ACTA MATHEMATICA VIETNAMICA
Volume 29, Number 1, 2004, pp. 23-39

CONNECTIONS BETWEEN IMPLICIT DIFFERENCE EQUATIONS AND DIFFERENTIAL-ALGEBRAIC EQUATIONS

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Abstract. Recently, a notion of index-1 linear implicit difference equations (LIDEs) has been introduced and the solvability of initial value problems (IVPs) as well as multipoint boundary-value problems (MPBVPs) for index-1 LIDEs has been studied. In this note, we show that the explicit Euler method (EEM) applied to linear transferable differential-algebraic equations (DAEs) leads to index-1 LIDEs. Besides, we discuss the convergence of solutions of IVPs (MPBVPs) for index-1 LIDEs to the solutions of the corresponding problems for transferable DAEs.

1. Introduction

Linear implicit difference equation

$$A_n x_{n+1} = B_n x_n + q_n \quad (n = 0, 1, 2, \dots), \tag{1}$$

where $A_n,\ B_n \in \mathbb{R}^{m\times m}$, $q_n \in \mathbb{R}^m$ are given and the matrices A_n are all singular, may be regarded as discrete analogues of certain linear DAEs.

According to [9], LIDEs (1) is said to be of index-1 if

(i) $\operatorname{rank} A_n \equiv r$ $(0 < r < m)$ for all $n = 0, 1, 2, \dots$

(ii) the matrices $A_n + B_n V_{n-1} Q^* V_n^\top$ are nonsingular for $n \geq 0$,

where $A_n = U_n \Sigma_n V_n^\top$ is a singular-value decomposition (SVD) of A_n,

$$\Sigma_n = \operatorname{diag}(\sigma_n^{(1)}, \dots, \sigma_n^{(r)}, 0, \dots, 0)$$

is a diagonal matrix with singular values $\sigma_n^{(1)} \geq \sigma_n^{(2)} \geq \dots \geq \sigma_n^{(r)} > 0$ on the main diagonal. Further, U_n (V_n) are orthogonal matrices, whose columns are left (right) singular vectors of A_n, respectively. Finally,

$$Q^* = \operatorname{diag}(O_r, I_{m-r}),$$

where O_k and I_k $(k = \overline{1, m})$ stand for the $k \times k$-zero matrix and the $k \times k$-identity matrix, respectively. For $k = m$ we simply put $O_m = O$ and $I_m = I$. For

Received May 23, 2003; in revised form November 18, 2003.
1991 *Mathematics Subject Classification.* 15A18, 34A09, 34B05, 39A10, 65L05.
Key words and phrases. Index-1 linear implicit difference equation, differential-algebraic equation, initial-value problem, multipoint boundary-value problem, singular-value decomposition.

definiteness, we set V_{-1} to be an appropriate orthogonal matrix ($V_{-1} = I$ for example).

It has been shown that the index of LIDEs does not depend on the choice of SVDs of A_n. Furthermore, the unique solvability of some IVPs for index-1 LIDEs has been established in [9].

The MPBVP

$$A_n x_{n+1} = B_n x_n + q_n \quad (n = 0, \ldots, N-1) \tag{2}$$

$$\sum_{n=0}^{N} D_n x_n = \gamma, \tag{3}$$

when N becomes large, represents a large-scale system of $m(N+1)$ linear equations. Some necessary and sufficient conditions for the solvability and the unique solvability of problems (2)-(3) have been derived in [2] and [9]. As a direct consequence of these results, a Fredholm alternative for the problems (2) and (3) was obtained.

A close examination of the definition of index-1 LIDEs suggests that instead of setting $V_{-1} = I$ one can simply let $V_{-1} := V_0$ and all results of [9] remain true. This fact will be useful later when we deal with discretization schemes for DAEs.

Consider the DAE

$$A(t)x' + B(t)x = q(t), \quad t \in J := [t_0, T], \tag{4}$$

where $A, B \in C(J, \mathbb{R}^{m\times m})$, $q \in C(J, \mathbb{R}^m)$ and the matrix $A(t)$ is singular for every $t \in J$.

Following Griepentrog and März [5], the DAE (4) is called transferable (or index-1 tractable) if

(i) there exists a smooth projection $Q \in C^1(J, \mathbb{R}^{m\times m})$ onto $\mathrm{Ker}A(t)$, i.e., $Q^2(t) = Q(t)$ and $\mathrm{Im}\, Q(t) = \mathrm{Ker}A(t)$ for any $t \in J$, and

(ii) the matrix $G(t) := A(t) + B(t)Q(t)$ is nonsingular for any $t \in J$.

It should be noted that the transferability of linear DAEs is independent of the choice of a smooth projection $Q(t)$ onto $\mathrm{Ker}A(t)$.

The aim of this note is to reveal a connection between linear transferable DAEs and index-1 LIDEs, i.e., to describe some discretization methods for DAEs (4) that lead to index-1 LIDEs (1) as well as to show that under certain conditions solutions of IVPs and MPBVPs for index-1 LIDEs will converge to the solutions of the corresponding problems for transferable DAEs. Due to the limitation of space, all the discussions on descriptor systems will be omitted.

For the numerical solution of DAEs, we refer the interested reader to a huge literature devoted to numerical methods for solving DAEs (see [4], [5] and therein references).

2. Discretization of linear DAEs

We begin with the following useful lemma.

Lemma 2.1. *Suppose that the linear DAE* (4) *is transferable. Let* $Q \in C^1(J, \mathbb{R}^{m\times m})$ *be an arbitrary projection onto* $\mathrm{Ker}A(t)$*. Then for every* $t \in J$ *and sufficiently small* $\tau > 0$*, the matrices* $G(t,\tau) := A(t) - \tau B(t)Q(t)$ *and* $H(t,\tau) := A(t) - \tau[B(t) - A(t)P'(t)]Q(t)$*, where* $P(t) = I - Q(t)$*, are both nonsingular. Moreover, there hold the estimates*

$$\|G^{-1}(t,\tau)\| \le \frac{C_1}{\tau}\,, \tag{5}$$

and

$$\|H^{-1}(t,\tau)\| \le \frac{C_2}{\tau}\,, \tag{6}$$

where C_1 *and* C_2 *are positive constants.*

Proof. By definition, the matrix $G(t) := A(t) + B(t)Q(t)$ is nonsingular for all $t \in J$. Noting that

$$\begin{aligned} G(t,\tau) &= A(t) + B(t)Q(t) - (1+\tau)B(t)Q(t) \\ &= G(t)[I - (1+\tau)G^{-1}(t)B(t)Q(t)], \end{aligned}$$

and taking into account the fact that $G^{-1}(t)B(t)Q(t) = Q(t)$ we get

$$G(t,\tau) = G(t)[I - (1+\tau)Q(t)] = G(t)[P(t) - \tau Q(t)].$$

Since $[P(t) - \tau Q(t)]^{-1} = P(t) - \frac{1}{\tau}Q(t) = \frac{1}{\tau}[\tau P(t) - Q(t)]$, it follows that

$$G^{-1}(t,\tau) = \frac{1}{\tau}[\tau P(t) - Q(t)]G^{-1}(t).$$

From the continuity of A, B, P, Q and the last relation, we have (5).

Observing that $H(t,\tau) = G(t,\tau)[I + \tau P(t)P'(t)Q(t)]$, hence,

$$H^{-1}(t,\tau) = [I - \tau P(t)P'(t)Q(t)]G^{-1}(t,\tau),$$

we come to the estimate (6). □

In what follows we assume that the singular matrix $A(t)$ with a constant $\mathrm{rank}A(t) \equiv r$ possesses a SVD

$$A(t) = U(t)\Sigma(t)V^\top(t), \tag{7}$$

where $U \in C(J, \mathbb{R}^{m\times m})$, $V \in C^1(J, \mathbb{R}^{m\times m})$ are orthogonal matrices, i.e.,

$$U^\top(t)U(t) = V^\top(t)V(t) = I.$$

Furthermore, $\Sigma \in C(J, \mathbb{R}^{m\times m})$ is a diagonal matrix with singular values $\sigma_1(t) \ge \sigma_2(t) \ge \cdots \ge \sigma_r(t) > 0$ on its main diagonal. If $A \in C^1(J, \mathbb{R}^{m\times m})$ then the decomposition (7) with a smooth matrix $V(t)$ is followed from similar results for Hermitian matrix $A^\top(t)A(t)$ in [3, Corollary 3], see also [7, Section II.6.2]. However, the relation (7) can be valid for non-smooth matrices $A(t)$. For example, if $A(t)=\operatorname{diag}(\sigma_1(t),\dots,\sigma_r(t),0,\dots,0)$, where $\sigma_i(t)\in C(J,\mathbb{R})$ $(i=\overline{1,r})$, then we can choose $V(t)\equiv I$.

Denoting $Q(t)=V(t)Q^*V^\top(t)$, where $Q^*=\operatorname{diag}(O_r,I_{m-r})$ is as before, we see that $Q(t)$ is a smooth projection onto $\operatorname{Ker}A(t)$.

Now let $J_\tau=\{t_0<t_1<\dots<t_N=T\}$ be an uniform partition of J, i.e., $t_n=t_0+n\tau$ $(n=\overline{0,N})$ and $\tau=(T-t_0)/N$.

Putting $A_n=A(t_n)$, $B_n=B(t_n)$, $q_n=q(t_n)$, $Q_n=Q(t_n)$ and $V_n=V(t_n)$ $(n=\overline{0,N})$, $V_{-1}=V_0$, we have $A_n=U_n\Sigma_nV_n^\top$ and $Q_n=V_nQ^*V_n^\top$ $(n=\overline{0,N})$.

Applying the explicit Euler method to (4) we get

$$A_n\frac{x_{n+1}-x_n}{\tau}+B_nx_n=q_n \qquad (n=\overline{0,N-1}) \tag{8}$$

or equivalently,

$$A_nx_{n+1}=(A_n-\tau B_n)x_n+\tau q_n \qquad (n=\overline{0,N-1}). \tag{9}$$

Theorem 2.2. *The explicit Euler method applied to linear transferable DAEs gives index-1 LIDEs.*

Proof. To prove that (9) is an index-1 LIDE, it suffices to show the non-singularity of the matrix $\overline{G}_n(\tau):=A_n+(A_n-\tau B_n)V_{n-1}Q^*V_n^\top$.

By Lemma 2.1 the matrix $G_n(\tau):=A_n-\tau B_nQ_n=A_n-\tau B_nV_nQ^*V_n^\top$ is nonsingular and $\|G_n^{-1}(\tau)\|\le c_1/\tau$ $(n=\overline{0,N-1})$. We can rewrite $\overline{G}_n(\tau)$ as

$$\begin{aligned}\overline{G}_n(\tau)&=A_n-\tau B_nV_nQ^*V_n^\top+\tau B_n(V_n-V_{n-1})Q^*V_n^\top+A_n(V_{n-1}-V_n)Q^*V_n^\top\\&=G_n(\tau)\{I+\tau G_n^{-1}(\tau)B_n(V_n-V_{n-1})Q^*V_n^\top+G_n^{-1}(\tau)A_n(V_{n-1}-V_n)Q^*V_n^\top\}.\end{aligned}$$

Since $V\in C^1(J,\mathbb{R}^{m\times m})$ and $V_n=V(t_n)$, it follows that $\|V_n-V_{n-1}\|=O(\tau)$. Furthermore, the relations $G_n(\tau)P_n=(A_n-\tau B_nQ_n)P_n=A_n$, imply $G_n^{-1}(\tau)A_n=P_n$. Thus

$$\begin{aligned}&\|\tau G_n^{-1}(\tau)B_n(V_n-V_{n-1})Q^*V_n^\top+G_n^{-1}(\tau)A_n(V_{n-1}-V_n)Q^*V_n^\top\|\\&=\|(\tau G_n^{-1}(\tau)B_n-P_n)(V_n-V_{n-1})Q^*V_n^\top\|\\&\le(c_1\|B_n\|+\|P_n\|)O(\tau)\|Q^*\|\|V_n^\top\|\\&\le c_3\tau,\end{aligned}$$

where c_3 is a positive constant. From the last inequality it follows that the matrix $I+G_n^{-1}(\tau)(\tau B_n-A_n)(V_n-V_{n-1})Q^*V_n^\top$ is invertible for τ sufficiently small, and hence, $\overline{G}_n(\tau)$ is nonsingular, which proves the theorem. $\square$

Suppose we are interested in finding a solution of (4) satisfying the initial condition

$$P(t_0)(x(t_0)-x^0)=0. \tag{10}$$

Clearly, the corresponding initial condition for LIDE (8) should be

$$P_0(x_0-x^0)=0. \tag{11}$$

For the sake of convenience, we set $Q_{-1} := Q_0$ and $P_{-1} := I - Q_{-1}$. The following theorem not only proves the convergence of EEM but also shows that the discretization process only concerns the differentiable part of solutions of equation (4). In what follows, we suppose, if necessary, that the DAE (4) is transferable on a larger segment containing $[t_0,\ T+\tau]$.

Theorem 2.3. *The explicit Euler method for the IVP associated with linear transferable DAEs is convergent.*

Proof. Let $\overline{G}_n = A_n + B_n V_{n-1} Q^* V_n^\top$. The transferability of DAE (4) ensures the non-singularity of the matrix $G_n = A_n + B_n Q_n$, where $Q_n = V_n Q^* V_n^\top$ is a projection onto $\mathrm{Ker} A_n$. Since $\overline{G}_n = G_n + B_n(V_{n-1} - V_n) Q^* V_n^\top$ and $\|V_{n-1} - V_n\| = O(\tau)$, it follows that $\overline{G}_n$ is also nonsingular.

Applying $P_n \overline{G}_n^{-1}$ and $Q_n \overline{G}_n^{-1}$ to both sides of (8) and taking into account the relations $P_n \overline{G}_n^{-1} A_n = P_n$; $Q_n \overline{G}_n^{-1} A_n = O$; $\overline{G}_n^{-1} B_n V_{n-1} Q^* V_n^\top = Q_n$ we find

$$P_n\Big(\frac{x_{n+1} - x_n}{\tau}\Big) + P_n \overline{G}_n^{-1} B_n x_n = P_n \overline{G}_n^{-1} q_n \tag{12}$$

and

$$Q_n \overline{G}_n^{-1} B_n x_n = Q_n \overline{G}_n^{-1} q_n. \tag{13}$$

Observing that

$$\begin{aligned} P_n \overline{G}_n^{-1} B_n Q_{n-1} &= P_n \overline{G}_n^{-1} (A_n + B_n V_{n-1} Q^* V_n^\top) V_n Q^* V_n^\top V_n V_{n-1}^\top \\ &= P_n \overline{G}_n^{-1} \overline{G}_n Q_n V_n V_{n-1}^\top = P_n Q_n V_n V_{n-1}^\top = O \end{aligned}$$

and

$$P_n x_n = (P_n - P_{n-1}) P_{n-1} x_n + (P_n - P_{n-1}) Q_{n-1} x_n + P_{n-1} x_n,$$

we can rewrite relation (12) as

$$\begin{aligned} P_n x_{n+1} = P_{n-1} x_n &+ \{(P_n - P_{n-1}) - \tau P_n \overline{G}_n^{-1} B_n\} P_{n-1} x_n \\ &+ (P_n - P_{n-1}) Q_{n-1} x_n + \tau P_n \overline{G}_n^{-1} q_n \quad (n \geq 0). \end{aligned} \tag{14}$$

Using the fact that

$$\begin{aligned} Q_n \overline{G}_n^{-1} B_n Q_{n-1} x_n &= Q_n \overline{G}_n^{-1} \{A_n + B_n V_{n-1} Q^* V_n^\top\} V_n Q^* V_n^\top V_n V_{n-1}^\top x_n \\ &= Q_n V_n V_{n-1}^\top x_n = V_n Q^* V_n^\top V_n V_{n-1}^T x_n \\ &= V_n V_{n-1}^\top Q_{n-1} x_n, \end{aligned}$$

we can transform (13) into

$$Q_{n-1} x_n = V_{n-1} V_n^\top \{Q_n \overline{G}_n^{-1} q_n - Q_n \overline{G}_n^{-1} B_n P_{n-1} x_n\}. \tag{15}$$

Putting $u_n = P_{n-1} x_n \ \ (n \geq 0)$ and taking into account (14) and (15) we come to the relation

$$\begin{aligned} u_{n+1} = u_n &+ \{(P_n - P_{n-1})[I - V_{n-1} Q^* V_n^\top \overline{G}_n^{-1} B_n] - \tau P_n \overline{G}_n^{-1} B_n\} u_n \\ &+ \tau P_n \overline{G}_n^{-1} q_n + (P_n - P_{n-1}) V_{n-1} Q^* V_n^\top \overline{G}_n^{-1} q_n, \end{aligned}$$

or

$$u_{n+1} = M_n u_n + r_n, \quad u_0 = P_{-1}x^0 = P_0 x^0, \tag{16}$$

where $M_n = I + (P_n - P_{n-1})[I - V_{n-1}Q^* V_n^\top \overline{G}_n^{-1} B_n] - \tau P_n \overline{G}_n^{-1} B_n$ and $r_n = \tau P_n \overline{G}_n^{-1} q_n + (P_n - P_{n-1})V_{n-1}Q^* V_n^\top \overline{G}_n^{-1} q_n$ $(n \geq 0)$. Now let $\overline{u}_n$ satisfy the difference equations

$$\overline{u}_{n+1} = \overline{M}_n \overline{u}_n + \overline{r}_n, \overline{u}_0 = P_0 x^0, \tag{17}$$

where $\overline{M}_n = I + \tau P_n'(I - Q_n G_n^{-1} B_n) - \tau P_n G_n^{-1} B_n$ and $\overline{r}_n = \tau P_n G_n^{-1} q_n + \tau P_n' Q_n G_n^{-1} q_n$ $(n \geq 0)$.

Obviously,

$$\begin{cases} (\overline{u}_{n+1} - \overline{u}_n)/\tau = [P_n'(I - Q_n G_n^{-1} B_n) - P_n G_n^{-1} B_n]\overline{u}_n \\ \qquad\qquad\qquad\qquad + P_n G_n^{-1} q_n + P_n' Q_n G_n^{-1} q_n, \\ \overline{u}_0 = P_0 x^0, \end{cases} \tag{18}$$

is obtained by applying the explicit Euler method to the ODE

$$\begin{cases} u' &= [P'(I - QG^{-1}B) - PG^{-1}B]u + PG^{-1}q + P'QG^{-1}q \\ u(t_0) &= P(t_0)x^0. \end{cases} \tag{19}$$

It has been proved [5] that $u(t) = P(t)x(t)$, where $x(t)$ is a unique solution of the IVP (4), (10). Moreover,

$$x(t) = (I - QG^{-1}B)u(t) + QG^{-1}q. \tag{20}$$

It is clear that $\|\overline{u}_n - u(t_n)\| = O(\tau)$ and $\|\overline{x}_n - x(t_n)\| = O(\tau)$, where

$$\overline{x}_n = (I - Q_n G_n^{-1} B_n)\overline{u}_n + Q_n G_n^{-1} q_n. \tag{21}$$

Using the decomposition $x_n = P_{n-1}x_n + Q_{n-1}x_n$ $(n \geq 0)$ and relation (15) we get

$$x_n = (I - V_{n-1}Q^* V_n^\top \overline{G}_n^{-1} B_n)u_n + V_{n-1}Q^* V_n^\top \overline{G}_n^{-1} q_n. \tag{22}$$

From (21), (22) it follows that

$$\begin{aligned} x_n - \overline{x}_n = {} & (I - V_{n-1}Q^* V_n^\top \overline{G}_n^{-1} B_n)(u_n - \overline{u}_n) \\ & + (Q_n G_n^{-1} - V_{n-1}V_n^\top Q_n \overline{G}_n^{-1})(B_n \overline{u}_n - q_n). \end{aligned} \tag{23}$$

Since $\overline{u}_n$ is bounded, $\|I - V_{n-1}V_n^\top\| = \|(V_n - V_{n-1})V_n^\top\| = O(\tau)$ and $\|\overline{G}_n^{-1} - G_n^{-1}\| = \|\overline{G}_n^{-1}(\overline{G}_n - G_n)G_n^{-1}\| = O(\tau)$, we come to the conclusion that if $\|u_n - \overline{u}_n\| \to 0$ $(\tau \to 0)$ then $\|x_n - \overline{x}_n\| \to 0$ $(\tau \to 0)$, and hence, $\|x_n - x(t_n)\| \to 0$ $(\tau \to 0)$.

Let $\xi_n := \|u_n - \overline{u}_n\|$ and $\gamma_n = \|M_n - \overline{M}_n\|\|\overline{u}_n\| + \|r_n - \overline{r}_n\| = o(\tau)$. From (16), (17) we get

$$\xi_{n+1} \leq \|M_n\|\xi_n + \gamma_n \quad (n \geq 0). \tag{24}$$

Using (24) and taking into account that $\xi_0 = 0$ we get the estimate

$$\xi_{n+1} \leq \sum_{k=0}^{n-1} \Big(\prod_{i=k+1}^{n} \|M_i\| \Big) \gamma_k + \gamma_n \qquad (n \geq 0).$$

Since $\|M_i\| \leq 1 + \tau c$, where c is a positive constant for $i = \overline{0, n}$ we have

$$\prod_{i=k+1}^{n} \|M_i\| \leq (1 + \tau c)^{n-k} \leq (1 + \tau c)^n \leq e^{n\tau c} \leq e^{c(T-t_0)}.$$

Thus we come to the relation

$$\xi_{n+1} \leq e^{c(T-t_0)} \cdot n \cdot \max_k \gamma_k + \gamma_n = \frac{o(\tau)}{\tau}, \ \text{ i.e., } \ \|u_n - \bar{u}_n\| \to 0 \quad (\tau \to 0),$$

as desired. □

Now we propose another discretization scheme for DAE (4) that also leads to an index-1 LIDE. The convergence of solutions of new discretized equations is faster than that of (8).

Lemma 2.4. *The explicit Euler method applied to IVP for a linear transferable DAE with the constant kernel* Ker$A(t)$,*i.e.,* Ker$A(t)$ *does not depend on* t, *is convergent. Moreover, there holds the estimate* $\|x_n - x(t_n)\| = O(\tau)$.

Proof. Let Q be a certain projection onto Ker$A(t)$ and $P = I - Q$. In this case, the initial condition is $P(x_0 - x^0) = 0$. Applying PG_n^{-1}, where $G_n = A_n + B_n Q$ is nonsingular due to the transferability of (4), to both sides of (8) and taking into account the relations $PG_n^{-1}A_n = P; PG_n^{-1}B_n = PG_n^{-1}B_n P$, we find

$$\frac{u_{n+1} - u_n}{\tau} = -PG_n^{-1}B_n u_n + PG_n^{-1} q_n, \tag{25}$$

where $u_n := Px_n$.

Furthermore, performing QG_n^{-1} to both sides of (8) and noting that $QG_n^{-1}A_n = O$; $QG_n^{-1}B_n = Q + QG_n^{-1}B_n P$ we get $Qx_n = QG_n^{-1}q_n - QG_n^{-1}B_n u_n$. Thus the unique solution of (8) is given by

$$x_n = (I - QG_n^{-1}B_n)u_n + QG_n^{-1}q_n, \tag{26}$$

where u_n is defined by (25) and $u_0 = Px^0$. On the other hand, a transferable DAE (4) with the constant kernel Ker$A(t)$ has a unique solution of the form (see [5, 10])

$$x(t) = [I - QG^{-1}(t)B(t)]u(t) + QG^{-1}(t)q(t), \tag{27}$$

where $u(t)$ is a solution of the IVP

$$\begin{cases} u' = -PG^{-1}(t)B(t)u + PG^{-1}(t)q \\ u(t_0) = Px^0. \end{cases} \tag{28}$$

Clearly, (25) is the explicit Euler scheme applied to IVP (28), hence $\|u_n - u(t_n)\| = O(\tau) \ \ (n \leq N)$. From (26), (27) it follows that $\|x_n - x(t_n)\| = O(\tau)$. The lemma is proved. □

Theorem 2.5. *Given a linear transferable DAE* (4) *with a SVD* (7)*. Then*
(i) *the LIDE*

$$A_n V_n V_{n+1}^\top x_{n+1} = [A_n - \tau(B_n - A_n V_n {V'}_n^\top)]x_n + \tau q_n, \tag{29}$$

$$P_0(x_0 - x^0) = 0, \tag{30}$$

where $P_0 = V_0 P^* V_0^\top$ *and* $P^* = \operatorname{diag}(I_r, O_{m-r})$*, is of index-1.*
(ii) *the IVP* (29), (30) *has a unique solution* x_n *which converges to the unique solution of* (4), (10)*. Moreover, there holds an estimate* $\|x_n - x(t_n)\| = O(\tau)$ $(n \leq N)$*.*

Proof. Using decomposition (7) and substituting $x(t) = V(t)y(t)$, we can reduce (4) to the form

$$\Sigma(t)y' + (U^\top(t)B(t) - \Sigma(t){V'}^\top(t))V(t)y = U^\top(t)q. \tag{31}$$

Obviously, $\operatorname{Ker}\Sigma(t)$ does not depend on t and $Q^* = \operatorname{diag}(O_r, I_{m-r})$ is a projection onto $\operatorname{Ker}\Sigma(t)$. Since

$$\begin{aligned}\widetilde{F}(t) &:= \Sigma(t) + [U^\top(t)B(t) - \Sigma(t){V'}^\top(t)]V(t)Q^* \\ &= U^\top(t)[A(t) + B(t)Q(t) - A(t)V(t){V'}^\top(t)Q(t)]V(t),\end{aligned}$$

where $Q(t) = V(t)Q^*V^\top(t)$, it implies

$$\begin{aligned}\widetilde{F}(t) &= U^\top(t)[G(t) - A(t)V(t){V'}^\top(t)Q(t)]V(t) \\ &= U^\top(t)G(t)[I - G^{-1}(t)A(t)V(t){V'}^\top(t)Q(t)]V(t),\end{aligned}$$

where $G(t) = A(t) + B(t)Q(t)$ as before. Observing that $G^{-1}(t)A(t) = P(t)$ and $[I - P(t)V(t){V'}^\top(t)Q(t)]^{-1} = I + P(t)V(t){V'}^\top(t)Q(t)$ we get

$$\widetilde{F}^{-1}(t) = V^\top(t)[I + P(t)V(t){V'}^\top(t)Q(t)]G^{-1}(t)U(t).$$

Thus the transferability of DAE (31) is established. By Lemma 2.4, the explicit Euler method for (31)

$$\Sigma_n \frac{y_{n+1} - y_n}{\tau} + (U_n^\top B_n - \Sigma_n {V'_n}^\top)V_n y_n = U_n^\top q_n, \tag{32}$$

$$P^*(y_0 - y^0) = 0, \tag{33}$$

where $P^* = I - Q^*, y^0 = V_0^\top x^0$, is convergent. Moreover, $\|y_n - y(t_n)\| = O(\tau)$ $(n \leq N)$. Now setting $x_n = V_n y_n$ one can easily reduce (29), (30) from (32), (33) and get the estimate $\|x_n - x(t_n)\| \leq \|V_n\|\|y_n - y(t_n)\| = O(\tau)$. We complete the proof by showing that (29) is an index-1 LIDE. For this purpose, let us consider the matrix $\overline{A}_n = A_n V_n V_{n+1}^\top = U_n \Sigma_n V_n^\top V_n V_{n+1}^\top = U_n \Sigma_n \overline{V}_n^\top$, where $\overline{V}_n := V_{n+1}$. To verify that (29) is index-1, we have to prove the nonsingularity of the matrix

$$\begin{aligned}\overline{F}_n &= \overline{A}_n + [A_n - \tau(B_n - A_nV_n{V'}_n^{\top})]\overline{V}_{n-1}Q^*\overline{V}_n^{\top}\\ &= A_nV_nV_{n+1}^{\top} + [A_n - \tau(B_n - A_nV_n{V'}_n^{\top})]V_nQ^*V_{n+1}^{\top}\\ &= F_nV_nV_{n+1}^{\top},\end{aligned}$$

where F_n denotes the matrix $A_n+[A_n-\tau(B_n-A_nV_n{V'}_n^{\top})]Q_n$ and $Q_n = V_nQ^*V_n^{\top}$. Clearly, $F_n = A_n - \tau(B_n - A_nV_n{V'}_n^{\top})Q_n$ is nonsingular if and only if $\overline{F}_n$ is nonsingular. Using the fact that the matrix $G_n(\tau) = A_n - \tau B_nQ_n$ is nonsingular by Lemma 2.1 and

$$F_n = G_n(\tau)\{I + \tau G_n^{-1}(\tau)A_nV_n{V'}_n^{\top}Q_n\} = G_n(\tau)\{I + \tau P_nV_{n-1}{V'}_n^{\top}Q_n\}$$

we can conclude that for sufficiently small τ, the matrix F_n, and hence, the matrix $\overline{F}_n$ is nonsingular. □

3. Connection between MPBVPs for linear transferable DAEs and index-1 LIDEs

This section deals with a relation between MPBVPs for linear transferable DAEs and index-1 LIDEs. For the sake of simplicity we shall restrict our consideration to the following two-point boundary-value problem (TPBVP): find a solution of the DAE (4) satisfying the two-point boundary condition

$$C_0x(t_0) + C_Tx(T) = \gamma, \tag{34}$$

where $\gamma \in \mathbb{R}^m$ and $C_0, C_T \in \mathbb{R}^{m\times m}$ are given vector and matrices, respectively.

The corresponding discretized problem for (4), (34) which will be treated here is of the form

$$A_nx_{n+1} = (A_n - \tau B_n)x_n + \tau q_n \quad (n = \overline{0, N-1}) \tag{35}$$

$$C_0x_0 + C_Tx_N = \gamma. \tag{36}$$

Theorem 2.2 ensures that (35) is an index-1 LIDE. It is known (cf. [5, Theorem 25, p.48], see also [1, Corollary 2.1]), that the TPBVP (4), (34) is uniquely solvable for any $q \in C(J, \mathbb{R}^m)$ and $\gamma \in \mathrm{Im}(C_0, C_T)$ if and only if the shooting matrix $D := C_0X(t_0) + C_TX(T)$, where $X(t)$ is the fundamental solution matrix, satisfying

$$A(t)X' + B(t)X = O, \quad P(t_0)(X(t_0) - I) = O,$$

has the properties

$$\mathrm{Ker}D = \mathrm{Ker}A(t_0), \quad \mathrm{Im}D = \mathrm{Im}(C_0, C_T). \tag{37}$$

On the other hand, denoting

$$B_n(\tau) := A_n - \tau B_n,$$
$$\widetilde{P}_n(\tau) := I - V_{n-1}V_n^\top Q_n \overline{G}_n^{-1}(\tau)B_n(\tau),$$
$$M_{n-1}^{(n)} := \prod_{k=0}^{n-1} \overline{G}_{n-k-1}^{-1}(\tau)B_{n-k-1}(\tau) \quad (n = \overline{1,N})$$

and defining the shooting matrix

$$\widetilde{D}(\tau) := C_0\widetilde{X}_0(\tau) + C_T\widetilde{X}_N(\tau),$$

where $\widetilde{X}_0(\tau) := \widetilde{P}_0(\tau)$, $\widetilde{X}_n(\tau) := \widetilde{P}_n(\tau)M_{n-1}^{(n)}$ $(n = \overline{1, N-1})$, $\widetilde{X}_N(\tau) := P_{N-1}M_{N-1}^{(N)}$, we come to the following necessary and sufficient condition for the unique solvability of the MPBVP (35), (36) (see [2, Theorem 1]):

$$\dim\mathrm{Ker}(\widetilde{D}(\tau), C_TQ_{N-1}) = m. \tag{38}$$

Unfortunately, there are many examples showing that the unique solvability of the continuous problem (4), (34) does not necessarily imply the unique solvability of the discretized problem (35), (36). Thus, let

$$A(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ B(t) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ C_0 = C_T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{39}$$

A simple calculation shows that $X(t) = e^{t-t_0}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $D = C_0X(t_0) + C_TX(T) = (1 + e^{T-t_0})\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Clearly, condition (37) holds, because Ker$D$= Span$\{(0,1)^\top\}$=Ker$A(t_0)$ and ImD=Span$\{(1,1)^\top\}$=Im(C_0, C_T). Therefore the TPBVP (4), (34) with the given data (39) is uniquely solvable for any $q \in C(J, \mathbb{R}^2)$ and $\gamma \in$ Span$\{(1,1)^\top\}$. On the other hand, for the corresponding discretized problem (35), (36) with data (39), we have $\widetilde{D}(\tau) = \begin{pmatrix} 1+(1+\tau)^N & 0 \\ 1+(1+\tau)^N & 0 \end{pmatrix}$, therefore

$$\dim\mathrm{Ker}\,(\widetilde{D}(\tau), C_TQ_{N-1}) = \dim\mathrm{Ker}\begin{pmatrix} 1+(1+\tau)^N & 0 & 0 & 1 \\ 1+(1+\tau)^N & 0 & 0 & 1 \end{pmatrix} = 3 > m = 2,$$

i.e., condition (38) does not hold. Thus, the MPBVP (35), (36) with the given data (39) is not uniquely solvable for some $\{q_n\}$ and $\gamma \in$ Span$\{(1,1)^\top\}$.

A device to overcome this difficulty is to add to (35) the equation

$$A_N x_{N+1} = B_N(\tau)x_N + \tau q_N. \tag{40}$$

The last equation taken together with (35) allows us to determine x_n $(n = \overline{0,N})$ knowing P_0x_0. Note that x_{N+1} is not uniquely defined by (40).

Clearly, if $\{x_n\}_{n=0}^{N+1}$ is a solution of (35), (36) and (40), then its first $(N+1)$ values $\{x_n\}_{n=0}^{N}$ form a solution of (35) and (36).

Definition 3.1. The augmented problem (35), (36), (40) is said to be uniquely solvable w.r.t. the first $(N+1)$ components if for any $\{q_n\}_{n=0}^{N}$ and $\gamma \in \mathrm{Im}(C_0, C_T)$, it possesses a solution $\{x_n\}_{n=0}^{N+1}$, and, moreover, the first $(N+1)$ values $\{x_n\}_{n=0}^{N}$ are uniquely determined, i.e., if $\{y_n\}_{n=0}^{N+1}$ is an another solution of (35), (36), (40) then $x_n = y_n$ $(n = \overline{0, N})$.

Rewrite the augmented problem (35), (36), (40) as follows

$$A_n(x_{n+1} - x_n)/\tau + B_n x_n = q_n \quad (n = \overline{0, N}) \tag{41}$$

$$C_0 x_0 + C_T x_N = \gamma. \tag{42}$$

Acting as in the proof of Theorem 2.3 we can split system (41) into subsystems, each of them consists of one pair of equations

$$\begin{cases} P_n x_{n+1} = P_n(P_{n-1}x_n + Q_{n-1}x_n) - \tau P_n \overline{G}_n^{-1} B_n P_{n-1} x_n \\ \qquad\qquad + \tau P_n \overline{G}_n^{-1} q_n \\ Q_{n-1} x_n = V_{n-1} V_n^{\top} (Q_n \overline{G}_n^{-1} q_n - Q_n \overline{G}_n^{-1} B_n P_{n-1} x_n). \end{cases} \quad (n = \overline{0, N})$$

Thus, the IVP for the last subsystems is reduced to the following equations

$$\begin{cases} u_{n+1} = \widetilde{M}_n u_n + \widetilde{r}_n \quad (n = \overline{0, N}), \\ u_0 = u_0^* (:= P_0 x^0) \\ x_n = \widetilde{P}_n u_n + V_{n-1} Q^* V_n^{\top} \overline{G}_n^{-1} q_n, \end{cases} \tag{43}$$

where $\widetilde{M}_n := P_n \widetilde{P}_n - \tau P_n \overline{G}_n^{-1} B_n$, $\widetilde{P}_n := I - V_{n-1} Q^* V_n^{\top} \overline{G}_n^{-1} B_n$ and

$$\widetilde{r}_n := P_n V_{n-1} Q^* V_n^{\top} \overline{G}_n^{-1} q_n + \tau P_n \overline{G}_n^{-1} q_n \quad (n = \overline{0, N-1}).$$

With the same notations as in Theorem 2.3, it is easy to see that $M_n = \widetilde{M}_n + Q_{n-1}$ and $r_n = \widetilde{r}_n$ $(n = \overline{0, N-1})$. Let $X_0 := \widetilde{P}_0$, $X_n := \widetilde{P}_n \widetilde{M}_{n-1} \ldots \widetilde{M}_0$, $n = 1, \ldots, N$, and $D(\tau) := C_0 X_0 + C_T X_N$. Clearly, $\{X_n\}_{n=0}^{N}$ is the fundamental solution of the following system

$$\begin{cases} A_n(X_{n+1} - X_n)/\tau + B_n X_n = O \quad (n = \overline{0, N}) \\ P_0(X_0 - I) = O. \end{cases}$$

Since the first $(N+1)$ components $\{x_n\}_{n=0}^{N}$ of the solution of the problem (41), (42) are given (cf. [2, Theorem 2]) by the formula:

$$x_n = X_n x^0 + z_n, \quad n = \overline{0, N},$$

where $z_0 := V_{-1} Q^* V_0^{\top} \overline{G}_0^{-1} q_0$, $z_n := \widetilde{P}_n(\widetilde{r}_{n-1} + \widetilde{M}_{n-1}\widetilde{r}_{n-2} + \cdots + \widetilde{M}_{n-1} \ldots \widetilde{M}_1 \widetilde{r}_0) + V_{n-1} Q^* V_n^{\top} \overline{G}_n^{-1} q_n$ $(n = \overline{1, N})$, and x^0 satisfies the algebraic system

$$D(\tau) x^0 = \gamma^*$$

with $\gamma^* := \gamma - C_0 z_0 - C_T z_N$.

Theorem 3.1. *The augmented problem* (41), (42) *is uniquely solvable w.r.t. the first* $(N+1)$ *components if and only if the shooting matrix* $D(\tau)$ *satisfies the conditions*

$$\text{Ker}D(\tau) = \text{Ker}A(t_0), \quad \text{Im}D(\tau) = \text{Im}(C_0, C_T). \tag{44}$$

Proof. Suppose that the shooting matrix $D(\tau)$ satisfies the conditions (44). Since $\text{Im}D(\tau)$=$\text{Im}(C_0, C_T)$, it follows that for $\gamma \in \text{Im}(C_0, C_T)$ there exists x_0^* such that $D(\tau)x_0^* = \gamma^*$. Thus $x_n = X_n x_0^* + z_n$, $n = \overline{0, N}$, form a solution of the problem (41), (42). Now, suppose that $\{x_n\}_{n=0}^N$ and $\{\bar{x}_n\}_{n=0}^N$ are the first $(N+1)$ values of two solutions of equations (41), (42), i.e., there exist x_0^* and $\bar{x}_0^*$ such that

$$x_n = X_n x_0^* + z_n, \quad \bar{x}_n = X_n \bar{x}_0^* + z_n, \quad n = \overline{0, N},$$

and moreover, $D(\tau)x_0^* = \gamma^*$, $D(\tau)\bar{x}_0^* = \gamma^*$. Thus we get $D(\tau)(x_0^* - \bar{x}_0^*) = 0$, or $x_0^* - \bar{x}_0^* \in \text{Ker}D(\tau)$. From $\text{Ker}D(\tau)$=$\text{Ker}A(t_0)$=$\text{Ker}P_0$, it implies that $x_0^* - \bar{x}_0^*$ belongs to $\text{Ker}P_0$. Observing that $x_n - \bar{x}_n = X_n(x_0^* - \bar{x}_0^*)$ $(n = \overline{0, N})$ and using the fact that $X_n P_0 = X_n$ for $n = 0, \ldots, N$, we find that $x_n - \bar{x}_n = 0$ $(n = \overline{0, N})$. Therefore, the augmented problem (41), (42) is uniquely solvable w.r.t. the first $(N+1)$ components.

Conversely, suppose that the augmented problem is uniquely solvable w.r.t. the first $(N+1)$ components. Then the corresponding homogenous system

$$\begin{cases} A_n(x_{n+1} - x_n)/\tau + B_n x_n = 0 \quad (n = \overline{0, N}) \\ C_0 x_0 + C_T x_N = 0 \end{cases}$$

has a solution whose first $(N+1)$ components are uniquely determined and equal to zero. Letting $\bar{x}_0^* \in \text{Ker}D(\tau)$ and putting $x_n^* = X_n \bar{x}_0^*$, $n = \overline{0, N}$, we find that $\{x_n\}_{n=0}^N$ are the first $(N+1)$ values of a solution of the homogenous system. From the assumption it follows that $x_n^* = 0$ $(n = \overline{0, N})$. In particular, $x_0 = X_0 \bar{x}_0^* = 0$; hence $P_0 \bar{x}_0^* = 0$, therefore $\bar{x}_0^* \in \text{Ker}P_0$. Thus, $\text{Ker}D(\tau) \subset \text{Ker } P_0$. Since (41), (42) has a solution, for $q_n = 0$, $n = \overline{0, N-1}$, and $\gamma \in \text{Im}(C_0, C_T)$ there exists x_0^* such that $D(\tau)x_0^* = \gamma^*$. Noting that $\gamma^* = \gamma - C_0 z_0 - C_T z_N = \gamma$, we get $\gamma \in \text{Im}D(\tau)$. Hence $\text{Im}(C_0, C_T) \subset \text{Im}D(\tau)$.

Since $D(\tau) = C_0 X_0 + C_T X_N$ and $X_n = X_n P_0$, $n = \overline{0, N}$, it implies that $\text{Ker}P_0 \subset \text{Ker}D(\tau)$ and $\text{Im}D(\tau) \subset \text{Im}(C_0, C_T)$. Thus, we come to the relations (44). The proof is complete. □

Theorem 3.2. *Suppose that the continuous MPBVP* (4), (34) *is uniquely solvable for any* $q \in C(J, \mathbb{R}^m)$ *and* $\gamma \in \text{Im}(C_0, C_T)$. *Then*

(i) *for sufficiently small* $\tau > 0$*, the augmented problem* (41), (42) *is uniquely solvable w.r.t. the first* $(N+1)$ *components.*

(ii) *the discretization method is convergent, i.e.,*

$$\|x_n - x(t_n)\| \to 0 \quad \text{as } \tau \to 0,$$

where $\{x_n\}_{n=0}^N$ *are the first* $(N+1)$ *values of the solution of the agumented problem and* $x(t)$ *is the unique solution of the continuous MPBVP.*

Proof. Firstly, suppose that the continuous problem (4), (34) is uniquely solvable for any $q \in C(J, \mathbb{R}^m)$ and $\gamma \in \mathrm{Im}(C_0, C_T)$. Then the shooting matrix D satisfies condition (37). By a proof similar to that of Theorem 2.3, we can show that $\|X(t_0) - X_0\| \to 0$ and $\|X(T) - X_N\| \to 0$ as $\tau \to 0$. Thus

$$D(\tau) = C_0X_0 + C_TX_N = C_0X(t_0) + C_TX(T) + E(\tau) = D + E(\tau),$$

where $\|E(\tau)\| \to 0$ as $\tau \to 0$. Since the determinant is continuous in τ, the number of independent vectors of $D(\tau)$ is not less than that of D. This implies that $\mathrm{rank}D(\tau) \geq \mathrm{rank}D$. Hence

$$\mathrm{dimIm}D(\tau) \geq \mathrm{dimIm}D, \quad \mathrm{dimKer}D(\tau) \leq \mathrm{dimKer}D. \tag{45}$$

Taking (37) into account and using (45) we come to the desired relations (44). Theorem 3.1 ensures the unique solvability w.r.t. the first $(N + 1)$ values of the augmented problem (41), (42).

Now denoting $P_s(t) := I - Q(t)G^{-1}(t)B(t)$, $C(t) := P'(t)P_s(t) - P(t)G^{-1}(t)B(t)$, $h(t) := P(t)G^{-1}(t)q(t) + P'(t)Q(t)G^{-1}(t)q(t)$, and taking into account the boundary condition (34) and relations (19), (20) we get

$$u'(t) = C(t)u(t) + h(t), \quad t \in J, \tag{46}$$

$$\bar{C}_0u(t_0) + \bar{C}_Nu(T) = \beta, \quad Q(t_0)u(t_0) = 0, \tag{47}$$

$$x(t) = P_s(t)u(t) + Q(t)G^{-1}(t)q(t),$$

where $\bar{C}_0 := C_0P_s(t_0)$, $\bar{C}_N := C_TP_s(T)$ and

$$\beta := \gamma - C_0Q(t_0)G^{-1}(t_0)q(t_0) - C_TQ(T)G^{-1}(T)q(T).$$

According to [8], condition (47) is equivalent to

$$\bar{C}_0u(t_0) + \bar{C}_Nu(T) + KQ(t_0)u(t_0) = \beta, \tag{48}$$

where $K \in \mathbb{R}^{m\times m}$ is a matrix such that

$$\mathrm{Im}K \cap \mathrm{Im}(C_0, C_T) = \{0\}, \quad \mathrm{Ker}K \cap \mathrm{Ker}A(t_0) = \{0\}.$$

Therefore, equations (46), (47) are equivalent to (46), (48). Besides, it has been proved that [8] the shooting matrix $S := \bar{C}_0 + \bar{C}_NY(T) + KQ(t_0)$ is nonsingular, where Y is the fundamental solution matrix of the following equation

$$Y' = C(t)Y, \quad Y(t_0) = I.$$

So problem (46), (48) possesses a unique solution.

Proceeding as in the proof of Theorem 2.3, we find that the augmented problem (41), (42) is equivalent to the system of equations

$$u_{n+1} = M_nu_n + r_n \quad (n = \overline{0, N}), \tag{49}$$

$$\widetilde{C}_0u_0 + \widetilde{C}_Nu_N = \widetilde{\beta}, \quad Q_0u_0 = 0, \tag{50}$$

$$x_n = \widetilde{P}_nu_n + V_{n-1}Q^*V_n^\top\overline{G}_n^{-1}q_n,$$

where $\widetilde{C}_0 := C_0\widetilde{P}_0$, $\widetilde{C}_N := C_T\widetilde{P}_N$ and

$$\widetilde{\beta} := \gamma - C_0V_{-1}Q^*V_0^\top\overline{G}_0^{-1}q_0 - C_TV_{N-1}Q^*V_N^\top\overline{G}_N^{-1}q_N.$$

Now, let $\{\bar{u}_n\}_{n=0}^N$ satisfy the following TPBVP:

$$\bar{u}_{n+1} = \bar{M}_n\bar{u}_n + \bar{r}_n \quad (n = \overline{0, N-1}), \tag{51}$$

$$\bar{C}_0\bar{u}_0 + \bar{C}_N\bar{u}_N = \beta, \quad Q_0\bar{u}_0 = 0, \tag{52}$$

or equivalently,

$$\begin{cases} (\bar{u}_{n+1} - \bar{u}_n)/\tau = (P_n'P_s(t_n) - P_nG_n^{-1}B_n)\bar{u}_n \\ \qquad\qquad +P_nG_n^{-1}q_n + P_n'Q_nG_n^{-1}q_n \quad (n = \overline{0, N-1}), \\ \bar{C}_0\bar{u}_0 + \bar{C}_N\bar{u}_N = \beta, \quad Q_0\bar{u}_0 = 0. \end{cases}$$

Since the last system is obtained as an application of the EEM to the TPBVP (46), (47), we get (see [6, Theorem 1, p. 429]):

$$\|\bar{u}_n - u(t_n)\| = O(\tau), \quad n = \overline{0, N}.$$

Furthermore, arguing as in the proof of Theorem 2.3 and putting

$$\bar{x}_n := (I - Q_nG_n^{-1}B_n)\bar{u}_n - Q_nG_n^{-1}q_n \quad (n = \overline{0, N}),$$

we have $\|\bar{x}_n - x(t_n)\| = O(\tau), \;\; n = \overline{0, N}$. Noting that

$$\|V_nQ^*V_n^\top G_n^{-1} - V_{n-1}Q^*V_n^\top\overline{G}_n^{-1}\| = O(\tau) \quad \text{and}$$

$$x_n - \bar{x}_n = \widetilde{P}_n(u_n - \bar{u}_n) + (V_nQ^*V_n^\top G_n^{-1} - V_{n-1}Q^*V_n^\top\overline{G}_n^{-1})(B_n\bar{u}_n - q_n), \; n = \overline{0, N},$$

as well as $\|x_n - x(t_n)\| \le \|x_n - \bar{x}_n\| + \|\bar{x}_n - x(t_n)\|$, we come to the conclusion that if $\|u_n - \bar{u}_n\| \to 0 \;\; \tau \to 0$ then $\|x_n - \bar{x}_n\| \to 0$ as $\tau \to 0 \;\; (n = \overline{0, N})$, and hence, $\|x_n - x(t_n)\| \to 0$ as $\tau \to 0 \;\; (n = \overline{0, N})$.

Coming back to the problem (49), (50), we observe that its solution is a solution of the IVP

$$\begin{cases} u_{n+1} = M_nu_n + r_n, \quad n = \overline{0, N}, \\ u_0 = u_0^*, \end{cases}$$

where u_0^* satisfies the condition

$$S_1u_0^* = \beta_1, \tag{53}$$

where $S_1 := \widetilde{C}_0 + \widetilde{C}_NM_{N-1}\dots M_0 + KQ_0$ and

$$\beta_1 := \widetilde{\beta} - \widetilde{C}_N(r_{N-1} + M_{N-1}r_{N-2} + \dots + M_{N-1}\dots M_1r_0).$$

Similary, the unique solution of (51), (52) satisfies the IVP

$$\begin{cases} \bar{u}_{n+1} = \bar{M}_n\bar{u}_n + \bar{r}_n, \quad n = \overline{0, N-1}, \\ \bar{u}_0 = \bar{u}_0^* \end{cases}$$

where $\bar{u}_0^*$ is determined by

$$\bar{S}_1\bar{u}_0^* = \bar{\beta}_1 \tag{54}$$

with $\bar{S}_1 := \bar{C}_0 + \bar{C}_N \bar{M}_{N-1} \dots \bar{M}_0 + KQ_0$ and

$$\bar{\beta}_1 := \beta - \bar{C}_N(\bar{r}_{N-1} + \bar{M}_{N-1}\bar{r}_{N-2} + \dots + \bar{M}_{N-1} \dots \bar{M}_1\bar{r}_0).$$

Since

$$S_1 := \tilde{C}_0 + \tilde{C}_N M_{N-1} \dots M_0 + KQ_0,$$
$$\bar{S}_1 := \bar{C}_0 + \bar{C}_N \bar{M}_{N-1} \dots \bar{M}_0 + KQ_0$$

it follows that

$$\|\bar{S}_1 - S_1\| \le \|\bar{C}_0 - \tilde{C}_0\| + \|\bar{C}_N\bar{M}_{N-1} \dots \bar{M}_0 - \tilde{C}_N M_{N-1} \dots M_0\|.$$

Noting that $\|M_n - \bar{M}_n\| = o(\tau)$ for $n = \overline{0, N-1}$, we can write $\bar{M}_n = M_n + E_n$ $(n = \overline{0, N-1})$, where $\|E_n\| = o(\tau)$ for $n = \overline{0, N-1}$. Thus

$$\begin{aligned}\bar{M}_{N-1} \dots \bar{M}_0 &= (M_{N-1} + E_{N-1})\bar{M}_{N-2} \dots \bar{M}_0 \\ &= M_{N-1}\bar{M}_{N-2} \dots \bar{M}_0 + E_{N-1}\bar{M}_{N-2} \dots \bar{M}_0.\end{aligned}$$

Observing that

$$\|\prod_{i=k}^{N-1} \bar{M}_{N-1-i}\| \le \prod_{i=k}^{N-1} \|\bar{M}_{N-1-i}\| \le e^{\bar{c}(T-t_0)} = \text{const}, \quad \forall\, k = \overline{0, N-1}$$

and putting $\bar{E}_{N-1} := E_{N-1}\bar{M}_{N-2} \dots \bar{M}_0$, we get $\bar{M}_{N-1} \dots \bar{M}_0 = M_{N-1}\bar{M}_{N-2} \dots \bar{M}_0 + \bar{E}_{N-1}$, where $\|\bar{E}_{N-1}\| = o(\tau)$. By the same argument we have

$$\bar{M}_{N-1} \dots \bar{M}_0 = M_{N-1}M_{N-2}\bar{M}_{N-3} \dots \bar{M}_0 + \bar{E}_{N-1} + \bar{E}_{N-2},$$

where $\|\bar{E}_{N-2}\| = o(\tau)$, etc.. Finally, we come to the relation

$$\bar{M}_{N-1} \dots \bar{M}_0 = M_{N-1} \dots M_0 + \bar{E}_{N-1} + \dots + \bar{E}_0, \ \|\bar{E}_n\| = o(\tau) \quad (n = \overline{0, N-1}).$$

From this we get

$$\begin{aligned}&\|\bar{C}_N\bar{M}_{N-1} \dots \bar{M}_0 - \tilde{C}_N M_{N-1} \dots M_0\| \\ &= \|(\bar{C}_N - \tilde{C}_N)M_{N-1} \dots M_0 + \bar{C}_N(\bar{E}_{N-1} + \dots + \bar{E}_0)\| \\ &\le \|\bar{C}_N - \tilde{C}_N\|\|M_{N-1} \dots M_0\| + \|\bar{C}_N\| \sum_{n=0}^{N-1} \|\bar{E}_n\|.\end{aligned}$$

Noting that $\|\bar{E}_n\| = o(\tau) \quad (n = \overline{0, N-1})$ and $N = \dfrac{T - t_0}{\tau}$ we find $\sum_{n=0}^{N-1} \|\bar{E}_n\| = \dfrac{o(\tau)}{\tau} = o(1)$. Combining this estimate with $\|\bar{C}_N - \tilde{C}_N\| = O(\tau)$ and $\|M_{N-1} \dots M_0\| \le e^{c(T-t_0)}$ =const, we obtain $\|\bar{C}_N\bar{M}_{N-1} \dots \bar{M}_0 - \tilde{C}_n M_{N-1} \dots M_0\| = o(1)$; therefore $\|S_1 - \bar{S}_1\| = o(1)$.

Now we shall prove that $\|\beta_1 - \bar{\beta}_1\| = o(1)$, where

$$\beta_1 := \tilde{\beta} - \tilde{C}_N\beta_1^*, \quad \bar{\beta}_1 := \beta - \bar{C}_N\bar{\beta}_1^*, \tag{55}$$

and

$$\beta_1^* := r_{N-1} + M_{N-1}r_{N-2} + \cdots + M_{N-1}\dots M_1 r_0,$$
$$\bar{\beta}_1^* := \bar{r}_{N-1} + \bar{M}_{N-1}\bar{r}_{N-2} + \cdots + \bar{M}_{N-1}\dots \bar{M}_1 \bar{r}_0.$$

Consider the IVPs:

$$\begin{cases} v_{n+1} &= M_n v_n + r_n, \quad n = \overline{0, N-1} \\ v_0 &= v^0 \end{cases}$$

and

$$\begin{cases} \bar{v}_{n+1} &= \bar{M}_n \bar{v}_n + \bar{r}_n, \quad n = \overline{0, N-1} \\ \bar{v}_0 &= v^0. \end{cases}$$

Note that these IVPs were considered in Theorem 2.3 and their solutions were denoted by u_n and $\bar{u}_n$, respectively. Besides, from the proof of Theorem 2.3, it follows that $\|\bar{v}_N - v_N\| = o(1)$. Since $\beta_1^* = v_N - M_{N-1}\dots M_0 v^0$, $\bar{\beta}_1^* = \bar{v}_N - \bar{M}_{N-1}\dots \bar{M}_0 v^0$, and $\|M_{N-1}\dots M_0 - \bar{M}_{N-1}\dots \bar{M}_0\| = o(1)$, $\|v_N - \bar{v}_N\| = o(1)$, it follows that $\|\beta_1^* - \bar{\beta}_1^*\| = o(1)$.

From (55) we have

$$\begin{aligned} \|\beta_1 - \bar{\beta}_1\| &\le \|\tilde{\beta} - \beta\| + \|\bar{C}_N \bar{\beta}_1^* - \tilde{C}_N \beta_1^*\| \\ &\le \|\tilde{\beta} - \beta\| + \|\bar{C}_N\|\|\bar{\beta}_1^* - \beta_1^*\| + \|\bar{C}_N - \tilde{C}_N\|\|\beta_1^*\|. \end{aligned}$$

Using the fact that $\|\tilde{\beta} - \beta\| = O(\tau)$, $\|\bar{C}_N - \tilde{C}_N\| = O(\tau)$, $\|\bar{\beta}_1^* - \beta_1^*\| = o(1)$ and noting that $\|\bar{C}_N\| \le \|C_T\|\|P_s(T)\|$ =const, $\|\beta_1^*\| \le \|v_N\| + \|M_{N-1}\dots M_0 v^0\| \le$ const, we can conclude that $\|\bar{\beta}_1 - \beta_1\| = o(1)$. Thus

$$\|\overline{S}_1 - S_1\| = o(1) \text{ and } \|\beta_1 - \bar{\beta}_1\| = o(1) \text{ for sufficiently small } \tau. \tag{56}$$

As mentioned above, $Y(t)$ is the fundamental solution matrix of the IVP $Y'(t) = C(t)Y(t),\ t \in J;\quad Y(t_0) = I$. Applying the EEM to this problem and noting that

$$\begin{aligned} C(t) &:= P'(t)P_s(t) - P(t)G^{-1}(t)B(t) \\ &= P'(t)(I - Q(t)G^{-1}(t)B(t)) - P(t)G^{-1}(t)B(t), \text{ for } t \in J; \\ \bar{M}_n &:= I + \tau P_n'(I - Q_n G_n^{-1} B_n) - \tau P_n G_n^{-1} B_n, \quad n = \overline{0, N-1}, \end{aligned}$$

we get

$$\begin{cases} Y_{n+1} &= \bar{M}_n Y_n, \quad n = \overline{0, N-1}, \\ Y_0 &= I. \end{cases}$$

Therefore, $Y_N = \bar{M}_{N-1}\dots \bar{M}_0$. Thus we can rewrite $\bar{S}_1$ as $\bar{S}_1 = \bar{C}_0 + \bar{C}_N Y_N + KQ_0$. Since $S := \bar{C}_0 + \bar{C}_N Y(T) + KQ(t_0)$ and $\|Y_N - Y(T)\| = O(\tau)$, we come to the conclusion that $\|\bar{S}_1 - S\| = O(\tau)$. The last equality and relation (56) imply that $S_1 \to S$ as $\tau \to 0$. Since the shooting matrix S of the continuous problem is nonsingular, the matrices $\bar{S}_1^{-1}$ and S_1^{-1} do exist and are uniformly bounded for τ small enough. Hence from (53), (54), (56) it follows that $\|u_0^* - \bar{u}_0^*\| = o(1)$.

Proceeding as in the proof of Theorem 2.3 and noting that $\xi_0 := \|u_0 - \bar{u}_0\| = \|u_0^* - \bar{u}_0^*\| = o(1)$ we obtain $\xi_n \leq o(1)$ for all $n = \overline{0, N}$. It implies that $\|u_n - \bar{u}_n\| \to 0$ as $\tau \to 0$, hence $\|x_n - \bar{x}_n\| \to 0$. Therefore $\|x_n - x(t_n)\| \to 0$ $(n = \overline{0, N})$ as $\tau \to 0$. □

Acknowledgement

The authors would like to thank the referee for his (her) careful reading and valuable remarks which improve the presentation of the paper.

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