I. Since I is pure and has no embedded prime ideal, all prime ideals belonging to I have the same height, say $= h$. Let $Q_\lambda = (x_{p_1}^{\lambda_1}, \ldots, x_{p_h}^{\lambda_h})$ with $1 \le p_1 < \cdots < p_h \le n$ and with each $\lambda_j > 0$. Since $I^{pol} = \cap_\lambda Q_\lambda^{pol}$ and since

$$Q_\lambda^{pol} = \bigcap_{1 \le i_1 \le \lambda_1, \ldots, 1 \le i_h \le \lambda_h} (x_{p_1 i_1}, x_{p_2 i_2}, \ldots, x_{p_h i_h}),$$

it follows that I^{pol} is pure, as desired. □

Let $I \subset A$ be an ideal which is pure and has no embedded prime ideal, and write h for the height of I. We say that I is *strongly connected* if, for any minimal primes P and P' of I, there is a sequence of minimal prime ideals

$$(P =) \, P_0, P_1, \ldots, P_m \, (= P')$$

of I such that $\text{height}(P_{i-1} + P_i) = h + 1$ for all $i = 1, \ldots, m$.

Lemma 1.2. *Let $I \subset A$ be a monomial ideal with no embedded prime ideal and suppose that $\sqrt{I}$ is pure and strongly connected. Then I^{pol} is pure and strongly connected.*

Proof. Since $\sqrt{I}$ is pure, it follows that I is pure. Then Lemma 1.1 says that I^{pol} is pure. Let $h = \text{height}(I) = \text{height}(\sqrt{I}) = \text{height}(I^{pol})$.

In order to see why I^{pol} is strongly connected, let $P = (x_{i_1j_1}, \dots, x_{i_hj_h})$ and $Q = (x_{k_1\ell_1}, \dots, x_{k_h\ell_h})$ be minimal prime ideals of I^{pol}. It follows from the proof of Lemma 1.1 that $P' = (x_{i_1}, \dots, x_{i_h})$ and $Q' = (x_{k_1}, \dots, x_{k_h})$ are minimal prime ideals of $\sqrt{I}$. Since $\sqrt{I}$ is strongly connected, there is a sequence of minimal prime ideals

$$(P' =)\, P'_0, P'_1, \dots, P'_m\, (= Q')$$

of $\sqrt{I}$ such that $\text{height}(P'_{\nu-1} + P'_\nu) = h + 1$ for $\nu = 1, \dots, m$. Let $P'_\nu = (x_{i_{\nu_1}}, \dots, x_{i_{\nu_h}})$ and $P_\nu = (x_{i_{\nu_1}1}, \dots, x_{i_{\nu_h}1})$. It is clear that the sequence of minimal prime ideals

$$\begin{aligned} P = (x_{i_1j_1}, \dots, x_{i_hj_h}), (x_{i_11}, x_{i_2j_2}, \dots, x_{i_hj_h}), (x_{i_11}, x_{i_21}, x_{i_3j_3}, \dots, x_{i_hj_h}), \\ \dots, (x_{i_11}, \dots, x_{i_h1}) = P_0, P_1, P_2, \dots, P_m = (x_{k_11}, \dots, x_{k_h1}), \\ (x_{k_11}, \dots, x_{k_{h-1}1}, x_{k_h\ell_h}), (x_{k_11}, \dots, x_{k_{h-2}1}, x_{k_{h-1}\ell_{h-1}}, x_{k_h\ell_h}), \\ \dots, (x_{k_1\ell_1}, \dots, x_{k_h\ell_h}) = Q \end{aligned}$$

of I^{pol} satisfies the condition for I^{pol} to be strongly connected. □

We now come to the main theorem of the present paper.

Theorem 1.1. *Let $A = K[x_1, \dots, x_n]$ be the polynomial ring over a field K and $P \subset A$ a homogeneous prime ideal which contains no linear form. Suppose that there is a monomial order $<$ on A such that $in_<(P)$ has no embedded prime ideal. Then*

$$\text{reg}(P) \le e(A/P) - \text{codim}(A/P) + 1.$$

Proof. It is known [7, Theorem 1] that $\sqrt{in_<(P)}$ is pure and strongly connected. Since $in_<(P)$ has no embedded prime ideal, Lemma 1.2 guarantees that $in_<(P)^{pol}$ is pure and strongly connected.

Now, by virtue of [13, Theorem 3.2], it follows that

$$\text{reg}(in_<(P)^{pol}) \le e(B/in_<(P)^{pol}) - \text{codim}(B/in_<(P)^{pol}) + 1,$$

where B is a polynomial ring with $in_<(P)^{pol} \subset B$. Thus

$$\begin{aligned} \text{reg}(P) &\le \text{reg}(in_<(P)) \\ &= \text{reg}(in_<(P)^{pol}) \\ &\le e(B/in_<(P)^{pol}) - \text{codim}(B/in_<(P)^{pol}) + 1 \\ &= e(A/in_<(P)) - \text{codim}(A/in_<(P)) + 1 \\ &= e(A/P) - \text{codim}(A/P) + 1, \end{aligned}$$

as desired. □

Remark. Let geom-deg(A/I) (resp. arith-deg(A/I)) denote the geometric degree (resp. arithmetic degree) of A/I, where I is a homogeneous ideal of A. Consult e.g., [1] for the definition of the geometric degree and arithmetic degree of A/I. If $I \subset A$ is a homogeneous (but, not necessarily prime) ideal such that $in_<(I)$ has no embedded prime ideal for some monomial order $<$ on A, then $\mathrm{reg}(I) \leq$ geom-deg(A/I). If, in addition, I is pure, then $\mathrm{reg}(I) \leq e(A/I)$. In fact, it follows from [6, Theorem 1.1], [3, Theorem 3.8] and [11, Proposition 4.1] that

$$\begin{aligned} \mathrm{reg}(I) &\leq \mathrm{reg}(in_<(I)) \\ &\leq \text{arith-deg}(A/in_<(I)) \\ &= \text{geom-deg}(A/in_<(I)) \\ &\leq \text{geom-deg}(A/I). \end{aligned}$$

2. Examples

Let, as before, $A = K[x_1, \ldots, x_n]$ denote the polynomial ring over a field K and $P \subset A$ a homogeneous prime ideal with no linear form. It is a simple fact that the inequality (1) is satisfied if the quotient ring A/P is Cohen–Macaulay. Thus we are interested in prime ideals $P \subset A$, for which A/P is *not* Cohen–Macaulay, such that P possesses an initial ideal with no embedded prime.

Example 2.1. Let K be a field of characteristic 2. In [12, Theorem 20], Terai succeeded in constructing a homogeneous prime ideal P of the polynomial ring $A = K[x_1, \ldots, x_{14}]$ such that (i) the quotient ring A/P is *not* Cohen–Macaulay, and (ii) with respect to a reverse lexicographic monomial order $<_{rev}$ on A, the initial ideal $in_{<_{rev}}(P)$ is generated by squarefree quadratic monomials. (In particular $in_{<_{rev}}(P)$ possesses *no* embedded prime.) The prime ideal is generated by 42 quadratic polynomials and Krull-dim $A/P = 4$. Moreover, $A/in_{<_{rev}}(P)$ is isomorphic to the polynomial ring in one variable over the Stanley–Reisner ring of a triangulation of the real projective plane.

We do not know if there exists a prime ideal P of the polynomial ring $A = K[x_1, \ldots, x_n]$ over a field K of characteristic 0 such that (i) A/P is not Cohen–Macaulay, and (ii) P possesses an initial ideal which is generated by squarefree quadratic monomials.

We conclude the present paper with an example which was discovered by Laura Matusevich. (This example was originally in [14, Example 1.2].)

Example 2.2. Let R denote the affine semigroup ring generated by the monomials $t_1, t_1t_3, t_1t_3^3, t_1t_2, t_1t_2t_3, t_1t_2t_3^3$ and $I \subset K[x_1, \ldots, x_6]$ its toric ideal. Then R is not Cohen–Macaulay and I is generated by the binomials $x_5^3 - x_4^2x_6, x_3x_5 - x_2x_6, x_3x_4 - x_1x_6, x_1x_5 - x_2x_4, x_1x_4x_6 - x_2x_5^2, x_1^2x_6 - x_2^2x_5$ and $x_1^2x_3 - x_2^3$. Let $<_{rev}$ be the reverse lexicographic order on $K[x_1, \ldots, x_6]$ induced by the ordering $x_1 > x_3 > x_4 > x_5 > x_6 > x_2$. Then the initial ideal $in_{<_{rev}}(I)$ is generated by the monomials $x_5^3, x_3x_5, x_3x_4, x_1x_5, x_1x_4x_6, x_1^2x_6$ and $x_1^2x_3$, and its irreducible primary decomposition is $in_{<_{rev}}(I) = (x_1, x_3, x_5^3) \cap (x_1^2, x_4, x_5) \cap (x_3, x_5, x_6)$. Thus

$in_{<_{rev}}(I)$ has no embedded prime ideal. (The authors are grateful to K. Yanagawa for informing them this example.)

References

[1] D. Bayer and D. Mumford, *What can be computed in algebraic geometry?*, *in* "Computational Algebraic Geometry and Commutative Algebra (D. Eisenbud and L. Robbiano, Eds.)," Cambridge University Press, 1993, pp. 1 – 48.

[2] D. Eisenbud and S. Goto, *Linear free resolutions and minimal multiplicity*, J. Algebra **88** (1984), 89–133.

[3] A. Frübis-Krüger and N. Terai, *Bounds for the regularity of monomial ideals*, Mathematiche (Catania) **53** (1998), 83–97.

[4] L. Gruson, R. Lazarsfeld and C. Peskine, *On a theorem of Castelnuovo, and the equations defining space curves*, Invent. Math. **72** (1983), 491–506.

[5] J. Herzog and T. Hibi, *Castelnuovo–Mumford regularity of simplicial semigroup rings with isolated singularity*, Proc. Amer. Math. Soc. **131** (2003), 2641–2647.

[6] L. T. Hoa and N. V. Trung, *On the Castelnuovo–Mumford regularity and the arithmetic degree of monomial ideals*, Math. Z. **229** (1998), 519–537.

[7] M. Kalkbrener and B. Sturmfels, *Initial complexes of prime ideals*, Adv. Math. **116** (1995), 365–376.

[8] R. Lazarsfeld, *A sharp Castelnuovo bound for smooth surfaces*, Duke Math. J. **55** (1987), 423–429.

[9] H. Ohsugi and T. Hibi, *Quadratic initial ideals of root systems*, Proc. Amer. Math. Soc. **130** (2002), 1913–1922.

[10] J. Stückrad and W. Vogel, *"Buchsbaum Rings and Applications"*, Springer–Verlag, Berlin, 1986.

[11] B. Sturmfels, N. V. Trung and W. Vogel, *Bounds on degrees of projective schemes*, Math. Ann. **302** (1995), 417–432.

[12] N. Terai, *Some remarks on algebras with straightening laws*, J. Pure and Appl. Algebra **95** (1994), 87–101.

[13] N. Terai, *Eisenbud–Goto inequality for Stanley–Reisner rings*, *in* "Geometric and Combinatorial Aspects of Commutative Algebra" (J. Herzog and G. Restuccia, Eds.), Lect. Notes in Pure and Appl. Math., 217, Dekker, New York, 2001, pp. 379–391.

[14] N. V. Trung and L. T. Hoa, *Affine semigroups and Cohen–Macaulay rings generated by monomials*, Trans. Amer. Math. Soc. **298** (1986), 145–167.

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