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A CONVERSE OF THE JENSEN INEQUALITY FOR CONVEX MAPPINGS OF SEVERAL VARIABLES AND APPLICATIONS

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Abstract. In this paper we point out a converse result of the celebrated Jensen inequality for differentiable convex mappings of several variables and apply it to counterpart well-known analytic inequalities. Applications to Shannon's and Rényi's entropy mappings are also given.

1. Introduction

Let $f : X \to \mathbb{R}$ be a convex mapping defined on the linear space X and $x_i \in X$, $p_i \geq 0$ $(i = 1, ..., m)$ with $P_m := \sum_{i=1}^{m} p_i > 0$.

The following inequality is known in the literature as Jensen's inequality

$$f\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i\right) \leq \frac{1}{P_m}\sum_{i=1}^{m} p_i f(x_i). \tag{1.1}$$

There are many well known inequalities which are particular cases of Jensen's inequality such as the weighted arithmetic mean-geometric mean-harmonic mean inequality, the Ky Fan inequality, the Hölder inequality, etc. For a comprehensive list of recent results on the Jensen inequality, see the book [1] and the papers [2]-[14] where further results are given.

In this paper, we point out a converse inequality for Jensen's result (1.1) for the case of differentiable convex mappings whose partial derivatives are bounded. Applications for some particular inequalities and for the Shannon and Rényi entropy mappings are also given.

2. A converse inequality

The following converse of Jensen's inequality holds.

Theorem 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex mapping and $x_i \in \mathbb{R}^n$, $i = 1, ..., m$. Suppose that there exist $\psi, \phi \in \mathbb{R}^n$ satisfying $\psi \leq x_i \leq \phi$ (the order is considered on the co-ordinates) and $m, M \in \mathbb{R}^n$ are such that* $m \leq \dfrac{\partial f(x_i)}{\partial x} \leq M$

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for all $i \in \{1, ..., m\}$*. Then, for all* $p_i \geq 0$ $(i = 1, ..., m)$ *with* $P_m := \sum_{i=1}^{m} p_i > 0$*, it holds*

$$0 \leq \frac{1}{P_m} \sum_{i=1}^{m} p_i f(x_i) - f\Big(\frac{1}{P_m} \sum_{i=1}^{m} p_i x_i\Big) \leq \frac{1}{4} \|\Phi - \phi\| \|M - m\|, \tag{2.1}$$

where $\|\cdot\|$ *is the usual Euclidean norm on* $\mathbb{R}^n$*.*

Proof. For the sake of completeness, we first prove the following inequality for convex functions which was obtained by Dragomir and Goh in [14]:

$$\begin{aligned} 0 &\leq \frac{1}{P_m} \sum_{i=1}^{m} p_i f(x_i) - f\Big(\frac{1}{P_m} \sum_{i=1}^{m} p_i x_i\Big) \\ &\leq \frac{1}{P_m} \sum_{i=1}^{m} p_i \langle x_i, \nabla f(x_i) \rangle - \Big\langle \frac{1}{P_m} \sum_{i=1}^{m} p_i x_i, \frac{1}{P_m} \sum_{i=1}^{m} p_i \nabla f(x_i) \Big\rangle, \end{aligned} \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$ and

$$\nabla f(x) = \frac{\partial f(x)}{\partial x} := \Big(\frac{\partial f(x)}{\partial x^1}, ..., \frac{\partial f(x)}{\partial x^n}\Big)$$

is the vector of the partial derivatives, $x = (x^1, ..., x^n) \in \mathbb{R}^n$.

As $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable convex, we have

$$f(x) - f(y) \geq \langle \nabla f(y), x - y \rangle, \text{ for all } x, y \in \mathbb{R}^n. \tag{2.3}$$

Substituting $x = \frac{1}{P_m} \sum_{i=1}^{m} p_i x_i$ and $y = x_j$ to (2.3) yields

$$f\Big(\frac{1}{P_m} \sum_{i=1}^{m} p_i x_i\Big) - f(x_j) \geq \Big\langle \nabla f(x_j), \frac{1}{P_m} \sum_{i=1}^{m} p_i x_i - x_j \Big\rangle \tag{2.4}$$

for all $j \in \{1, ..., n\}$.

Multiplying (2.4) by $p_j \geq 0$ and summing over j from 1 to m, we obtain

$$\begin{aligned} &P_m f\Big(\frac{1}{P_m} \sum_{i=1}^{m} p_i x_i\Big) - \sum_{j=1}^{m} p_j f(x_j) \\ \geq\ &\frac{1}{P_m} \Big\langle \sum_{j=1}^{m} p_j \nabla f(x_j), \sum_{i=1}^{m} p_i x_i \Big\rangle - \sum_{j=1}^{m} p_j \langle \nabla f(x_j), x_j \rangle. \end{aligned}$$

Dividing this inequality by $P_m > 0$, we obtain (2.2).

A simple calculation shows that

$$\begin{aligned} &\frac{1}{P_m} \sum_{i=1}^{m} p_i \langle x_i, \nabla f(x_i) \rangle - \Big\langle \frac{1}{P_m} \sum_{i=1}^{m} p_i x_i, \frac{1}{P_m} \sum_{i=1}^{m} p_i \nabla f(x_i) \Big\rangle \\ = &\frac{1}{2P_m^2} \sum_{i,j=1}^{m} p_i p_j \langle x_i - x_j, \nabla f(x_i) - \nabla f(x_j) \rangle. \end{aligned} \tag{2.5}$$

Taking the modulus in both parts of (2.5), and noting that the left hand side is positive (by (2.2)), by Schwartz's inequality we obtain

$$\begin{aligned} (2.6) \qquad & \frac{1}{P_m}\sum_{i=1}^{m} p_i \langle x_i, \nabla f(x_i)\rangle - \Big\langle \frac{1}{P_m}\sum_{i=1}^{m} p_i x_i, \frac{1}{P_m}\sum_{i=1}^{m} p_i \nabla f(x_i) \Big\rangle \\ \leq\ & \frac{1}{2P_m^2}\sum_{i,j=1}^{m} p_i p_j \left|\langle x_i - x_j, \nabla f(x_i) - \nabla f(x_j)\rangle\right| \\ \leq\ & \frac{1}{2P_m^2}\sum_{i,j=1}^{m} p_i p_j \left\| x_i - x_j\right\| \left\|\nabla f(x_i) - \nabla f(x_j)\right\|. \end{aligned}$$

Using the Cauchy-Buniakowsky-Schwartz inequality for double sums, we can state that

$$\begin{aligned} (2.7) \quad & \frac{1}{2P_m^2}\sum_{i,j=1}^{m} p_i p_j \left\| x_i - x_j\right\| \left\|\nabla f(x_i) - \nabla f(x_j)\right\| \\ \leq\ & \Big(\frac{1}{2P_m^2}\sum_{i,j=1}^{m} p_i p_j \left\| x_i - x_j\right\|^2\Big)^{\frac{1}{2}} \times \Big(\frac{1}{2P_m^2}\sum_{i,j=1}^{m} p_i p_j \left\|\nabla f(x_i) - \nabla f(x_j)\right\|^2\Big)^{\frac{1}{2}}. \end{aligned}$$

As a simple calculation shows that

$$\frac{1}{2P_m^2}\sum_{i,j=1}^{m} p_i p_j \left\| x_i - x_j\right\|^2 = \frac{1}{P_m}\sum_{i=1}^{m} p_i \left\| x_i\right\|^2 - \Big\|\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i\Big\|^2$$

and

$$\begin{aligned} & \frac{1}{2P_m^2}\sum_{i,j=1}^{m} p_i p_j \left\|\nabla f(x_i) - \nabla f(x_j)\right\|^2 \\ =\ & \frac{1}{P_m}\sum_{i=1}^{m} p_i \left\|\nabla f(x_i)\right\|^2 - \Big\|\frac{1}{P_m}\sum_{i=1}^{m} \nabla f(x_i)\Big\|^2. \end{aligned}$$

By (2.6) and (2.7), we can assert that

$$\begin{aligned} (2.8) \qquad & \frac{1}{P_m}\sum_{i=1}^{m} p_i \langle x_i, \nabla f(x_i)\rangle - \Big\langle \frac{1}{P_m}\sum_{i=1}^{m} p_i x_i, \frac{1}{P_m}\sum_{i=1}^{m} p_i \nabla f(x_i) \Big\rangle \\ \leq\ & \Big(\frac{1}{P_m}\sum_{i=1}^{m} p_i \left\| x_i\right\|^2 - \Big\|\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i\Big\|^2\Big)^{\frac{1}{2}} \\ & \times\Big(\frac{1}{P_m}\sum_{i=1}^{m} p_i \left\|\nabla f(x_i)\right\|^2 - \Big\|\frac{1}{P_m}\sum_{i=1}^{m} \nabla f(x_i)\Big\|^2\Big)^{\frac{1}{2}}. \end{aligned}$$

Now, let us observe that

$$\begin{aligned} (2.9)\qquad & \frac{1}{P_m}\sum_{i=1}^{m}p_i\left\|x_i\right\|^2-\left\|\frac{1}{P_m}\sum_{i=1}^{m}p_ix_i\right\|^2 \\ = & \left\langle \phi-\frac{1}{P_m}\sum_{i=1}^{m}p_ix_i,\frac{1}{P_m}\sum_{i=1}^{m}p_ix_i-\psi\right\rangle-\frac{1}{P_m}\sum_{i=1}^{m}p_i\left\langle \phi-x_i,x_i-\psi\right\rangle . \end{aligned}$$

As $\psi\leq x_i\leq\phi$ $(i\in\{1,...,m\})$, we have $\langle\phi-x_i,x_i-\psi\rangle\geq 0$ for all $i\in\{1,...,m\}$; hence

$$\sum_{i=1}^{m}p_i\left\langle \phi-x_i,x_i-\psi\right\rangle\geq 0$$

and, by (2.9), we obtain

$$\begin{aligned} (2.10)\qquad & \frac{1}{P_m}\sum_{i=1}^{m}p_i\left\|x_i\right\|^2-\left\|\frac{1}{P_m}\sum_{i=1}^{m}p_ix_i\right\|^2 \\ \leq & \left\langle \phi-\frac{1}{P_m}\sum_{i=1}^{m}p_ix_i,\frac{1}{P_m}\sum_{i=1}^{m}p_ix_i-\psi\right\rangle. \end{aligned}$$

It is well known that if $y,z\in\mathbb{R}^n$ then

$$(2.11)\qquad 4\left\langle z,y\right\rangle\leq\left\|z+y\right\|^2,$$

where the equality holds iff $z=y$.

Now, applying (2.11) for $z=\phi-\frac{1}{P_M}\sum_{i=1}^{m}p_ix_i$ and $y=\frac{1}{P_m}\sum_{i=1}^{m}p_ix_i-\psi$, we have

$$\left\langle \phi-\frac{1}{P_m}\sum_{i=1}^{m}p_ix_i,\frac{1}{P_m}\sum_{i=1}^{m}p_ix_i-\psi\right\rangle\leq\frac{1}{4}\left\|\phi-\psi\right\|^2.$$

Then, from (2.9) and (2.10) we deduce that

$$(2.12)\qquad \frac{1}{P_m}\sum_{i=1}^{m}p_i\left\|x_i\right\|^2-\left\|\frac{1}{P_m}\sum_{i=1}^{m}p_ix_i\right\|^2\leq\frac{1}{4}\left\|\phi-\psi\right\|^2.$$

Similarly, we can state that

$$(2.13)\qquad \frac{1}{P_m}\sum_{i=1}^{m}p_i\left\|\nabla f\left(x_i\right)\right\|^2-\left\|\frac{1}{P_m}\sum_{i=1}^{m}\nabla f\left(x_i\right)\right\|^2\leq\frac{1}{4}\left\|M-m\right\|^2.$$

Finally, by (2.8), (2.12) and (2.13) we have

$$\begin{aligned} (2.14)\qquad & \frac{1}{P_m}\sum_{i=1}^{m}p_i\left\langle x_i,\nabla f\left(x_i\right)\right\rangle-\left\langle\frac{1}{P_m}\sum_{i=1}^{m}p_ix_i,\frac{1}{P_m}\sum_{i=1}^{m}p_i\nabla f\left(x_i\right)\right\rangle \\ \leq & \frac{1}{4}\left\|\Phi-\phi\right\|\left\|M-m\right\|, \end{aligned}$$

which, by (2.2), gives the desired inequality (2.1). □

Remark 1. A similar result for integrals can be stated, but we omit the details.

Remark 2. The conditions

$$\psi \leq x_i \leq \phi,\ m \leq \frac{\partial f(x_i)}{\partial x} \leq M \ \ (i = 1, ..., m) \tag{2.15}$$

can be replaced by the more general conditions

$$\sum_{i=1}^{m} p_i \langle \phi - x_i, x_i - \psi \rangle \geq 0 \text{ and } \sum_{i=1}^{m} p_i \Big\langle M - \frac{\partial f(x_i)}{\partial x}, \frac{\partial f(x_i)}{\partial x} - m \Big\rangle \geq 0 \tag{2.16}$$

and the conclusion (2.1) is valid.

Remark 3. Even if the new inequality (2.1) is not as sharp as the inequality (2.2), it may be more useful in practice when only some bounds of the partial derivatives $\frac{\partial f}{\partial x}$ and of the vectors x_i $(i = 1, ..., m)$ are known. Namely, it provides the opportunity to estimate the difference

$$\frac{1}{P_m}\sum_{i=1}^{m} p_i f(x_i) - f\Big(\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i\Big) =: \Delta(f, x, p)$$

when the quantities $\|\phi - \psi\|$ and $\|M - m\|$ are known. For example, if the partial derivatives $\frac{\partial f}{\partial x}$ are bounded, i.e., there exists $m, M \in \mathbb{R}^n$ such that $m \leq \frac{\partial f}{\partial x} \leq M$ on the co-ordinates, and if we choose the vector x_i $(i = 1, ..., m)$ not "very far" from a constant vector x_0, i.e., $\|\phi - \psi\| \leq \frac{4\varepsilon}{\|M - m\|}$, $\varepsilon > 0$ then, by (2.1), we can conclude that

$$0 \leq \Delta(f, x, p) \leq \varepsilon.$$

The case of convex mappings of a real variable can be stated as follows [20].

Corollary 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable convex mapping and $x_i \in I$ for all $i \in \{1, ..., m\}$. Then*

$$\begin{aligned} 0 &\leq \frac{1}{P_m}\sum_{i=1}^{m} p_i f(x_i) - f\Big(\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i\Big) \\ &\leq \frac{1}{4}(M - m)\big(f'(M) - f'(m)\big), \end{aligned} \tag{2.17}$$

where $p_i > 0$ $(i = 1, ..., m)$ and $P_m := \sum_{i=1}^{m} p_i > 0$.

The proof follows from the above findings because the mapping f' is monotonic nondecreasing, and then $f'(m) \leq f'(x_i) \leq f'(M)$ for all $i \in \{1, ..., m\}$.

3. Applications for weighted means

Consider the classical weighted means

$$\begin{aligned} A_n(\bar{p},x) &:= \sum_{i=1}^{n} p_i x_i \text{ - the arithmetic mean,} \\ G_n(\bar{p},x) &:= \prod_{i=1}^{n} x_i^{p_i} \text{ - the geometric mean,} \\ H_n(\bar{p},x) &:= \frac{1}{\sum_{i=1}^{n} \frac{p_i}{x_i}} \text{ - the harmonic mean,} \end{aligned}$$

provided that $x_i > 0$ $\left(i=\overline{1,n}\right)$ and p_i $(i=1,...,n)$ is a probability distribution, i.e., $\sum_{i=1}^{n} p_i = 1$.

The following inequality is known in the literature as the *arithmetic mean-geometric mean-harmonic mean inequality*

$$A_n(\bar{p},x) \geq G_n(\bar{p},x) \geq H_n(\bar{p},x), \tag{3.1}$$

where the equalities hold iff $x_1 = ... = x_n$ (for $p_i > 0,\ i=1,...,n$).

Corollary 2. *Let $0 < m \leq x_i \leq M < \infty$, $p_i > 0$ $(i=1,...,n)$. Then*

$$1 \leq \frac{A_n(\bar{p},x)}{G_n(\bar{p},x)} \leq \exp\left[\frac{(M-m)^2}{4mM}\right]. \tag{3.2}$$

Equalities hold in (3.2) *simultaneously iff $x_1 = ... = x_n$.*

The proof follows from (2.17) where $f(x) = -\ln x$, $x > 0$.

Corollary 3. *Let $0 < m \leq y_i \leq M < \infty$, $p_i > 0$ $(i=1,...,n)$. Then*

$$1 \leq \frac{G_n(\bar{p},\bar{y})}{H_n(\bar{p},\bar{y})} \leq \exp\left[\frac{(M-m)^2}{4mM}\right]. \tag{3.3}$$

Equalities hold iff $y_1 = ... = y_n$.

Corollary 4. *Let $p \geq 1$ and $0 \leq m \leq x_i \leq M < \infty$, $p_i > 0$ $(i=1,..,n)$. Then*

$$0 \leq \sum_{i=1}^{n} p_i x_i^p - \left(\sum_{i=1}^{n} p_i x_i\right)^p \leq \frac{p}{4}(M-m)\left(M^{p-1} - m^{p-1}\right). \tag{3.4}$$

Equalities hold iff $x_1 = ... = x_n$.

The proof follows from applying (2.17) to the mapping $f(x) = x^p$, $p \geq 1$, $x \geq 0$.

Finally, we have

Corollary 5. *Let p_i, x_i be as in Corollary 3. Then*

$$1 \leq \frac{\prod_{i=1}^{n} x_i^{p_i x_i}}{\left[A_n\left(\bar{p}, x\right)\right]^{A_n(\bar{p},x)}} \leq \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}. \tag{3.5}$$

Equalities hold iff $x_1 = ... = x_n$.

The proof follows from (2.1) if we choose $f(x) = x \ln x$, $x > 0$.

4. Applications for Shannon's entropy

Let X be a random variable with the range $R = \{x_1, ..., x_n\}$ and the probability distribution $p_1, ..., p_n$ ($p_i > 0$, $i = 1, ..., n$). Define the Shannon entropy mapping

$$H(X) := -\sum_{i=1}^{n} p_i \ln p_i.$$

The following well known theorem concerns the maximum possible value of $H(X)$ in terms of the size of R [15, p. 27].

Theorem 2. *Let X be defined as above. Then*

$$0 \leq H(X) \leq \ln n. \tag{4.1}$$

Furthermore, $H(X) = 0$ iff $p_i = 1$ for some i and $H(X) = \ln n$ iff $p_i = \frac{1}{n}$ for all $i \in \{1, ..., n\}$.

In [14], Dragomir and Goh proved the following counterpart result.

Theorem 3. *Let X be defined as above. Then*

$$0 \leq \ln n - H(X) \leq \sum_{1 \leq i < j \leq n} \left(p_i - p_j\right)^2. \tag{4.2}$$

Equalities hold simultaneously in both inequalities iff $p_i = \frac{1}{n}$ for all $i \in \{1, ..., n\}$.

Choosing $f(x) = -\ln x$ and $x_i = \xi_i$, $i = 1, ..., n$, we can deduce the next lemma from Corollary 1.

Lemma 1. *Let $0 < m \leq \xi_i \leq M < \infty$, $p_i > 0$ $(i = 1, ..., n)$ with $\sum_{i=1}^{n} p_i = 1$. Then*

$$0 \leq \ln\left(\sum_{i=1}^{n} p_i \xi_i\right) - \sum_{i=1}^{n} p_i \ln \xi_i \leq \frac{(M-m)^2}{4mM}. \tag{4.3}$$

Lemma 1 provides the following converse inequality for the Shannon entropy mapping (see also [17]).

Theorem 4. *Let* X *be as above and let* $p := \min_{i=\overline{1,n}} p_i$ *and* $P := \max_{i=\overline{1,n}} p_i$. *Then we have*

$$0 \le \ln n - H(X) \le \frac{(P-p)^2}{4pP}. \tag{4.4}$$

Proof. Choose in the above lemma $\xi_i = \frac{1}{p_i} \in \left[\frac{1}{P}, \frac{1}{p}\right]$ and $m = \frac{1}{P}$, $M = \frac{1}{p}$ to get the desired inequality (4.4). □

Another analytic inequality which can be applied for the entropy mapping is embodied in the following lemma.

Lemma 2. *Let* $0 < m \le \xi_i \le M < \infty$, $p_i > 0$ $(i = 1, ..., n)$ *with* $\sum_{i=1}^{n} p_i = 1$. *Then*

$$\begin{aligned} 0 &\le \sum_{i=1}^{n} p_i \xi_i \ln \xi_i - \sum_{i=1}^{n} p_i \xi_i \ln\left(\sum_{i=1}^{n} p_i \xi_i\right) \\ &\le \frac{1}{4}(M-m)(\ln M - \ln m) \\ &\le \frac{1}{4} \cdot \frac{(M-m)^2}{\sqrt{mM}}. \end{aligned} \tag{4.5}$$

Proof. The first inequality follows from Corollary 1, if we choose $f(x) = x \ln x$, which is a convex mapping on $(0, \infty)$, and $x_i = \xi_i$, $i = 1, ..., n$.

The second inequality follows from the celebrated inequality between the *geometric mean* $G(a,b) := \sqrt{ab}$ and the *logarithmic mean*

$$L(a,b) := \begin{cases} a & \text{if } b = a \\ \dfrac{b-a}{\ln b - \ln a} & \text{if } b \ne a \end{cases} \qquad (a, b > 0)$$

which states that

$$G(a,b) \le L(a,b), \quad a, b > 0,$$

i.e.,

$$\frac{\ln b - \ln a}{b - a} \le \frac{1}{\sqrt{ab}}.$$

Choosing $b = M$, $a = m$, we obtain

$$\begin{aligned} \frac{1}{4}(M-m)(\ln M - \ln m) &\le \frac{1}{4}\frac{(M-m)(\ln M - \ln m)}{\sqrt{Mm}} \\ &= \frac{1}{4\sqrt{Mm}}(M-m)^2. \end{aligned}$$

□

Lemma 2 provides the following converse inequality for the entropy mapping [18].

Theorem 5. *Let X be as in Theorem 4. Then we have*

$$0 \leq \ln n - H(X) \leq \frac{n}{4}(P-p)(\ln P - \ln p) \leq \frac{n}{4} \cdot \frac{(P-p)^2}{\sqrt{pP}} \cdot \tag{4.6}$$

Proof. Firstly, let us chose $p_i = \frac{1}{n}$ in (4.5) to get

$$\begin{aligned} 0 &\leq \frac{1}{n}\sum_{i=1}^{n}\xi_i \ln \xi_i - \frac{1}{n}\sum_{i=1}^{n}\xi_i \ln\Big(\frac{1}{n}\sum_{i=1}^{n}\xi_i\Big) \\ &\leq \frac{1}{4}(M-m)(\ln M - \ln m) \\ &\leq \frac{1}{4\sqrt{Mm}}(M-m)^2 . \end{aligned} \tag{4.7}$$

Now, if in (4.7) we assume that $\xi_i = p_i \in [p, P]$, then we obtain

$$0 \leq \frac{1}{n}\ln n - \frac{1}{n}H(X) \leq \frac{1}{4}(P-p)(\ln P - \ln p) \leq \frac{1}{4} \cdot \frac{(P-p)^2}{\sqrt{pP}},$$

which yields (4.6). □

5. Applications for Rényi's entropy

The Rényi entropy of order α, $\alpha \in (0,1) \cup (1,\infty)$, is defined as follows (see [19]):

$$H_\alpha(X) := \frac{1}{1-\alpha}\ln\Big(\sum_{i=1}^{n} p_i^\alpha\Big). \tag{5.1}$$

Jensen's inequality for convex mappings applied for $f(x) = -\ln(x)$ yields

$$\ln\Big(\sum_{i=1}^{n} p_i x_i\Big) \geq \sum_{i=1}^{n} p_i \ln x_i,\ x_i,\ p_i > 0\ \ (i = 1, ..., n),\ \sum_{i=1}^{n} p_i = 1. \tag{5.2}$$

If we choose $x_i := p_i^{\alpha-1}$ $(i = 1, ..., n)$ in (5.2), then we have

$$\ln\Big(\sum_{i=1}^{n} p_i^\alpha\Big) \geq (\alpha - 1)\sum_{i=1}^{n} p_i \ln p_i,$$

which is equivalent to

$$(1-\alpha)[H_\alpha(X) - H(X)] \geq 0. \tag{5.3}$$

Now, if $\alpha \in (0,1)$, then $H_\alpha(X) \leq H(X)$ and if $\alpha > 1$, then $H_\alpha(X) \geq H(X)$.

Equality holds in (5.3) iff $(p_i)_{i=\overline{1,n}}$ is a uniform distribution and this fact follows by the strict convexity of $-\ln(\cdot)$.

We can now point out a counterpart result for (5.3).

Theorem 6. *Under the above assumptions, it holds*

$$(1-\alpha)\left[H_{\alpha}(X)-H(X)\right] \leq \frac{1}{4}\frac{\left(P^{\alpha-1}-p^{\alpha-1}\right)^{2}}{p^{\alpha-1}P^{\alpha-1}}, \tag{5.4}$$

provided that $0<p\leq p_i\leq P<1 \quad (i=1,...,n)$.

Proof. We use Lemma 1 for $\xi_i := p_i^{\alpha-1}$ and take into account that, for $\alpha \in (0,1)$, we have $P^{\alpha-1} \leq \xi_i \leq p^{\alpha-1}$, $(i=1,...,n)$ and, for $\alpha \in (1,\infty)$, we have $p^{\alpha-1} \leq \xi_i \leq P^{\alpha-1}$ $(i=1,...,n)$. Choosing in the first case $m=P^{\alpha-1}$, $M=p^{\alpha-1}$ and in the second case $m=p^{\alpha-1}$, $M=P^{\alpha-1}$, we obtain the same upper bound

$$\frac{(M-m)^2}{mM}=\frac{\left(P^{\alpha-1}-p^{\alpha-1}\right)^{2}}{p^{\alpha-1}P^{\alpha-1}}.$$

□

Now, let us remark that a particular case of (4.3) for $p_i=\frac{1}{n}$ $(i=1,...,n)$ states that

$$0\leq \ln\left(\frac{1}{n}\sum_{i=1}^{n}\xi_i\right)-\frac{1}{n}\sum_{i=1}^{n}\ln(\xi_i)\leq \frac{(M-m)^2}{4mM}, \tag{5.5}$$

provided that $0<m\leq \xi_i\leq M<\infty$ $(i=1,...,n)$.

This inequality allows us to prove the following result for the Rényi entropy.

Theorem 7. *Under the assumptions of Theorem 6, it holds*

$$\begin{aligned} 0 &\leq (1-\alpha)H_{\alpha}(X)-\ln n-\alpha\ln G_n(p) \\ &\leq \frac{n}{4}\cdot\frac{(P^{\alpha}-p^{\alpha})^2}{p^{\alpha}P^{\alpha}}, \end{aligned} \tag{5.6}$$

where $G_n(p)$ *is the geometric mean of* p_i $(i=1,...,n)$, *i.e.,* $G_n(p)=\left(\prod_{i=1}^{n}p_i\right)^{\frac{1}{n}}$.

Proof. We choose in the inequality (5.6) $\xi_i=np_i^{\alpha}$ $(i=1,...,n)$ and observe that $np^{\alpha}\leq\xi_i\leq nP^{\alpha}$. Then we have $m=np^{\alpha}$ and $M=nP^{\alpha}$ in (5.5), and the desired inequality follows. □

If we assume that $\alpha\in(0,1)$ and apply Corollary 1 for the convex mapping $f(x)=-x^{\alpha}$, then we have

$$0\leq\left(\sum_{i=1}^{n}p_ix_i\right)^{\alpha}-\sum_{i=1}^{n}p_ix_i^{\alpha}\leq\frac{\alpha}{4}(M-m)\left(m^{\alpha-1}-M^{\alpha-1}\right), \tag{5.7}$$

provided that $0<m\leq x_i\leq M<\infty$, $p_i>0$ $(i=1,...,n)$ and $\sum_{i=1}^{n}p_i=1$.

If in (5.7) we put $p_i = \frac{1}{n}$ $(i = 1, ..., n)$, then we obtain

$$0 \le \frac{1}{n^\alpha}\Big(\sum_{i=1}^{n} x_i\Big)^\alpha - \frac{1}{n}\sum_{i=1}^{n} x_i^\alpha \le \frac{\alpha}{4}(M-m)\left(m^{\alpha-1} - M^{\alpha-1}\right). \tag{5.8}$$

The following result for the α-Rényi entropy holds.

Theorem 8. *If* $0 < p \le p_i \le P < 1$ $(i = 1, ..., n)$ *and* $\alpha \in (0,1)$, *then*

$$0 \le n^{1-\alpha} - \exp\left[(1-\alpha) H_\alpha(X)\right] \le \frac{\alpha}{4} n (P-p)\left(p^{\alpha-1} - P^{\alpha-1}\right). \tag{5.9}$$

Proof. Choosing $x_i = p_i$ $(i = 1, ..., n)$ in (5.8), we deduce

$$0 \le \frac{1}{n^\alpha} - \frac{1}{n}\sum_{i=1}^{n} p_i^\alpha \le \frac{\alpha}{4}(P-p)\left(p^{\alpha-1} - P^{\alpha-1}\right). \tag{5.10}$$

Taking into account that $\sum_{i=1}^{n} p_i^\alpha = \exp\left[(1-\alpha) H_\alpha(X)\right]$, from (5.10) we deduce the desired inequality (5.9). □

Now, if we define by $E(X) := \sum_{i=1}^{n} p_i^2$, the informational energy of the random variable X, then we have the following theorem.

Theorem 9. *If* p_i *and* α *are as in Theorem 8, then*

$$0 \le E^\alpha(X) - \exp\left[-\alpha H_{\alpha+1}(X)\right] \le \frac{\alpha}{4}(P-p)\left(p^{\alpha-1} - P^{\alpha-1}\right). \tag{5.11}$$

The proof follows from the inequality (5.7), if we choose $x_i = p_i$, $i = 1, ..., n$.

Now assume that $\alpha \in (1, \infty)$. Applying Corollary 1 to the convex mapping $f(x) = x^\alpha$, we deduce the following inequality

$$0 \le \sum_{i=1}^{n} p_i x_i^\alpha - \Big(\sum_{i=1}^{n} p_i x_i\Big)^\alpha \le \frac{\alpha}{4}(M-m)\left(m^{\alpha-1} - M^{\alpha-1}\right), \tag{5.12}$$

provided that $0 < m \le x_i \le M < \infty$, $p_i > 0$ $(i = 1, ..., n)$ and $\sum_{i=1}^{n} p_i = 1$.

Finally, using an argument similar to the above one, we can establish the next theorem.

Theorem 10. *Let* $\alpha \in (1, \infty)$ *and* $0 < p \le p_i \le P < 1$ $(i = 1, ..., n)$. *Then*

$$0 \le \exp\left[-\alpha H_\alpha(X)\right] - E^\alpha(X) \le \frac{\alpha}{4}(P-p)\left(P^{\alpha-1} - p^{\alpha-1}\right)$$

and

$$0 \le \exp\left[(1-\alpha) H_\alpha(X)\right] - n^{1-\alpha} \le \frac{\alpha}{4} n (P-p)\left(P^{\alpha-1} - p^{\alpha-1}\right).$$

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