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LYAPUNOV REGULARITY OF LINEAR DIFFERENTIAL ALGEBRAIC EQUATIONS OF INDEX 1

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Abstract. In this paper, we introduce a concept of Lyapunov regularity of linear differential algebraic equations (DAEs) based on the notion of Lyapunov exponents of DAEs. It was proved that under certain conditions a DAE of index 1 is Lyapunov regular if and only if the corresponding ordinary differential equation is Lyapunov regular.

1. Introduction

In science and practical applications there are numerous problems such as the problem of description of dynamic systems, electric circuit systems or problems in cybernetics etc... requiring investigation of solutions of differential equations of the type $Ax' + Bx = 0$, where A, B are constant or continuous time-dependent matrices of order m with $\det A = 0$; such equations are called differential algebraic equations (DAEs).

Investigation of DAEs was carried out intensively by many researchers around the world (see [1, 2, 4, 7, 8, 10] and the references therein). Many results on stability properties of DAEs were obtained: asymptotic and exponential stability of DAEs which are of index 1 and 2, a criterion for stability of a DAEs of index 1, stability of periodic DAEs (see [9, 10, 12, 13]). The method used for the above papers is based on the reduction of the investigation of a DAE to the investigation of a corresponding ordinary differential equation (ODE). For qualitative theory of linear ordinary differential equations (ODEs) Lyapunov introduced the notion of regularity. It is well known that a regular linear ODE has many good asymptotic properties such as all their solutions have exact Lyapunov exponents and their stability is robust under small nonlinear perturbations of order higher than 1 (see Bylov et al. [3]).

This paper continues the investigation of asymptotic properties of linear DAEs based on the method of Lyapunov exponents which started in our papers [5, 14]. In [5, 14] we introduced a concept of the Lyapunov spectrum of a linear DAE. In this paper, we shall develop a concept of Lyapunov regularity for linear DAEs which is similar to that of linear ODEs. Based on the notion of the Lyapunov spectrum of linear DAEs and the notion of the adjoint equation of DAEs we will define Lyapunov regularity of linear DAEs and derive asymptotic properties of

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regular DAEs. One of the main result of this paper is a theorem stating that under some assumptions a DAEs is regular if and only if a corresponding linear ODE is regular.

In the next section we recall some results from the theory of ODEs and DAEs which are needed in subsequent sections. In Section 3 we introduce the notion of Lyapunov regularity of a linear DAE which is based on the Lyapunov spectrum and the adjoint equation of a linear DAE. In Section 4 we derive a criterion for Lyapunov regularity of a linear DAE which states that under some assumptions a DAE is regular if and only if the corresponding ODE is regular. In the last section we prove some asymptotic properties of a regular DAE which are similar to those of a regular ODE.

2. Preliminaries

In this paper we will consider a linear DAE

$$A(t)x' + B(t)x = 0, \tag{1}$$

where A, $B : \mathbb{R}^+ \to L(\mathbb{R}^m, \mathbb{R}^m)$ are bounded continuous $m \times m$-matrix functions, $\operatorname{rank} A(t) = r$, r is a fixed integer, $r < m$, $N(t) := \ker A(t)$ is of the constant dimension $m - r$ for all $t \in \mathbb{R}^+$. *We will always assume that* (1) *is of index* 1, i.e. there exists a C^1-smooth projector $Q \in C^1(\mathbb{R}^+, L(\mathbb{R}^m, \mathbb{R}^m))$ onto $\ker A(t)$, such that the matrix $G(t) := A(t) + B(t)Q(t)$ has bounded inverse on each interval $[t_0, T] \subset \mathbb{R}^+$ (see [8, 9]). For definition of a solution $x(t)$ of the DAE (1) one does not require $x(t)$ to be C^1-smooth but only some of its coordinates are smooth. Namely, we introduce the space

$$C^1_A(\mathbb{R}^+, \mathbb{R}^m) := \{x(t) : \mathbb{R}^+ \to \mathbb{R}^m,\ x(t) \text{ is continuous and } P(t)x(t) \in C^1(\mathbb{R}^+, \mathbb{R}^m)\},$$

where $P(t) := I - Q(t)$ and I is the identity operator of $\mathbb{R}^m$, and note that the function space C^1_A does not depend on the choice of the C^1-smooth projector $Q(t)$ onto $\ker A(t)$.

Definition 2.1. Assume that $N(t)$ is smooth, i.e. there exists a differentiable projector function Q onto $N(t)$. A function $x \in C^1_A(\mathbb{R}^+, \mathbb{R}^m)$ is said to be a *solution* of (1) on $\mathbb{R}^+$ if the identity

$$Ax' + Bx = A[(Px)' - P'x] + Bx = 0,$$

where $P(t) := I - Q(t)$ and I is the identity operator of $\mathbb{R}^m$, is satisfied for all $t \in \mathbb{R}^+$.

The following proposition on the existence and uniqueness of the solution of an initial value problem (IVP) for the DAE (1) of index 1 was proved in Griepentrog and März [8, p. 36], Balla and März [1].

Proposition 2.1. *For each given* $x^0 \in \mathbb{R}^m$ *the IVP*

$$A(t)x' + B(t)x = 0, \quad x(t_0) - x^0 \in N(t_0), \tag{2}$$

is uniquely solvable on each interval $[t_0, T] \subset \mathbb{R}^+$. *The solution is defined by the state variable system*

$$u'(t) = P'(t)u(t) - P(t)(I + P'(t))G^{-1}(t)B(t)u(t), \quad u(t_0) = P(t_0)x^0, \tag{3}$$

$$x(t) = u(t) - Q(t)G^{-1}(t)B(t)u(t); \tag{4}$$

furthermore, $u(t) = P(t)x(t)$.

Note that, using the projector $P_s(t) := I - Q(t)G^{-1}(t)B(t)$ onto

$$S(t) := \{x \in \mathbb{R}^m : B(t)x \in \text{ im } A(t)\},$$

the formulas (3), (4) can be rewritten as

$$u'(t) = [P'(t)P_s(t) - P(t)G^{-1}(t)B(t)]u(t), \tag{5}$$

$$x(t) = P_s(t)u(t), \tag{6}$$

or (see [1, 9])

$$u'(t) = \big(P'(t) - P(t)A_1^{-1}(t)B_0(t)\big)u(t), \tag{7}$$

where

$$B_0 := B - AP', \quad A_1 := A + B_0Q.$$

Definition 2.2. The ODE (5) (or (7)) is called the *corresponding (under P) ordinary differential equation* (ODE) of the DAE (1) of index 1.

Note that (5) and (7) are two different versions of the state variable equation of the DAE (1) (see [1, 8, 9]). Though derived from the same DAE (1) the ODEs (5) and (7) are different ODEs in the space $\mathbb{R}^m$ because to derive (5) and (7) from (1) we use G and A_1 which are, in general, different matrices. However, restricted to the invariant solution space $\operatorname{im} P$ the ODEs (5) and (7) are the same and their solutions are $u = Px$ with x being solutions of (1). The matrices A_1 and G are related by the formula $A_1 = G - AP'Q = G(I - PP'Q)$. Hence, if $P'Q = 0$, which is the case if $\ker A(t)$ does not depend on t, then $A_1 = G$ and (7) coincides with (5).

Definition 2.3. A square matrix $X(t)$ of order m is called a *fundamental solution matrix* (FSM) of (1) if its first r vector-columns are linearly independent solutions of (1) and the last $m - r$ vector-columns of $X(t)$ are zero.

Definition 2.4. For a real function $f : \mathbb{R}^+ \to \mathbb{R}$, the number

$$\lambda(f) := \limsup_{t\to\infty} \frac{1}{t} \ln |f(t)|$$

(which may be $\pm\infty$) is called the *Lyapunov exponent* of f. If in the above formula the limit exists, i.e. we can replace lim sup by lim, then we say that f has *exact*

Lyapunov exponent. The Lyapunov exponent of a matrix function $F(t) = [f_{ik}(t)]$ $(j, k = 1, \dots, m)$ is

$$\lambda(F) := \max_{j,k} \lambda(f_{jk}).$$

Clearly, if G^{-1} is bounded on $\mathbb{R}^+$ and the projector P has bounded continuous derivative P' then (5) is an ODE with bounded continuous coefficients, hence it has a finite Lyapunov spectrum, which implies that (1) has a finite Lyapunov spectrum (see [5]). Note that for a given DAE (1) there are many different bounded and differentiable projectors P which lead to different corresponding ODEs under these projectors. Among these projectors there may be some with bounded derivatives and some with unbounded derivatives (for a simple example, consider a DAE (1) with constant $\ker A(t) \equiv \ker A(t_0)$). Those with bounded derivatives lead to ODE (5) with bounded coefficients, hence the classical theory of Lyapunov exponents is applicable (see [3]). Meanwhile, those projectors P with unbounded derivative may lead to ODE (5) with unbounded coefficients for which we still do not have a well developed theory of Lyapunov exponents, hence we do not benefit from reduction of (1) to (5). From now on we consider only C^1-smooth projectors with bounded derivatives. Moreover, *we will also assume that G^{-1} is bounded on* $\mathbb{R}^+$ (this is an assumption on P), which implies that $P = G^{-1}A$ is bounded.

Definition 2.5. A FSM

$$X(t) = [x_1(t), \dots, x_r(t), 0, \dots, 0]$$

of (1) is called *normal* if the sum

$$\sigma := \sum_{i=1}^{r} \lambda(x_i)$$

attains its minimum in the set of all FSMs of (1).

It is known (see [5]) that a FSM

$$X(t) = [x_1(t), \dots, x_r(t), 0, \dots, 0]$$

of (1) is normal if and only if the system $x_1(t), \dots, x_r(t)$ has the property of incompressibility, i.e. for any linear combination

$$y(t) = \sum_{i=1}^{r} c_i x_i(t),$$

we have $\lambda(y) = \max_{i\in\{1,\dots,r\},\, c_i\neq 0} \lambda(x_i)$.

Let $X(t) = [x_1(t), \dots, x_r(t), 0, \dots, 0]$ be a normal FSM of (1) and $\lambda(x_i) = \lambda_i$. We may always order $x_1(t), \dots, x_r(t)$ by increasing values of their Lyapunov exponents

$$\lambda_1 \leq \cdots \leq \lambda_r.$$

Note that two normal FSMs have the same set of Lyapunov exponents $\lambda_1 \leq \cdots \leq \lambda_r$, which is called the *Lyapunov spectrum of* (1) (see [5]). We may also identify, as some authors do, distinct Lyapunov exponents $\widetilde{\lambda}_1 < \cdots < \widetilde{\lambda}_d$ of a normal FSM of (1) and their multiplicities $n_1, \ldots, n_d$ in the set $\lambda_1 \leq \cdots \leq \lambda_r$, and call the set of pairs (λ_i, n_i), $i = 1, \ldots, d$, the Lyapunov spectrum of (1) as well. However, in this paper we prefer to use the definition of Lyapunov spectrum of (1) as the set $\lambda_1 \leq \cdots \leq \lambda_r$.

In the sequel, we will need the notion of adjoint equation of a DAE. We cite here the definition and some properties of adjoint equations from Balla and März [1] and refer to [1] for more details.

Definition 2.6. The equation

$$(A^*\varphi)' - B^*\varphi = 0 \tag{8}$$

is called the *adjoint equation of* (1).

In this paper we use the notation A^* for the conjugate matrix of a matrix A which coincides with the transposed matrix A^T of A since we are considering the real case. It is known that if the equation (1) is of index 1, then the adjoint equation (8) is of index 1 too, i.e. $\dim A^*(t) = r < m$ for all $t \in \mathbb{R}^+$, the pencil of matrices $\{A^*, B^*\}$ has index 1 and there exists a projector $Q_* \in C^1(\mathbb{R}^+, L(\mathbb{R}^m, \mathbb{R}^m))$ such that $\operatorname{im} A^*(t) = \ker Q_*(t)$ for all $t \in \mathbb{R}^+$ (see [1]).

Similarly to the case of the DAE (1) a solution φ of (8) is not necessarily C^1-smooth, it should only belong to the function space

$$C^1_{*A}(\mathbb{R}^+, \mathbb{R}^m) := \{\varphi \in C(\mathbb{R}^+, \mathbb{R}^m) : A^*\varphi \in C^1(\mathbb{R}^+, \mathbb{R}^m)\}.$$

Note that the function spaces $C^1_{A^*}(\mathbb{R}^+, \mathbb{R}^m)$ and $C^1_{*A}(\mathbb{R}^+, \mathbb{R}^m)$ are different. However, if A is C^1-smooth, then $C^1_{A^*}(\mathbb{R}^+, \mathbb{R}^m) = C^1_{*A}(\mathbb{R}^+, \mathbb{R}^m)$.

For an adjoint DAE of a DAE of index 1, the IVP also has a unique solution. To formulate the theorem, we denote by A^+ the Moore-Penrose inverse of a matrix A, i.e. $A^+ \in L(\mathbb{R}^m, \mathbb{R}^m)$ and

$$\begin{aligned} &A^+y = x \in \operatorname{im}(A^*) \quad \text{for} \quad y \in \operatorname{im} A \quad \text{with} \quad Ax = y, \\ &A^+y = 0 \quad \text{for} \quad y \in \ker A^*. \end{aligned}$$

Then AA^+ and A^+A are orthoprojectors onto $\operatorname{im} A$ and $\operatorname{im} A^*$ along $\ker A^*$ and $\ker A$.

Theorem 2.1. [1] *For an arbitrary $\varphi^0 \in \mathbb{R}^m$, there exists and unique a solution $\varphi \in C^1_{*A}(\mathbb{R}^+, \mathbb{R}^m)$ of the IVP*

$$\begin{cases} (A^*\varphi)' - B^*\varphi = 0, \\ A^*(t_0)(\varphi(t_0) - \varphi^0) = 0, \end{cases}$$

and this solution is of the form

$$\varphi(t) = P_{*s}(t)A^{+*}(t)P^*(t)v(t),$$
$$\varphi(t_0) = P_{*s}(t_0)\varphi^0,$$

where $P_{*s} := A_1^{*-1}A^*$ *and* $v(t)$ *is the solution of the IVP*

$$\begin{cases} v' &= (B_0^* A_1^{*-1} P^* - P^{*'})v, \\ v(t_0) &= A_1^*(t_0)\varphi^0. \end{cases}$$

Note that, any solution φ of (8) satisfies the condition $\varphi \in \operatorname{im} P_{*s}$, hence $S_* := \operatorname{im} P_{*s}$ is the solution space for (8). We have $\dim S_* = r$, $S_* \oplus \ker A^*(t) = \mathbb{R}^m$.

Theorem 2.2. [1] *Suppose that*

$$X \in C_A^1(\mathbb{R}^+, L(\mathbb{R}^m, \mathbb{R}^m)) \text{ and } \Phi \in C_{*A}^1(\mathbb{R}^+, L(\mathbb{R}^m, \mathbb{R}^m))$$

are FSMs of (1) *and* (8), *then*

$$(\Phi^* A X)' = 0. \tag{9}$$

Note that if we truncate the zero-columns in the definition of FSM (Definition 2.3) then we get a $(m \times r)$-matrix function which is a minimal fundamental solution in the sense of [1]. Given a FSM $X(t) = [x_1(t), \dots, x_r(t), 0, \dots, 0]$ in the sense of Definition 2.3, any FSM in the sense of [1] is related to X by a simple above truncation and a multiplication by a constant matrix from the right (see [1], Theorem 3 and Lemmas 3, 4). We choose this particular definition of FSM (Definition 2.3) since the zero-columns added to the solutions $x_1(t), \dots, x_r(t)$ while make X a square matrix do not affect the Lyapunov exponent of X, hence the latter is determined by $x_1(t), \dots, x_r(t)$. By this choice of definition, a FSM X, in general, does not have the group property unlike a FSM of an ODE or a so-called normalized maximal FSM of a DAE (see [1], Theorem 7). Since we do not need the group property of a FSM in this paper but we are interested mainly in Lyapunov exponents of solutions and of FSMs of DAEs, Definition 2.3 is sufficient for our aim.

3. Lyapunov regularity of linear differential algebraic equations

In this section, we introduce a concept of Lyapunov regularity of linear DAEs, which is an application of the classical Lyapunov regularity of ODEs to the DAEs. The Lyapunov regularity is an important property of linear ODEs, which was introduced by Lyapunov in the 19^{th} century. A regular system has a good asymptotic behavior, which was exploited intensively in the qualitative theory of ODEs.

Let us recall the classical notion of Lyapunov regularity of an ODE. Consider a linear ODE

$$x' = A(t)x, \quad x \in \mathbb{R}^m, \quad t \in \mathbb{R}^+, \tag{10}$$

where $A \in C(\mathbb{R}^+, L(\mathbb{R}^m, \mathbb{R}^m))$ and $\sup_{0 \le t < \infty} \|A(t)\| < +\infty$. The solution space of (10) is of dimension m. A *fundamental solution matrix* (FSM) of (10) is a square $(m \times m)$-matrix composed by m linearly independent column-vector functions which are solutions of (10). A FSM is called *normal* if the sum of Lyapunov exponents of its column-vectors attains the minimal possible value among that for all FSMs of (10). Denote by σ_X the sum of all Lyapunov exponents of solutions of (10) in a normal FSM $X(t)$ of (10). Note that this quantity does not depend on the choice of the normal FSM $X(t)$.

Definition 3.1. The equation (10) is called *Lyapunov regular* if the following equality holds

$$\sigma_X = \liminf_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \operatorname{trace} A(t_1) dt_1.$$

It is known (see [6]) that (10) is Lyapunov regular if and only if

(i) there exists the limit

$$S = \lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \operatorname{trace} A(t_1) dt_1$$

and

(ii) the Lyapunov equality $\sigma_X = S$ holds.

Based on considering the adjoint equation of (10), Perron had found an equivalent definition of Lyapunov regularity which allows us to avoid the use of Lyapunov inequality. Recall that the adjoint equation of (10) is the linear ODE

$$\dot{y} = -A^*(t)y, \tag{11}$$

where $A^*(t) = \overline{A^T(t)}$ is the complex conjugate of $A(t)$ and $A^*(t) = A^T(t)$ in the real case.

If X, Y are FSMs of (10) and (11), then we have the Lagrange equality

$$Y^* X = C,$$

where C is a constant matrix. Perron proved that (10) is regular if and only if the Lyapunov spectrum of (10)

$$\lambda_1 \le \cdots \le \lambda_m$$

and the Lyapunov spectrum of the adjoint ODE (11) ordered by decreasing values of Lyapunov exponents

$$\beta_1 \ge \cdots \ge \beta_m$$

satisfy the equality

$$\lambda_s + \beta_s = 0 \quad \text{for all} \quad s = 1, \dots, m.$$

This theorem of Perron gives us an alternative definition of Lyapunov regularity. Following the idea of Perron, in this paper we introduce a concept of Lyapunov regularity of the DAEs of index 1, which is based on the concept of Lyapunov spectrum of the DAE introduced in [5], and the concept of adjoint equation introduced in [1, 2].

Definition 3.2. The linear DAE (1) of index 1, is called *Lyapunov regular* if

$$\lambda_i + \beta_i = 0 \quad \text{for all} \quad i = 1, \dots, r,$$

where $\lambda_1 \leq \cdots \leq \lambda_r$ is the Lyapunov spectrum of (1) and $\beta_1 \geq \cdots \geq \beta_r$ is the Lyapunov spectrum ordered decreasingly of the adjoint equation (8) of (1).

4. Criterion for Lyapunov regularity of DAEs

In this section we derive a criterion for Lyapunov regularity of a linear DAE of index 1 by means of Lyapunov regularity of the corresponding ODEs.

Recall that for the DAE (1) we assume that the matrix $G(t) := A(t)+B(t)Q(t)$ has bounded inverse on $\mathbb{R}^+$, hence $P(t)$ is bounded and (5) has bounded coefficients on $\mathbb{R}^+$. Moreover, we shall assume that the nullspace of A does not depend on $t \in \mathbb{R}^+$, which implies that $P'Q \equiv 0$ (see [5]), hence $A_1 = G$ and (7) coincides with (5). We will need the following lemma.

Lemma 4.1. *Let* (1) *be a DAE of index 1. Assume that the coefficient matrices* A, B, A_1^{-1} *are bounded on* $\mathbb{R}^+$ *and the nullspace of* A *does not depend on* t*. Then there exists a normal FSM* $U(t)$ *of the corresponding ODE*

$$u' = (P' - PA_1^{-1}B_0)u \tag{7}$$

such that the first r *vector-columns of* $U(t)$ *are solutions of* (7)*, which belong to* $\operatorname{im} P(t)$ *for all* $t \in \mathbb{R}^+$ *and the remaining* $m - r$ *vector-columns of* $U(t)$ *are solutions of* (7)*, which belong to* $\ker P(t)$ *for all* $t \in \mathbb{R}^+$.

Proof. Let $U(t) = [u_1(t), \dots, u_m(t)]$ be a normal FSMs of (7) and

$$\lambda(u_1) \leq \cdots \leq \lambda(u_m).$$

Suppose that for some index k $(1 \leq k \leq m)$ and some $t_0 \in \mathbb{R}^+$ we have $u_k(t_0) \notin \operatorname{im} P(t_0)$ and $u_k(t_0) \notin \ker P(t_0)$. Then we can write

$$u_k(t_0) = P(t_0)u_k(t_0) + Q(t_0)u_k(t_0) = v_1(t_0) + v_2(t_0),$$

where $0 \neq v_1(t_0) \in \operatorname{im} P(t_0)$ and $0 \neq v_2(t_0) \in \ker P(t_0)$.

Since $\ker A(t) = N$ does not depend on t, we have $P'Q = 0$ (see [5]), therefore for a function u being a solution of (7) we have

$$(Pu)' = Pu' + P'u = (P' - PA_1^{-1}B_0)Pu + P'Qu = (P' - PA_1^{-1}B_0)(Pu),$$

and

$$(Qu)' = -P'Qu = 0.$$

Consequently, if $v_1(t)$ is the solution of the initial value problem (IVP)

$$\begin{cases} u' &= (P' - PA_1^{-1}B_0)u, \\ u(t_0) &= v_1(t_0), \end{cases}$$

then $v_1(t) \in \operatorname{im} P(t)$ for all $t \in \mathbb{R}^+$, and if $v_2(t)$ is the solution of the IVP

$$\begin{cases} u' &= (P' - PA_1^{-1}B_0)u, \\ u(t_0) &= v_2(t_0), \end{cases}$$

then $v_2(t) \equiv v_2(t_0)$, i.e. $v_2(t)$ is a nonzero constant.

Due to the linearity of the equation (7) we have

$$u_k(t) = v_1(t) + v_2(t) \quad \text{for all} \quad t \in \mathbb{R}^+.$$

Moreover, for all $t \in \mathbb{R}^+$ we have

$$v_1(t) = P(t)u(t) \quad \text{and} \quad v_2(t) = Q(t)u(t).$$

Since P is bounded, from this it follows that

$$\lambda_1(v_1) \le \lambda(P) + \lambda(u) \le \lambda(u),$$
$$\lambda(v_2) \le \lambda(Q) + \lambda(u) \le \lambda(u).$$

There are three possible cases.

Case 1: $\lambda(v_1) < \lambda(v_2) = 0$, hence $\lambda(u_k) = 0$. In this case v_1 may be represented as a linear combination of solutions $u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_m$ because otherwise the set U_1 of solutions $\{u_1(t), \dots, u_{k-1}(t), v_1(t), u_{k+1}(t), \dots, u_m(t)\}$ of (7) is a FSM of (7) which satisfies $\sigma_{U_1} < \sigma_U$ contradicting the assumption that U is normal. Therefore v_2 is linearly independent of the vectors $u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_m$ and we may replace u_k by v_2 in U without changing the sum of Lyapunov exponents σ_U. Thus, we obtain a new system, which is a normal FSM of (7) and has k-th vector $v_2 \in \ker P(t)$.

Case 2: $\lambda(v_1) > \lambda(v_2) = 0$, hence $\lambda(u_k) = \lambda(v_1)$. In this case v_2 may be represented as a linear combination of $u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_m$. Hence v_1 is linearly independent of $u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_m$ and we may replace u_k by v_1 in U to receive a new normal FSM, which has k-th vector $v_1 \in \operatorname{im} P(t)$.

Case 3: $\lambda(v_1) = 0 = \lambda(v_2)$, hence $\lambda(u_k) = 0$. In this case we may replace u_k either by v_1 or by v_2 provided v_1 or v_2 is linearly independent of $u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_m$ (note that at least one of v_1, v_2 must be linearly independent of $u_1, \dots, u_{k-1}$, $u_{k+1}, \dots, u_m$).

Thus, in any case, we can always change from a normal FSM $U(t)$ which has k-th column-vector belonging to neither $\ker P(t)$ nor $\operatorname{im} P(t)$ to a new normal FSM $\widetilde{U}(t)$ which has k-th column-vector belonging to either $\ker P(t)$ or $\operatorname{im} P(t)$ and has the same other column-vectors as $U(t)$.

Repeating this procedure at most m times we find a normal FSM of (7) which satisfies the conclusion of the lemma. □

Lemma 4.2. *Assume that* (1) *is a DAE of index 1, the coefficient matrices* A, B *are bounded on* $\mathbb{R}^+$*, the nullspace of* A *is independent on* $t \in \mathbb{R}^+$ *and the canonical projector* P_s *onto* $S(t)$ *along* $N = \ker A(t)$ *is bounded. If* P_1, P_2 *are* C^1*-smooth bounded projectors along* N*. Then the Lyapunov spectra of the corresponding ODEs of* (1) *under projectors* P_1 *and* P_2 *coincide.*

Proof. The corresponding ODE of (1) under P_1 is

$$u' = (P_1' - P_1 A_1^{-1} B_0)u, \tag{12}$$

where $A_1 = A + B_0 Q_1$, $Q_1 = I - P_1$, $B_0 = B - AP_1'$. The corresponding ODE of (1) under P_2 is

$$u' = (P_2' - P_2 \overline{A}_1^{-1} \overline{B}_0)u, \tag{13}$$

where $\overline{A}_1 = A + \overline{B}_0 Q_2$, $Q_2 = I - P_2, \overline{B}_0 = B - AP_2'$.

By Lemma 4.1 we can find a normal FSM

$$U = \big[u_1(t), \dots, u_r(t), u_{r+1}(t), \dots, u_m(t)\big]$$

of (12) such that $u_i(t) \in \operatorname{im} P_1(t)$, $i = 1, 2, \dots, r$, and $u_j(t) \in \ker P_1(t)$, $j = r+1, \dots, m$, for all $t \in \mathbb{R}^+$.

Put

$$x_i(t) := P_s(t) u_i(t), \quad i = 1, \dots, r,$$

and

$$\overline{u}_i(t) := P_2(t) x_i(t), \quad i = 1, \dots, r.$$

Then $x_i(t)$ are solutions of (1), $\overline{u}_i(t)$ are solutions of (13) and $\overline{u}_i(t) \in \operatorname{im} P_2(t)$, $i = 1, \dots, r$, for all $t \in \mathbb{R}^+$.

It is known that $\lambda(u_i) = \lambda(x_i) = \lambda(\overline{u}_i)$, $i = 1, \dots, r$. Since there is a one-to-one correspondence between u_i and x_i, as well as between x_i and $\overline{u}_i$, it is easily seen that $\overline{u}_1, \dots, \overline{u}_r$ are independent and they span $\operatorname{im} P_2(t)$. Take $m-r$ linearly independent solutions $\overline{u}_{r+1}, \dots, \overline{u}_m$ of (13), which belong to $\ker P_2$, and compose a FSM $\overline{U} = [\overline{u}_1, \dots, \overline{u}_m]$ of (13). We see that the sum of Lyapunov exponents of solutions of (13) from $\overline{U}(t)$ equals the sum of Lyapunov exponents of solutions of (12) from $U(t)$, because $\lambda(u_i) = \lambda(\overline{u}_i)$ for all $i = 1, \dots, r$ and $\lambda(u_j) = \lambda(\overline{u}_j) = 0$ for all $j = r+1, \dots, m$.

Since U is a normal FSM of (12) the above argument implies that the sum σ_1 of the Lyapunov exponents of a normal FSM of (12) is greater than or equal to the sum σ_2 of the Lyapunov exponents of a normal FSM of (13). Changing the role of (12) and (13) we find that $\sigma_2 \geq \sigma_1$. Hence $\sigma_1 = \sigma_2$. Moreover, the FSM $\overline{U}$ of (13) constructed above must be normal. Consequently, the Lyapunov spectra of (12) and (13) coincide. □

Lemma 4.2 states that the Lyapunov spectrum of the corresponding ODE of (1) under a projector does not depend on the choice of the projector from a certain class. Our next theorem will show that the Lyapunov regularity of the corresponding ODE does neither depend on the choice of the projector.

Theorem 4.1. *Suppose the conditions of Lemma 4.2 are satisfied. Then the corresponding ODEs* (12) *and* (13) *of* (1) *under* P_1 *and* P_2 *are either both Lyapunov regular or both nonregular.*

Proof. Let $U(t) = \big[u_1(t), \dots, u_r(t), u_{r+1}(t), \dots, u_m(t)\big]$ be a normal FSM of (12) such that $u_i(t) \in \operatorname{im} P_1(t)$, $i = 1, \dots, r$, $u_j \in \ker P_1$ $j = r+1, \dots, m$. Such a FSM exists by Lemma 4.1. Moreover, u_j are non-zero constants, $j = r+1, \dots, m$.

Put

$$
\begin{aligned}
x_i(t) &:= P_s(t) u_i(t), \quad i = 1, \dots, r, \\
\overline{u}_i(t) &:= P_2(t) x_i(t), \quad i = 1, \dots, r,
\end{aligned}
$$

where $P_s(t)$ denotes the canonical projector of (1). In the proof of Lemma 4.2, it was shown that

$$\lambda(u_i) = \lambda(x_i) = \lambda(\overline{u}_i), \quad i = 1, \dots, r.$$

Moreover, if we take a set of $m - r$ non-zero vectors

$$\overline{u}_{r+1}, \dots, \overline{u}_m \in \ker P_2(t) = N = \ker A(t),$$

which are linearly independent, then

$$\overline{U}(t) = \big[\overline{u}_1(t), \dots, \overline{u}_r(t), \overline{u}_{r+1}, \dots, \overline{u}_m\big]$$

is a normal FSM of (13). Furthermore,

$$\sigma_U = \sigma_{\overline{U}}.$$

Since $P_1(\cdot)$ is bounded, the angle between $\ker P_1(t)$ and $\operatorname{im} P_1(t)$ is separated away from 0 by a constant independent of $t \in \mathbb{R}^+$. Similarly for $\ker P_2(t)$ and $\operatorname{im} P_2(t)$.

Furthermore, the linear operators

$$
\begin{aligned}
P_2(t) P_s(t) : \operatorname{im} P_1(t) &\rightarrow \operatorname{im} P_2(t), \\
P_1(t) P_s(t) : \operatorname{im} P_2(t) &\rightarrow \operatorname{im} P_1(t)
\end{aligned}
$$

are bounded and are inverses of each other.

Therefore there exists a constant $K > 1$ independent of t such that

$$\frac{1}{K} |\det U(t)| \le |\det \overline{U}(t)| \le K |\det U(t)| \tag{14}$$

for all $t \in \mathbb{R}^+$.

Using Liouville's formula for ODEs (12) and (13) we see that due to (14) the Lyapunov equality for (12) and (13) can happen only simultaneously. The theorem is proved. □

Theorem 4.2. *Let* (1) *be a linear DAE of index 1. Assume that the matrices A, B, A_1^{-1} are bounded on $\mathbb{R}^+$ and the nullspace of $A(t)$ is independent of $t \in \mathbb{R}^+$. Then the DAE* (1) *is Lyapunov regular if and only if the corresponding ODE* (7) *of* (1) *under a C^1-smooth bounded projector P, is Lyapunov regular.*

Proof. By Theorem 4.1 we may choose Q to be the orthogonal projector onto $N = \ker A(t)$ and $P = I - Q$, hence P and Q are independent of t and $P' = Q' = 0$ because $N = \ker A(t)$ is independent of $t \in \mathbb{R}^+$.

Let (1) be Lyapunov regular. We show that the ODE (7) is regular. Since $P' = 0$, the adjoint equation of (7) is

$$v' = B_0^* A_1^{*-1} P^* v. \tag{15}$$

Denote by $v(t) \in C^1(\mathbb{R}^+, \mathbb{R}^m)$ the unique solution of the IVP

$$\begin{cases} v' &= B_0^* A_1^{*-1} P^* v, \\ v(0) &= v_0, \quad v_0 \in \mathbb{R}^m, \end{cases}$$

then $P^* v(t) \in C^1(\mathbb{R}^+, \mathbb{R}^m)$ is the solution of the IVP

$$\begin{cases} (P^* v)' &= P^* B_0^* A_1^{*-1} (P^* v), \\ P^* v(0) &= P^* v_0, \quad v_0 \in \mathbb{R}^m. \end{cases} \tag{16}$$

Take a normal FSM $\Phi = [\varphi_1(t), \ldots, \varphi_r(t)]$ of the DAE (8) such that $\lambda(\varphi_1) \geq \lambda(\varphi_2) \geq \cdots \geq \lambda(\varphi_r)$. Then φ_i are solutions of the IVP

$$\begin{cases} (A^* \varphi_i)' - B^* \varphi_i = 0, \\ A^*(0)(\varphi_i(0) - \varphi_i^0) = 0, \end{cases}$$

and by Theorem 2.1 they can be represented by solutions of the IVPs

$$\begin{cases} v_i' &= B_0^* A_1^{*-1} P^* v_i, \\ v_i(0) &= A_1^*(0) \varphi_i^0 \end{cases}$$

by the formula

$$\varphi_i = P_{*s} A^{+*} P^* v_i.$$

Since $P_{*s} = A_1^{*-1} A^*$, we get

$$\varphi_i = A_1^{*-1} A^* A^{+*} P^* v_i = A_1^{*-1} P^* v_i.$$

Therefore,

$$A_1^* \varphi_i = P^* v_i, \quad i = 1, 2, \ldots, r.$$

Since $P' = 0$ we have $B_0 = B$, $A_1 = A + BQ$ both bounded, hence for all $i = 1, \ldots, r$

$$\lambda(P^* v_i) = \lambda(A_1^* \varphi_i) \leq \lambda(A_1^*) + \lambda(\varphi_i) \leq \lambda(\varphi_i),$$

and, since A_1^{-1} is bounded by assumption,

$$\lambda(\varphi_i) = \lambda(A_1^{*-1} P^* v_i) \leq \lambda(A_1^{*-1}) + \lambda(P^* v_i) \leq \lambda(P^* v_i).$$

Thus

$$\lambda(\varphi_i) = \lambda(P^* v_i), \quad i = 1, 2, \dots, r.$$

Now we show that $\operatorname{im} P^*$ and $\ker P^*$ are invariant with respect to (15). Suppose that $v(t)$ is a solution of (15) with $v(0) = v_0 \in \operatorname{im} P^*$.

Since $Q^{*'} = 0$ we have

$$(Q^* v)' = Q^{*'} v + Q^* v' = Q^* v' = Q^* B_0^* A_1^{*-1} P^* v = 0$$

because

$$\begin{aligned} Q^* B_0^* A_1^{*-1} P^* &= (P A_1^{-1} B_0 Q)^* = (P A_1^{-1}(A_1 - A))^* \\ &= (P A_1^{-1} A_1 - P A_1^{-1} A)^* = (P - P)^* = 0. \end{aligned}$$

Since $Q^* v(0) = Q^* v_0 = 0$ we have $Q^* v(t) = 0$ for all $t \in \mathbb{R}^+$. Therefore,

$$v(t) = (P^* + Q^*) v(t) = P^* v(t) \in \operatorname{im} P^* \quad \text{for all} \quad t \in \mathbb{R}^+.$$

Thus $\operatorname{im} P^*$ is invariant with respect to (15).

Now suppose that $v(0) = v_0 \in \ker P^*$. By (16) and $P^* v(0) = P(v_0) = 0$ we have $P^* v(t) = 0$ for all $t \in \mathbb{R}^+$. Thus $\ker P^*(t)$ is invariant with respect to (15).

Moreover, if $v(0) \in \ker P^*$ then $v(t) = v(0) =$ constant for all $t \in \mathbb{R}^+$, hence $\lambda(v) = 0$.

By Lemma 4.1, we can find a normal FSM of (7)

$$U(t) = \left[u_1(t), \dots, u_r(t), u_{r+1}, \dots, u_m\right]$$

such that

$$\begin{aligned} \lambda(u_1) \le \cdots \le \lambda(u_r), \ \ u_1, \dots, u_r \in \operatorname{im} P, \\ \lambda(u_{r+1}) = \cdots = \lambda(u_m) = 0, \ \ u_{r+1}, \dots, u_m \in \ker P. \end{aligned}$$

Put

$$x_i(t) := P_s u_i(t), \quad i = 1, \dots, r,$$

where $P_s(t)$ is the canonical projector of (1). Then $X(t) := \left[x_1(t), \dots, x_r(t)\right]$ is a normal FSM of (1) and

$$\lambda(x_i) = \lambda(u_i), \quad i = 1, \dots, r.$$

Since (1) is regular we can find a normal FSM

$$\Phi(t) = \left[\varphi_1(t), \dots, \varphi_r(t)\right]$$

of the adjoint DAE (8) of (1) such that

$$\lambda(\varphi_1) \ge \cdots \ge \lambda(\varphi_r)$$

and

$$\lambda(\varphi_i) + \lambda(x_i) = 0 \quad \text{for all} \quad i = 1, \dots, r. \tag{17}$$

It is easily seen that the solutions $\varphi_i(t)$, $i = 1, \ldots, r$, correspond to the solutions $v_i(t)$ of (15) with $v_i(t) \in \operatorname{im} P^*(t)$ and $v_i(t)$ are linearly independent. Extend this set of solutions of (15) to a FSM

$$V(t) = \big[v_1(t), \ldots, v_r(t),\ \ v_{r+1}, \ldots, v_m\big]$$

by adding $m - r$ basis vector $v_{r+1}, \ldots, v_m$ of $\ker P^*$. As we have shown above, at the beginning of the proof,

$$\lambda(v_i) = \lambda(P^* v_i) = \lambda(\varphi_i), \quad i = 1, \ldots, r.$$

Therefore, by (17),

$$\lambda(u_i) + \lambda(v_i) = 0 \quad \text{for all} \quad i = 1, \ldots, m. \tag{18}$$

Since (15) is the adjoint equation of the ODE (7) and U is a normal FSM of (7), by the Theorem of Perron the relation (18) implies that (7) is Lyapunov regular.

Conversely, suppose that (7) is regular. Choose normal FSMs $U = [u_1, \ldots, u_r, u_{r+1}, \ldots, u_m]$ of (7) and $V = [v_1, \ldots, v_r,\ v_{r+1}, \ldots, v_m]$ of (15) such that

$$\begin{aligned}
&u_i(t) \in \operatorname{im} P,\ \ v_i(t) \in \operatorname{im} P^*, \quad i = 1, \ldots, r,\\
&u_j(t) \in \ker P,\ \ v_j(t) \in \ker P^*, \quad j = r+1, \ldots, m,\\
&\lambda(u_1) \le \ldots \le \lambda(u_r), \quad \lambda(v_1) \ge \ldots \ge \lambda(v_r),\\
&\lambda(u_k) + \lambda(v_k) = 0, \quad \text{for all} \quad k = 1, \ldots, m.
\end{aligned}$$

By U and V we find corresponding normal FSMs $X = [x_1, \ldots, x_r]$ of DAE (1) and $\Phi = [\varphi_1, \ldots, \varphi_r]$ of the adjoint equation (8) of (1), which satisfy the relations

$$\lambda(x_i) + \lambda(\varphi_i) = 0, \quad i = 1, \ldots, r,$$

hence (1) is Lyapunov regular. □

Theorem 4.2 gives us a characterization of Lyapunov regularity of DAE (1) via its corresponding ODE. However, the assumption of constant null space $\ker A(t)$ of (1) is restrictive. This assumption was used essentially in the proof of the theorem where in order to apply Lemma 4.1 we need the invariance with respect to the ODE (7) of the function space $\ker A(t) = \ker P(t)$. Now, to apply our result to a larger class of DAEs we reduce the general case of variable null space to the case of Theorem 4.2 by means of a change of variables.

Let us make a change of variable $x = F\bar{x}$ in the DAE (1), where $F : \mathbb{R}^+ \to L(\mathbb{R}^m, \mathbb{R}^m)$ is a nonsingular bounded differentiable $m \times m$-matrix function with bounded inverse and bounded derivative. The DAE (1) is transformed into the following equivalent DAE (see [10])

$$\bar{A}(t)\bar{x}' + \bar{B}(t)\bar{x} = 0, \tag{1'}$$

where $\bar{A}(t) := A(t)F(t)$, $\bar{B}(t) := B(t)F(t) + A(t)F'(t)$. It is easily seen that (1') is again a DAE of index 1 with bounded continuous coefficients, hence the theory of Lyapunov spectrum and Lyapunov regularity is applicable to (1'). We show that this kind of transformation does not change Lyapunov regularity of the DAE (1).

Theorem 4.3. *Let* (1) *be a linear DAE of index 1. Assume that the matrices* A*,* B*,* A_1^{-1} *are bounded on* $\mathbb{R}^+$ *and* F *is a nonsingular bounded differentiable* $m \times m$*-matrix function with bounded inverse and bounded derivative. Then the DAE* (1) *is Lyapunov regular if and only if the transformed DAE* (1') *of* (1) *by* F *is Lyapunov regular.*

Proof. Recall that the adjoint equation of (1) is

$$(A^*\varphi)' - B^*\varphi = 0 \tag{8}$$

and the adjoint equation of (1') is

$$(\bar{A}^*\psi)' - \bar{B}^*\psi = 0. \tag{8'}$$

Suppose φ is a solution of (8). Multiplying F^* from left in both sides of (8) we have

$$\begin{aligned} 0 &= F^*(A^*\varphi)' - F^*B^*\varphi \\ &= (F^*A^*\varphi)' - F^{*\prime}A^*\varphi - F^*B^*\varphi \\ &= ((AF)^*\varphi)' - (F^*B^* + F^{*\prime}A^*)\varphi \\ &= (\bar{A}^*\varphi)' - \bar{B}^*\varphi. \end{aligned}$$

Thus, φ is a solution of (8'). Conversely, if ψ is a solution of (8') then it is a solution of (8). Therefore the solution space of (8) and (8') coincide, hence their Lyapunov spectra coincide. Since (1') is the transformed equation of (1) by F and since F and F^{-1} are both bounded it is easily seen that the Lyapunov spectra of (1) and (1') coincide. Consequently, by Definition 3.2 the DAE (1) is Lyapunov regular if and only if so is (1'). □

To conclude this section we remark that Theorems 4.2 and 4.3 give us a criterion for Lyapunov regularity of index 1 DAEs via corresponding ODEs. This criterion is applicable to a fairly large class of DAEs with variable null space which can be reduced to DAEs with constant null space by bounded differentiable nonsingular transformation. It is well known that if we use a larger class of transformation then we can always reduce a DAE of index 1 into its Kronecker normal form which obviously has constant null space (see [10]). However, the class of transformations we use in Theorem 4.3 is smaller than that needed in [10] to reduce a general DAE to its Kronecker normal form. This restriction is needed because we are able to apply the theory of Lyapunov spectrum and Lyapunov regularity only to the DAEs and ODEs with *bounded coefficients*. One can easily see that if we do not require the boundedness assumption on F as in Theorem 4.3 then starting from a simple regular DAE (say, in Kronecker normal form and with constant coefficients) we may transform it into a DAE with unbounded coefficients, hence the above theory is not applicable.

5. Some properties of regular DAEs

Recall the linear DAE (1) of index 1 from Section 2

$$A(t)x' + B(t)x = 0,$$

where $A, B \in C(\mathbb{R}^+, L(\mathbb{R}^+, \mathbb{R}^m))$ are bounded on $\mathbb{R}^+$, $\dim \operatorname{im} A(t) = r < m$, and its adjoint equation

$$(A^*\varphi)' - B^*\varphi = 0. \tag{8}$$

Suppose that X, Φ are arbitrary FSMs of (1) and (8), respectively. By Theorem 2.2

$$(\Phi^* AX)' = 0,$$

hence $\Phi^* AX = C$, where C is a constant matrix. If x, φ are arbitrary (column-vector) solutions of (1) and (8), respectively, then we may include them into some FSMs of (1) and (8), hence we have

$$\varphi^* Ax = \text{const.} \tag{19}$$

(this identity can be proved directly, see [2]). Note that the solution space of (1) and (8) are $\operatorname{im} P_s(t)$ and $\operatorname{im} P_{*s}(t)$ which are of dimension r. Due to the theorem on existence and uniqueness of solutions of DAEs (see [1,8]) each solution $x(t)$ of (1) or $\varphi(t)$ of (8) is determined by its initial value $x(0) \in \operatorname{im} P_s(0)$ or $\varphi(0) \in \operatorname{im} P_{*s}(0)$. Consequently, since FSMs of (1) and (8) are composed by solutions they are determined by their initial values as well. By (19) and the linearity of (1) and (8) the abstract theory of exponents presented in Bylov et al. [3, §2.6 and §3.2] is applicable to the Lyapunov exponents of (1) and (8). Now we introduce some notions which are versions of those from the theory of ODEs (see [3] and [6]) adapted to the DAEs.

Definition 5.1. Two FSMs $X = [x_1, \dots, x_r, 0, \dots, 0]$ and $\Phi = [\varphi_1, \dots, \varphi_r, 0, \dots, 0]$ of (1) and (8) are called *conjugate* if

$$\varphi_i^* Ax_i = 1 \quad \text{for all} \quad i = 1, \dots, r. \tag{20}$$

Note that pairs of conjugate FSMs of (1) and (8) exist because by Corollary 3 of [2] for each pair X, Φ of maximal FSMs of (1) and (8) normalized at 0 we have $\Phi^* AX = A(0)$, hence $\operatorname{rank}(\Phi^* AX) = r$. From (20) it follows that if X and Φ are conjugate then, since A is bounded, for any $i = 1, \dots, r$ we have

$$\lambda(x_i) + \lambda(\varphi_i) \geq 0, \quad i = 1, \dots, r.$$

Definition 5.2. The number

$$\gamma'(X, \Phi) := \max_{1 \leq i \leq r} \{\lambda(x_i) + \lambda(\varphi_i)\},$$

is called the *defect* of the conjugate pair X, Φ.

Definition 5.3. The number

$$\gamma = \min \gamma'(X, \Phi),$$

where minimum is taken with respect to all conjugate pairs X, Φ of FSMs of (1) and (8), is called the *coefficient of irregularity* of (1) and (8).

Definition 5.4. The number

$$\pi := \max\{\lambda_i + \beta_i\}, \quad i = 1, 2, \dots, r,$$

where $\lambda_1 \leq \cdots \leq \lambda_r$ is the Lyapunov spectrum of (1) and $\beta_1 \geq \cdots \geq \beta_r$ is the Lyapunov spectrum of (8) ordered decreasingly, is called the *Perron coefficient* of (1) and (8).

For coefficients γ, γ' and π, similarly to the case of ODEs we have the following relations (cf. Bylov et al. [3], Lemma 3.2.5, p. 68).

Lemma 5.1. *The following inequalities hold*

(i) $0 \leq \pi \leq \gamma \leq \gamma'$,

(ii) $0 \leq \pi \leq \gamma \leq r\pi$.

From Lemma 5.1 and Definition 3.2 it follows immediately

Theorem 5.1. *The DAE* (1) *is Lyapunov regular if and only if its coefficient of irregularity and Perron coefficient vanish* ($\pi = \gamma = 0$).

Note that in Theorem 5.1 we do not assume that $\ker A(t)$ is constant.

Theorem 5.2. *Suppose that the assumptions of Theorem 4.2 are satisfied. If the linear DAE* (1) *is Lyapunov regular, then any solution of* (1) *has exact Lyapunov exponent.*

Proof. By Theorem 4.2, since (1) is regular the ODE (7) is regular, hence any solution of (7) has exact Lyapunov exponent (see [3], Theorem 22.1.1, p. 284). Let $x(t)$ be an arbitrary solution of (1). Then $u(t) := P(t)x(t)$ is a solution of (7) and $x(t) = P_s(t)u(t)$. From the assumptions of the theorem the projectors P, P_s are bounded, hence there exists a positive constant $c > 1$ such that

$$1 \leq \|P\| \leq c, \quad 1 \leq \|P_s\| \leq c.$$

Then we have for all $t \in \mathbb{R}^+$ the inequalities

$$\|u(t)\| \leq \|P(t)\| \cdot \|x(t)\| \leq c\|x(t)\|$$

and

$$\|x(t)\| \leq \|P_s(t)\| \cdot \|u(t)\| \leq c\|u(t)\|.$$

Therefore,

$$c^{-1}\|u(t)\| \leq \|x(t)\| \leq c\|u(t)\|,$$

hence x has exact Lyapunov exponent because u has exact Lyapunov exponent. The theorem is proved. □

Note that using Theorem 4.3 we can have a result for a larger class of DAEs with variable null space. Namely, suppose that the DAE (1) can be transformed into a DAE satisfying the assumptions of Theorem 4.2 by a transformation F which is a nonsingular differentiable $m \times m$-matrix function and is bounded together with F^{-1} and F'; if (1) is Lyapunov regular, then any solution of (1) has exact Lyapunov exponent.

Theorem 5.3. *Suppose that the assumptions of Theorem 4.2 are satisfied. If* (1) *is Lyapunov regular, then the following Lyapunov equality holds*

$$\sigma_X = \sum_{i=1}^{r} \lambda(x_i) = \lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \operatorname{trace}(P' - PA_1^{-1}B_0)(t_1)dt_1,$$

where $X = [x_1, \dots, x_r]$ *is a normal FSM of* (1).

Proof. It (1) is Lyapunov regular then, by Theorem 4.2, its corresponding ODE (7)

$$u' = (P' - PA_1^{-1}B_0)u$$

is also Lyapunov regular. Therefore, we have (see Demidovich [6, p. 166])

$$\sigma_U = \sum_{i=1}^{m} \lambda(u_i) = \lim \frac{1}{t} \int_{t_0}^{t} \operatorname{trace}(P' - PA_1^{-1}B_0)(t_1)dt_1,$$

where $U = [u_1, \dots, u_m]$ is a normal FSM of (7). On the other hand, since the nullspace N of $A(t)$ does not depend on t and the matrices A, B, A_1^{-1} are bounded (cf. proof of Theorem 4.2) we have

$$\sigma_U = \sigma_X.$$

Therefore,

$$\sigma_X = \sum_{i=1}^{r} \lambda(x_i) = \lim \frac{1}{t} \int_{t_0}^{t} \operatorname{trace}(P' - PA_1^{-1}B_0)(t_1)dt_1.$$

□

Theorem 5.4. *Suppose that the assumptions of Theorem 4.2 are satisfied. The linear DAE* (1) *is Lyapunov regular if and only if*

(i) *there exists limit*

$$S = \lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \operatorname{trace}(P' - PA_1^{-1}B_0)(t_1)dt_1$$

(ii) *and the Lyapunov equality*

$$\sigma_X = \sum_{i=1}^{r} \lambda(x_i) = S,$$

where $X = [x_1, \dots, x_r]$ *is a normal FSM of* (1)*, holds.*

Proof. Suppose that (1) is Lyapunov regular. By Theorem 4.2, the corresponding ODE (7) of (1) is Lyapunov regular, hence the exact limit

$$S = \lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \operatorname{trace}(P' - PA_1^{-1}B_0)(t_1)dt_1$$

exists and $\sigma_U = \sum_{i=1}^{m} \lambda(u_i) = S$, where $U = [u_1, \dots, u_m]$ is a normal FSM of (7).

On the other hand, from Lemmas 4.1, 4.2 and Theorem 4.1, it follows that $\sigma_U = \sigma_X$, hence $\sigma_X = \sigma_U = S$.

Conversely, suppose that there exists exact limit

$$S = \lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} (P' - PA_1^{-1}B_0)(t_1)dt_1$$

and

$$\sigma_X = \sum_{i=1}^{r} \lambda(x_i) = S.$$

Since $N = \ker A(t)$ does not depend on t and A, B, A_1^{-1} are bounded, we have $\sigma_X = \sigma_U$, where U is a normal FSM of (7), hence $\sigma_U = S$.

Therefore, the corresponding ODE (7) of (1) is Lyapunov regular, which implies that (1) is regular. □

Now we consider a DAE

$$A(t)x' + B(t)x = 0, \tag{21}$$

where

$$A(t) = \begin{pmatrix} W(t) & 0 \\ 0 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix},$$

the matrices W, B_{22} are invertible and are of the order r and $m - r$, and W, W^{-1}, B, B_{22}^{-1} are continuous and bounded on $\mathbb{R}^+$.

Theorem 5.5. *The DAE* (21) *is Lyapunov regular if and only if the ODE*

$$u_1' = W^{-1}(B_{12}B_{22}^{-1}B_{21} - B_{11})u_1 \tag{22}$$

is Lyapunov regular.

Proof. Obviously, $N = \ker\ A(t) = \mathbb{R}^{m-r}$ does not depend on t. We consider a projector $Q = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$ onto N, $P = I - Q = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. Then the corresponding (under P) ODE of (21) is

$$u' = \begin{pmatrix} W^{-1}(B_{12}B_{22}^{-1}B_{21} - B_{11}) & 0 \\ 0 & 0 \end{pmatrix} u \tag{23}$$

or, equivalently,

$$\begin{cases} u_1' &= W^{-1}(B_{12}B_{22}^{-1}B_{21} - B_{11})u_1, \\ u_2' &= 0, \end{cases}$$

here

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1 \in \mathbb{R}^r, \quad u_2 \in \mathbb{R}^{m-r}.$$

Clearly, the regularity of (22) and (23) is equivalent. By Theorem 4.2, the Lyapunov regularity of (21) and (23) is equivalent. Hence the Lyapunov regularity of (21) and (22) is equivalent. □

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