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UPPER AND LOWER ESTIMATES FOR A FRÉCHET NORMAL CONE

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ABSTRACT. This paper investigates upper and lower estimates for the Fréchet normal cone to the graph of a normal cone mapping related to a linear inequality system in a reflexive Banach space under right-hand side perturbations. Our results develop some material in the recent papers of Yao and Yen [8], and Nam [6]. In particular, by constructing suitable counterexamples, we solve the two open questions of [8] in the negative.

1. Introduction

Let X be a real Banach space with the dual denoted by X^*. Consider a set of indices $T = \{1, 2, \ldots, m\}$, a vector system $\{a_i^* \in X^* \mid i \in T\}$, and a polyhedral convex set

$$\Theta(b) = \Big\{x \in X \mid \langle a_i^*, x\rangle \leq \beta_i \text{ for all } i \in T\Big\} \tag{1.1}$$

depending on the parameter $b = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m$. The numbers $\beta_1, \ldots, \beta_m \in \mathbb{R}$ are interpreted as *right-hand side perturbations* of the linear inequality system

$$\langle a_i^*, x\rangle \leq \beta_i, \quad i \in T. \tag{1.2}$$

For a pair $(x, b) \in X \times \mathbb{R}^m$, we call

$$I(x, b) = \big\{i \in T \mid \langle a_i^*, x\rangle = \beta_i\big\} \tag{1.3}$$

the *active index set* of $\Theta(b)$ at x. For any $I \subset T$, let $\overline{I} = T\backslash I$. The symbol b_I (resp., $b_{\overline{I}}$) denotes the vector with the components β_i where $i \in I$ (resp., $i \in \overline{I}$).

Our aim is to obtain upper and lower estimates for the *Fréchet normal cone* [4, p. 4] (the definition will be recalled in the next section) to the graph of the multifunction $\mathcal{F} : X \times \mathbb{R}^m \rightrightarrows X^*$ with

$$\mathcal{F}(x, b) := N(x; \Theta(b)) \quad \forall (x, b) \in X \times \mathbb{R}^m, \tag{1.4}$$

where

$$N(x; \Theta(b)) = \Big\{x^* \in X^* \mid \langle x^*, u - x\rangle \leq 0 \text{ for all } u \in \Theta(b)\Big\}$$

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denotes the *normal cone* to $\Theta(b)$ at x in the sense of convex analysis. Such estimates of the Fréchet normal cone

$$\widehat{N}\big((\overline{x},\overline{b},\overline{x}^*);\mathrm{gph}\mathcal{F}\big), \tag{1.5}$$

where $(\overline{x},\overline{b},\overline{x}^*)$ belongs to the graph

$$\mathrm{gph}\mathcal{F} := \Big\{(x,b,x^*)\in X\times\mathbb{R}^m\times X^* \mid x^*\in\mathcal{F}(x,b)\Big\},$$

yield upper and lower estimates of the values of the *Fréchet coderivative* [4, p. 41] $\widehat{D}\mathcal{F}(\overline{x},\overline{b},\overline{x}^*)(\cdot)$ of the multifunction $\mathcal{F}$. If both X and X^* are Asplund spaces (this happens, for instance, when X is a reflexive Banach space or a separable Banach space) and the inequality system (1.2) satisfies the Slater condition, then each value of the *normal coderivative* (also called the *limiting coderivative* or the *Mordukhovich coderivative*) $D\mathcal{F}(\overline{x},\overline{b},\overline{x}^*)(\cdot)$ is defined as a Painlevé-Kuratowski limit of the values of a family of Fréchet coderivatives of $\mathcal{F}$ (see [4, Corollary 2.36]). Since the normal coderivative has an important role [4, 5, 7] in characterizing the local behavior of the multifunction under consideration, the investigation on the Fréchet normal cone $\widehat{N}\big((x,b,x^*);\mathrm{gph}\mathcal{F}\big)$ can be considered as a major step towards a comprehensive differentiation of the map $(x,b)\mapsto\mathcal{F}(x,b)$.

Extending Dontchev and Rockafellar's results [2], where b is fixed, to the case where b is changing, Yao and Yen [8] have found an *upper estimate* for the Fréchet normal cone (1.5) in the case X is a finite dimensional Euclidean space and b is a moving vector. In [9], the results of [8] are applied to the stability analysis of parametric variational inequalities whose constraint sets are perturbed polyhedra. Some arguments of [8] in employing active index sets for calculating Fréchet and limiting normal cones have been used in [3] for the second-order analysis of polyhedral systems in finite and infinite dimensional Banach spaces.

Recently, Nam [6] has studied the Lipschitzian stability of parametric variational inequalities on reflexive Banach spaces by invoking the Farkas lemma in [1], the generalized differentiation theory in [4, 5], and a technique in [8].

Concerning the above-mentioned upper estimate for (1.5) given in [8], two open questions were stated in the same paper. Imposing an additional assumption, Nam [6] has solved the first question in the affirmative. Namely, he proved that if the vectors a_i^*, $i\in I(\bar{x},\bar{b})$, are linearly independent, then the upper estimate holds as an equality; i.e. the Fréchet normal cone (1.5) can be computed exactly by an explicit formula. It is of interest to find out whether Nam's linear independence assumption is essential for that exact computation of the Fréchet normal cone. We will show that even if the vectors a_i^*, $i\in I(\bar{x},\bar{b})$, are *positively linearly independent* (see Section 3) and the *Slater condition* is satisfied for (1.2), the desired formula may not hold.

Since exact formulae for computing the cone in (1.5) are not available, one may wish to have some *lower estimate* for the Fréchet normal cone. Interestingly, Lemma 4.2 in [8] and its proof can lead us to such a lower estimate. This result is given in Section 4, where we also solve the second question of [8] in the negative by constructing a counterexample.

The rest of this paper has three sections. Section 2 recalls some basic definitions and notations from [4]. Section 3 discusses an upper estimate for the normal cone (1.5) and gives a complete solution to Question 1 from [8, p. 169]. A lower estimate for the cone (1.5) and our solution for Question 2 from [8, p. 169] are presented in Section 4.

2. Basic definitions and preliminaries

For a multifunction $\Psi : X \rightrightarrows X^*$, the expression $\operatorname*{Lim\,sup}_{x\to\overline{x}} \Psi(x)$ denotes the *sequential Kuratowski-Painlevé upper limit* with respect to the norm topology of X and the weak* topology of X^*, i.e.,

$$\operatorname*{Lim\,sup}_{x\to\overline{x}} \Psi(x) = \Big\{x^* \in X^* \mid \quad \exists \text{ sequences } x_k \to \overline{x},\ x_k^* \xrightarrow{w^*} x^*, \text{ with } x_k^* \in \Psi(x_k) \text{ for all } k = 1, 2, \dots\Big\}.$$

Normal cones to sets and coderivatives of multifunctions are defined [4] as follows. The set

$$\widehat{N}_\varepsilon(\overline{x}; \Omega) := \Big\{x^* \in X^* \mid \limsup_{x \xrightarrow{\Omega} \overline{x}} \frac{\langle x^*, x - \overline{x}\rangle}{||x - \overline{x}||} \le \varepsilon\Big\}, \tag{2.1}$$

where the notation $x \xrightarrow{\Omega} \overline{x}$ means $x \to \overline{x}$ and $x \in \Omega$, contains the Fréchet ε-normals to Ω at $\overline{x} \in \Omega$. For $\varepsilon = 0$, the set in (2.1) is a closed convex cone which is called the *Fréchet normal cone* to Ω at $\overline{x}$ and is denoted by $\widehat{N}(\overline{x}; \Omega)$. One puts $\widehat{N}_\varepsilon(\overline{x}; \Omega) = \emptyset$ for all $\varepsilon \ge 0$ when $\overline{x} \notin \Omega$. The cone

$$N(\overline{x}; \Omega) := \operatorname*{Lim\,sup}_{x\to\overline{x},\ \varepsilon\downarrow 0} \widehat{N}_\varepsilon(x; \Omega), \tag{2.2}$$

which is generally nonconvex and nonclosed [4, Example 1.7], is said to be the *limiting normal cone* (also called the *basic normal cone*, or the *Mordukhovich normal cone*) to Ω at $\overline{x}$. If $\overline{x} \notin \Omega$, then one puts $N(\overline{x}; \Omega) = \emptyset$.

If X is an Asplund space [4, Definition 2.17], the expression on the right-hand-side of (2.2) can be simplified. Namely, if X is Asplund and Ω is locally closed around $\overline{x}$ then, according to [4, Theorem 2.35],

$$N(\overline{x}; \Omega) = \operatorname*{Lim\,sup}_{x\to\overline{x}} \widehat{N}(x; \Omega). \tag{2.3}$$

From (2.1) and (2.2), it follows that $\widehat{N}(\overline{x}; \Omega) \subset N(\overline{x}; \Omega)$. If Ω is convex then, according to [4, Propositions 1.3 and 1.5],

$$\widehat{N}(\overline{x}; \Omega) = N(\overline{x}; \Omega) = \Big\{x^* \in X^* \mid \langle x^*, x - \overline{x}\rangle \le 0 \text{ for all } x \in \Omega\Big\}; \tag{2.4}$$

thus, both the Fréchet and the limiting normal cones coincide with the normal cone of convex analysis.

Let $F : X \rightrightarrows Y$ be a multifunction between Banach spaces. For any $(\overline{x}, \overline{y}) \in X \times Y$ and $\varepsilon \geq 0$, the *ε-coderivative* of F at $(\overline{x}, \overline{y})$ is the multifunction $\widehat{D}^*_\varepsilon F(\overline{x}, \overline{y}) : Y^* \rightrightarrows X^*$ defined by

$$\widehat{D}^*_\varepsilon F(\overline{x}, \overline{y})(y^*) := \Big\{x^* \in X^* \mid (x^*, -y^*) \in \widehat{N}_\varepsilon\big((\overline{x}, \overline{y}); \mathrm{gph}F\big)\Big\}. \tag{2.5}$$

When $\varepsilon = 0$, the construction in (2.5) is called the *Fréchet coderivative* of F at $(\overline{x}, \overline{y})$ and is denoted by $\widehat{D}^* F(\overline{x}, \overline{y})$. We put $\widehat{D}^*_\varepsilon F(\overline{x}, \overline{y})(y^*) = \emptyset$ for all $\varepsilon \geq 0$ and $y^* \in Y^*$ when $(\overline{x}, \overline{y}) \notin \mathrm{gph}F$. The multifunction $D^* F(\overline{x}, \overline{y}) : Y^* \rightrightarrows X^*$ defined by

$$D^* F(\overline{x}, \overline{y})(y^*) := \Big\{x^* \in X^* \mid (x^*, -y^*) \in N\big((\overline{x}, \overline{y}); \mathrm{gph}F\big)\Big\} \tag{2.6}$$

is said to be the *Mordukhovich coderivative* (also called the *normal coderivative*, or the *limiting coderivative*) of F at $(\overline{x}, \overline{y})$. We put $D^* F(\overline{x}, \overline{y})(y^*) = \emptyset$ for all $y^* \in Y^*$ when $(\overline{x}, \overline{y}) \notin \mathrm{gph}F$.

From (2.3) and (2.6), it is clear that the computation of the Fréchet normal cone to the graph of a multifunction between Asplund spaces is a crucial step towards a complete differentiation of that multifunction.

Computation/estimation of Fréchet normal cones will be our main concern in the subsequent sections.

3. Upper estimate

We are going to analyze the upper estimate for the Fréchet normal cone (1.5) which was obtained in [8] for the case $X = \mathbb{R}^n$ and in [6] for the case X is an arbitrary reflexive space.

From now on, we assume that the Banach space X is reflexive.

Recall that the *tangent cone* to a convex set Ω at $\overline{x} \in \Omega$ is the topological closure of the cone $\big\{\lambda(x - \overline{x}) \mid x \in \Omega,\ \lambda \geq 0\big\}$.

The first claim of the following proposition was proved in [6] by using a generalized version of the Farkas lemma [1].

Proposition 3.1. (See [6, Lemma 3.1]) *Let $\overline{b} \in \mathbb{R}^m$, $\Theta(\overline{b})$ be given by* (1.1)*, $\overline{x} \in \Theta(\overline{b})$, and $I(\overline{x}, \overline{b})$ defined by* (1.3)*. Then*

$$N(\overline{x}; \Theta(\overline{b})) = \mathrm{pos}\big\{a^*_i \mid i \in I(\overline{x}, \overline{b})\big\} := \left\{ \sum_{i \in I(\overline{x}, \overline{b})} \lambda_i a^*_i \mid \lambda_i \geq 0 \ \ \forall i \in I(\overline{x}, \overline{b} \right\} \tag{3.1}$$

and

$$T(\overline{x}; \Theta(\overline{b})) = \Big\{v \in X \mid \langle a^*_i, v\rangle \leq 0 \ \ \forall i \in I(\overline{x}, \overline{b})\Big\}. \tag{3.2}$$

For an element $(\overline{x}, \overline{b}, \overline{x}^*) \in \mathrm{gph}\mathcal{F}$, where $\mathcal{F}$ has been defined in (1.4), we have

$$\overline{x}^* \in \mathcal{F}(\overline{x}, \overline{b}) = N(\overline{x}; \Theta(\overline{b})).$$

Hence, according to Proposition 3.1, there exist multipliers $\lambda_i \geq 0$, $i \in I(\overline{x}, \overline{b})$, such that

$$\overline{x}^* = \sum_{i \in I(\overline{x}, \overline{b})} \lambda_i a_i^*.$$

We write

$$I_0(\overline{x}, \overline{b}, \overline{x}^*) = \{i \in I(\overline{x}, \overline{b}) \mid \lambda_i = 0\}.$$

In some cases, we abbreviate $I(\overline{x}, \overline{b})$ and $I_0(\overline{x}, \overline{b}, \overline{x}^*)$ to I and I_0, respectively. Let

$$E(\overline{x}, \overline{b}, \overline{x}^*) = \Big\{(x^*, b^*, v) \mid \begin{array}{l} x^* \in \big(T(\overline{x}; \Theta(\overline{b})) \cap \{\overline{x}^*\}^\perp\big)^*, \\ v \in T(\overline{x}; \Theta(\overline{b})) \cap \{\overline{x}^*\}^\perp, \\ x^* = -\sum_{i \in I} b_i^* a_i^*,\ b_{\overline{I}}^* = 0,\ b_{I_0}^* \leq 0 \end{array}\Big\}, \tag{3.3}$$

$$E_0(\overline{x}, \overline{b}, \overline{x}^*) = \Big\{(x^*, b^*, v) \mid \begin{array}{l} x^* \in \big(T(\overline{x}; \Theta(\overline{b})) \cap \{\overline{x}^*\}^\perp\big)^*, \\ v \in T(\overline{x}; \Theta(\overline{b})) \cap \{\overline{x}^*\}^\perp, \\ x^* = -\sum_{i \in I} b_i^* a_i^*,\ b_{\overline{I}}^* = 0,\ b_I^* \leq 0 \end{array}\Big\}, \tag{3.4}$$

where $b^* = (b_1^*, \ldots, b_m^*) \in \mathbb{R}^m$,

$$\Omega^* := \Big\{x^* \in X^* \mid \langle x^*, u \rangle \leq 0 \text{ for every } u \in \Omega\Big\}$$

for any $\Omega \subset X$, and

$$\{x^*\}^\perp := \{v \in X \mid \langle x^*, v \rangle = 0\}$$

for any $x^* \in X^*$.

Theorem 3.2. (See [6, Proposition 3.2]) *If* $(\overline{x}, \overline{b}, \overline{x}^*) \in \text{gph}\mathcal{F}$*, then*

$$\widehat{N}\big((\overline{x}, \overline{b}, \overline{x}^*); \text{gph}\mathcal{F}\big) \subset E(\overline{x}, \overline{b}, \overline{x}^*), \tag{3.5}$$

where $E(\overline{x}, \overline{b}, \overline{x}^*)$ *is given by* (3.3)*. Also, if the vectors* $\{a_i^* \mid i \in I\}$ *are linearly independent then*

$$\widehat{N}\big((\overline{x}, \overline{b}, \overline{x}^*); \text{gph}\mathcal{F}\big) = E(\overline{x}, \overline{b}, \overline{x}^*). \tag{3.6}$$

For the case $X = \mathbb{R}^n$, the upper estimate (3.5) was obtained in [8]. In our notation, Question 1 of [8] can be restated as follows.

Question 1. *Does the inclusion* (3.5) *always hold as an equality?*

The second assertion of Theorem 3.2 solves this question in the affirmative under the condition that $\{a_i^* \mid i \in I\}$ are linearly independent. We are going to show that, in general, the inclusion (3.5) does not hold as an equality.

Definition 1. Let $\{v_j\}_{j \in J}$ be a family of finitely many vectors of a vector space V over the real number. We say that $\{v_j\}_{j \in J}$ is *positively linearly independent* if from the conditions $\sum_{j \in J} \lambda_j v_j = 0$ and $\lambda_j \geq 0$ for all $j \in J$ it follows that $\lambda_j = 0$ for all $j \in J$.

Proposition 3.3. *The inclusion* (3.5) *may be strict in some cases.*

Proof. Let $X = \mathbb{R}^2$ and let $a_1^* = (0,1)$, $a_2^* = (0,2) \in X^* = \mathbb{R}^2$, $\overline{b} = (0,0) \in \mathbb{R}^2$, $\overline{x} = (0,0) \in X$. We have

$$\begin{aligned}
\Theta(\overline{b}) &= \Big\{x = (x_1, x_2) \in \mathbb{R}^2 \mid \langle a_i^*, x\rangle \le 0,\ i = 1,2\Big\} = \mathbb{R} \times (-\infty, 0],\\
I(\overline{x}, \overline{b}) &= \big\{i \mid \langle a_i^*, \overline{x}\rangle = 0\big\} = \{1,2\},\\
\mathcal{F}(\overline{x}, \overline{b}) &= N(\overline{x}; \Theta(\overline{b})) = \operatorname{pos}\{a_1^*, a_2^*\} = \Big\{\lambda_1 a_1^* + \lambda_2 a_2^* \mid \lambda_i \ge 0,\ i = 1,2\Big\}\\
&= \{0\} \times [0, +\infty),\\
T(\overline{x}; \Theta(\overline{b})) &= \big(N(\overline{x}; \Theta(\overline{b}))\big)^* = \Big\{v = (v_1, v_2) \in \mathbb{R}^2 \mid \langle a_i^*, v\rangle \le 0,\ i = 1,2\Big\}\\
&= \mathbb{R} \times (-\infty, 0].
\end{aligned}$$

For $\alpha > 0$, since $\overline{x}^* = (0,\alpha) \in \{0\} \times [0,+\infty) = \mathcal{F}(\overline{x}, \overline{b})$, we have $(\overline{x}, \overline{b}, \overline{x}^*) \in \operatorname{gph}\mathcal{F}$. To describe the set $E(\overline{x}, \overline{b}, \overline{x}^*)$, we observe that

$$\begin{aligned}
&\{\overline{x}^*\}^\perp = \{(0,\alpha)\}^\perp = \mathbb{R} \times \{0\},\\
&T(\overline{x}; \Theta(\overline{b})) \cap \{\overline{x}^*\}^\perp = \mathbb{R} \times \{0\},\\
&\big(T(\overline{x}; \Theta(\overline{b})) \cap \{\overline{x}^*\}^\perp\big)^* = \{0\} \times \mathbb{R}.
\end{aligned}$$

If we use the representation

$$\overline{x}^* = (0, \alpha) = \alpha a_1^* + 0 a_2^*,$$

then $I_0 = I_0(\overline{x}, \overline{b}, \overline{x}^*) = \{2\}$. Choose $b^* = (b_1^*, b_2^*) \in \mathbb{R}^2$, where $b_2^* \le 0$. (The value b_1^* will be determined later.) For every $\gamma \in \mathbb{R}$, we have

$$v = (\gamma, 0) \in T(\overline{x}; \Theta(\overline{b})) \cap \{\overline{x}^*\}^\perp.$$

Let

$$x^* = -(b_1^* a_1^* + b_2^* a_2^*) = (0, -b_1^* - 2b_2^*) \in \big(T(\overline{x}; \Theta(\overline{b})) \cap \{\overline{x}^*\}^\perp\big)^*.$$

Hence, $(x^*, b^*, v) \in E(\overline{x}, \overline{b}, \overline{x}^*)$.

We now show that (x^*, b^*, v) does not belong to the cone $\widehat{N}\big((\overline{x}, \overline{b}, \overline{x}^*); \operatorname{gph}\mathcal{F}\big)$. Consider the sequences $\{b^k\}$ and $\{x^k\}$, where $b^k = (\frac{1}{k}, \frac{1}{k})$ and $x^k = (\frac{1}{2k}, \frac{1}{2k})$, $k \in \mathbb{N} = \{1, 2, 3, \dots\}$. We see that $b^k \to \overline{b}$, $x^k \to \overline{x}$ as $k \to \infty$. For every $k \in \mathbb{N}$,

$$\Theta(b^k) = \Big\{x = (x_1, x_2) \in \mathbb{R}^2 \bigm| x_2 \le \frac{1}{k},\ 2x_2 \le \frac{1}{k}\Big\} = \mathbb{R} \times \Big(-\infty, \frac{1}{2k}\Big].$$

Then $x^k \in \Theta(b^k)$, $I(x^k, b^k) = \{2\}$, and

$$\mathcal{F}(x^k, b^k) = N(x^k; \Theta(b^k)) = \operatorname{pos}\{a_2^*\} = \{0\} \times [0, +\infty),$$

for all $k \in \mathbb{N}$. Now, consider the sequence $\{u_k^*\}_{k\in\mathbb{N}}$ where $u_k^* = (0, \alpha + \frac{1}{2k})$ for all $k \in \mathbb{N}$. For every $k \in \mathbb{N}$, we have $u_k^* \in \mathcal{F}(x^k, b^k)$, and $u_k^* \to (0, \alpha) = \overline{x}^*$ as

$k \to \infty$. Note that

$$\begin{aligned}
&\limsup_{(x,b,u^*) \xrightarrow{\text{gph}\mathcal{F}} (\overline{x},\overline{b},\overline{x}^*)} \frac{\langle x^*, x - \overline{x}\rangle + \langle b^*, b - \overline{b}\rangle + \langle v, u^* - \overline{x}^*\rangle}{||x - \overline{x}|| + ||b - \overline{b}|| + ||u^* - \overline{x}^*||} \\
&\geq \limsup_{k\to\infty} \frac{\langle x^*, x^k - \overline{x}\rangle + \langle b^*, b^k - \overline{b}\rangle + \langle v, u_k^* - \overline{x}^*\rangle}{||x^k - \overline{x}|| + ||b^k - \overline{b}|| + ||u_k^* - \overline{x}^*||} \\
&= \limsup_{k\to\infty} \frac{\big\langle (0, -b_1^* - 2b_2^*), (\frac{1}{2k}, \frac{1}{2k})\big\rangle + \big\langle (b_1^*, b_2^*), (\frac{1}{k}, \frac{1}{k})\big\rangle + \big\langle (\gamma, 0), (0, \frac{1}{2k})\big\rangle}{||(\frac{1}{2k}, \frac{1}{2k})|| + ||(\frac{1}{k}, \frac{1}{k})|| + ||(0, \frac{1}{2k})||} \\
&= \lim_{k\to\infty} \frac{\frac{-b_1^* - 2b_2^*}{2k} + \frac{b_1^* + b_2^*}{k}}{\frac{\sqrt{2}}{2k} + \frac{\sqrt{2}}{k} + \frac{1}{2k}} = \frac{b_1^*}{\sqrt{2} + 2\sqrt{2} + 1} =: \mu.
\end{aligned}$$

For $b_1^* := 1$, we have

$$\mu = \frac{1}{\sqrt{2} + 2\sqrt{2} + 1} > 0.$$

This implies that $(x^*, b^*, v) \notin \widehat{N}\big((\overline{x}, \overline{b}, \overline{x}^*); \text{gph}\mathcal{F}\big)$, and hence,

$$E(\overline{x}, \overline{b}, \overline{x}^*) \not\subset \widehat{N}\big((\overline{x}, \overline{b}, \overline{x}^*); \text{gph}\mathcal{F}\big).$$

The proof is complete. □

Remark 1. We have seen that without the linear independence of $\{a_i^* \mid i \in I(\overline{x}, \overline{b})\}$ the equality (3.6) may fail to hold. The vectors $\{a_1^*, a_2^*\}$ in the above proof are not linearly independent, but they are positively linearly independent. Thus we have shown that *the inclusion* (3.5) *may be strict even in the case where the vectors* $\{a_i^* \mid i \in I(\overline{x}, \overline{b})\}$ *are positively linearly independent.*

Remark 2. As usual, we say that the inequality system (1.2) satisfies the *Slater condition* if there exists $x^0 \in X$ with $\langle a_i^*, x^0\rangle < \beta_i$ for all $i \in T$. This condition is a significant sign of the stability of the given inequality system. One might hope that (3.6) holds when the Slater condition is satisfied. But the proof of Proposition 3.3 shows that this is not the case. Indeed, taking $x^0 = (0, -1)$ we have $\langle a_i^*, x^0\rangle < \overline{b}_i$ for $i = 1, 2$ but, as shown in the proof, $E(\overline{x}, \overline{b}, \overline{x}^*) \not\subset \widehat{N}\big((\overline{x}, \overline{b}, \overline{x}^*); \text{gph}\mathcal{F}\big)$.

4. Lower estimate

Using the set $E_0(\overline{x}, \overline{b}, \overline{x}^*)$ defined by (3.4), we now provide a lower estimate for the Fréchet normal cone $\widehat{N}\big((\overline{x}, \overline{b}, \overline{x}^*); \text{gph}\mathcal{F}\big)$. Our result is an extension of [8, Lemma 4.2], where it was assumed that $X = \mathbb{R}^n$.

Theorem 4.1. *If* $(\overline{x}, \overline{b}, \overline{x}^*) \in \text{gph}\mathcal{F}$, *then*

$$E_0(\overline{x}, \overline{b}, \overline{x}^*) \subset \widehat{N}\big((\overline{x}, \overline{b}, \overline{x}^*); \text{gph}\mathcal{F}\big). \tag{4.1}$$

Proof. Let $(x^*, b^*, v) \in E_0(\overline{x}, \overline{b}, \overline{x}^*)$. In order to show that

$$(x^*, b^*, v) \in \widehat{N}\big((\overline{x}, \overline{b}, \overline{x}^*); \mathrm{gph}\mathcal{F}\big),$$

we need to verify the inequality

$$\limsup_{(x,b,u^*)\overset{\mathrm{gph}\mathcal{F}}{\longrightarrow}(\overline{x},\overline{b},\overline{x}^*)} \frac{\langle x^*, x-\overline{x}\rangle + \langle b^*, b-\overline{b}\rangle + \langle v, u^*-\overline{x}^*\rangle}{||x-\overline{x}|| + ||b-\overline{b}|| + ||u^*-\overline{x}^*||} \le 0. \tag{4.2}$$

Given any sequence $(x^k, b^k, u_k^*) \overset{\mathrm{gph}\mathcal{F}}{\longrightarrow} (\overline{x}, \overline{b}, \overline{x}^*)$. Since $(x^k, b^k) \to (\overline{x}, \overline{b})$, we must have $I(x^k, b^k) \subset I(\overline{x}, \overline{b}) := I$ for all k sufficiently large. As

$$u_k^* \in \mathrm{pos}\big\{a_i^* \mid i \in I(x^k, b^k)\big\} \subset \mathrm{pos}\big\{a_i^* \mid i \in I(\overline{x}, \overline{b})\big\} = N(\overline{x}; \Theta(\overline{b})) = \mathcal{F}(\overline{x}, \overline{b}),$$

the condition $v \in T(\overline{x}; \Theta(\overline{b})) \cap \{\overline{x}^*\}^\perp$ implies that

$$\langle v, u_k^* - \overline{x}^*\rangle = \langle v, u_k^*\rangle \le 0. \tag{4.3}$$

From the equalities $x^* = -\sum_{i\in I} b_i^* a_i^*$ and $b_{\overline{I}}^* = 0$, we deduce that

$$\begin{aligned}
\langle x^*, x^k - \overline{x}\rangle + \langle b^*, b^k - \overline{b}\rangle &= \Big\langle -\sum_{i\in I} b_i^* a_i^*, x^k - \overline{x}\Big\rangle + \langle b_I^*, b^k - \overline{b}\rangle \\
&= \sum_{i\in I} b_i^*\big(\langle a_i^*, \overline{x}\rangle - \langle a_i^*, x^k\rangle\big) + \sum_{i\in I} b_i^*\big(b_i^k - \overline{b}_i\big) \\
&= \sum_{i\in I} b_i^*\Big(\langle a_i^*, \overline{x}\rangle - \overline{b}_i + b_i^k - \langle a_i^*, x^k\rangle\Big) \\
&= \sum_{i\in I} b_i^*\big(b_i^k - \langle a_i^*, x^k\rangle\big).
\end{aligned}$$

Since $b_I^* \le 0$ and $\langle a_i^*, x^k\rangle \le b_i^k$ for all $i \in I$, this implies that

$$\langle x^*, x^k - \overline{x}\rangle + \langle b^*, b^k - \overline{b}\rangle \le 0. \tag{4.4}$$

From (4.3) and (4.4), we obtain

$$\limsup_{k\to\infty} \frac{\langle x^*, x^k-\overline{x}\rangle + \langle b^*, b^k-\overline{b}\rangle + \langle v, u_k^*-\overline{x}^*\rangle}{||x^k-\overline{x}|| + ||b^k-\overline{b}|| + ||u_k^*-\overline{x}^*||} \le 0,$$

which yields (4.2) because the sequence $(x^k, b^k, u_k^*) \overset{\mathrm{gph}\mathcal{F}}{\longrightarrow} (\overline{x}, \overline{b}, \overline{x}^*)$ was given arbitrarily. The proof is complete. □

Our next goal is to solve the following question.

Question 2. (See [8]) *Does the inclusion* $(x^*, b^*, v) \in \widehat{N}\big((\overline{x}, \overline{b}, \overline{x}^*); \mathrm{gph}\mathcal{F}\big)$ *imply* $b_I^* \le 0$ *where* $I = I(\overline{x}, \overline{b})$*? In other words, does* (4.1) *always hold as an equality?*

Proposition 4.2. *The inclusion* (4.1) *may be strict in some cases.*

Proof. Let $X = \mathbb{R}^2$ and $a_1^* = (1,0),\ a_2^* = (0,1) \in X^* = \mathbb{R}^2$. Choose $\overline{b} = (0,0) \in \mathbb{R}^2$, $\overline{x} = (0,0) \in X$, and observe that

$$\begin{aligned}
\Theta(\overline{b}) &= \Big\{x = (x_1, x_2) \in \mathbb{R}^2 \mid \langle a_i^*, x\rangle \leq 0,\ i = 1,2\Big\} = (-\infty, 0] \times (-\infty, 0],\\
I(\overline{x}, \overline{b}) &= \big\{i \mid \langle a_i^*, \overline{x}\rangle = 0\big\} = \{1,2\},\\
\mathcal{F}(\overline{x}, \overline{b}) &= N(\overline{x}; \Theta(\overline{b})) = \mathrm{pos}\{a_1^*, a_2^*\} = [0, +\infty) \times [0, +\infty).
\end{aligned}$$

For any $\alpha > 0$, we have $\overline{x}^* = (0, \alpha) \in \mathcal{F}(\overline{x}, \overline{b})$. This means that $(\overline{x}, \overline{b}, \overline{x}^*) \in \mathrm{gph}\mathcal{F}$. We want to find a triplet $(x^*, b^*, v) \in \widehat{N}\big((\overline{x}, \overline{b}, \overline{x}^*); \mathrm{gph}\mathcal{F}\big)$ with $b_I^* \not\leq 0$, i.e., there exists $i \in I = I(\overline{x}, \overline{b})$ such that $b_i^* > 0$. Note that

$$\begin{aligned}
&\{\overline{x}^*\}^\perp = \{(0, \alpha)\}^\perp = \mathbb{R} \times \{0\},\\
&T(\overline{x}; \Theta(\overline{b})) = (-\infty, 0] \times (-\infty, 0],\\
&T(\overline{x}; \Theta(\overline{b})) \cap \{\overline{x}^*\}^\perp = (-\infty, 0] \times \{0\},\\
&\big(T(\overline{x}; \Theta(\overline{b})) \cap \{\overline{x}^*\}^\perp\big)^* = [0, +\infty) \times \mathbb{R}.
\end{aligned}$$

We have $\overline{x}^* = (0, \alpha) = 0a_1^* + \alpha a_2^*$. Hence $I_0 = I_0(\overline{x}, \overline{b}, \overline{x}^*) = \{1\}$. Choose

$$x^* = (1, -1) \in \big(T(\overline{x}; \Theta(\overline{b})) \cap \{\overline{x}^*\}^\perp\big)^*,$$

$b^* = (b_1^*, b_2^*)$ with $b_1^* = -1 \leq 0$, $b_2^* = 1$, and

$$v = (\gamma, 0) \in T(\overline{x}; \Theta(\overline{b})) \cap \{\overline{x}^*\}^\perp,$$

where $\gamma \leq 0$. Observe that $\{a_1^*, a_2^*\}$ are linearly independent. By the choice of (x^*, b^*, v), we have $x^* = -(b_1^* a_1^* + b_2^* a_2^*)$ and $b_{I_0}^* \leq 0$. According to Theorem 3.2, we can infer that $(x^*, b^*, v) \in \widehat{N}\big((\overline{x}, \overline{b}, \overline{x}^*); \mathrm{gph}\mathcal{F}\big)$. Since $b_2^* = 1 > 0$ and $2 \in I \backslash I_0$, we have shown that the inclusion $(x^*, b^*, v) \in \widehat{N}\big((\overline{x}, \overline{b}, \overline{x}^*); \mathrm{gph}\mathcal{F}\big)$ does not imply $b_I^* \leq 0$. The proof is complete. □

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