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# ARE SEVERAL RECENT GENERALIZATIONS OF EKELAND'S VARIATIONAL PRINCIPLE MORE GENERAL THAN THE ORIGINAL PRINCIPLE?

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Abstract. We show that several recent results proposed as generalizations of Ekeland's variational principle are in fact equivalent to the original principle.

In 1974, Ekeland [2] introduced a variational principle, which appears to be one of the most important results in nonlinear analysis during the last three decades and has numerous applications. Recently, several generalizations of this principle have been proposed (see [11, 12]). The purpose of the present note is to show that these seemingly more general results can be derived from the original Ekeland variational principle. Let us begin with the original principle.

**Theorem 1.** [2]. *Let $(X, d)$ be a complete metric space, $f : X \to R \cup \{+\infty\}$ be a lower semicontinuous functional, not identically $+\infty$ and bounded from below. Then, for every $e > 0, \lambda > 0$ and $x_0 \in X$ such that*

$$f(x_0) < \inf_{x\in X} f(x) + e,$$

*there exists $x^* \in X$ such that*

(a) $f(x^*) \le f(x_0)$;

(b) $d(x_0, x^*) \le \lambda$;

(c) $f(x) - f(x^*) + \dfrac{\varepsilon}{\lambda} d(x^*, x) > 0, \quad \forall x \neq x^*.$

For our purpose, the following version of [3] is more suitable.

**Theorem 2.** [3]. *Let $(X, d)$ and $f$ be as in Theorem 1, $x_0 \in domf$ and $\varepsilon > 0$ be fixed. Then there exists $x^* \in X$ such that*

(i) $f(x^*) - f(x_0) + \varepsilon d(x_0, x^*) \le 0$;

(ii) $f(x) - f(x^*) + \varepsilon d(x^*, x) > 0, \quad \forall x \neq x^*.$

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For the sake of completeness, we supply a proof of the equivalence between Theorems 1 and 2. Let the assumptions of Theorem 1 be satisfied. Setting $\varepsilon = \dfrac{e}{\lambda}$, we deduce from Theorem 2 that $d(x_0, x^*) \leq \dfrac{\lambda}{e}(f(x_0) - f(x^*)) \leq \dfrac{\lambda}{e}e = \lambda$. Hence the conclusion of Theorem 1 is valid. Now, suppose Theorem 1 holds. Let $x_0 \in domf$ and $\varepsilon > 0$ be given. Fix any $e > f(x_0) - \inf_{x \in X} f(x)$ and set $\lambda = \dfrac{e}{\epsilon}$. Consider

$$S := \{x \in X : f(x) - f(x_0) + \varepsilon d(x_0, x) \leq 0\}.$$

By the lower semicontinuity of $f$, $S$ is closed. Furthermore, $S$ is nonempty as $x_0 \in S$. Applying Theorem 1 for the chosen $e, \lambda$ and for $S$ instead of $X$ one finds $x^* \in S$ satisfying (a), (b) and

(c′) $f(x) - f(x^*) + \dfrac{e}{\lambda} d(x^*, x) > 0, \quad \forall x \in S \setminus \{x^*\}$.

Since $x^* \in S$, $(i)$ holds.

To show (ii) it is sufficent to check that (c′) is satisfied also for $x \notin S$. By the definition of $S$, the property $x \notin S$ means

$$f(x) - f(x_0) + \varepsilon d(x_0, x) > 0.$$

From this and (i) we easily deduce (ii) and then obtain Theorem 2. This shows that Theorem 2 holds true.

To derive the generalizations of the Ekeland variational principle given in [11, 12], we first prove the following intermediate theorem.

**Theorem 3.** *Let $(X, d)$, $f$ and $x_0$ be as in Theorem 2. Let $H : [0, +\infty) \to [0, +\infty)$ be a nonincreasing function such that $\int_0^s H(r)dr$ is a continuous function of $s$ and $\int_0^{+\infty} H(r)dr = +\infty$. Let $y_0 \in X$ be fixed. Then, there exists $x^* \in X$ such that*

(a) $f(x^*) \leq f(x_0)$;

(b) $d(y_0, x^*) \leq r_0 + \bar{r}$;

(c) $f(x) - f(x^*) + H(r(x^*))d(x^*, x) > 0, \quad \forall x \neq x^*$;

*where $r(.) = d(y_0, .)$, $r_0 = r(x_0)$ and $\bar{r}$ is such that*

$$\int_{r_0}^{r_0+\bar{r}} H(r)dr \geq f(x_0) - \inf_{x \in X} f(x).$$

*Proof.* . For $x \in X$ set

$$E(x) = \Big\{ y \in X |\ r(x) \leq r(y) \quad \text{and} \quad f(y) - f(x) + \int_{r(x)}^{r(y)} H(r)dr \leq 0 \Big\}$$

Clearly, $x \in E(x)$. Besides, if $y \in E(x)$ then it is easily seen that $E(y) \subseteq E(x)$. Furthermore, $E(x)$ is closed. To see this, let $x_n \in E(x)$, $x_n \to \bar{x}$. Then

$$r(x) \le r(x_n),$$

$$f(x_n) - f(x) + \int_{r(x)}^{r(x_n)} H(r)dr \le 0.$$

By the assumed continuity and lower semicontinuity, letting $n \to \infty$ we obtain

$$r(x) \le r(\bar{x}),$$

$$f(\bar{x}) - f(x) + \int_{r(x)}^{r(\bar{x})} H(r)dr \le 0,$$

which shows that $\bar{x} \in E(x)$. So $E(x)$ is a complete metric space.

For every $y \in E(x_0)$, by the definitions of $E(x_0)$ and $\bar{r}$, we can estimate the distance $d(y_0, y) = r(y)$ as follows

$$\int_{r_0}^{r(y)} H(r)dr \le f(x_0) - f(y) \le f(x_0) - \inf_{y \in X} f(y) \le \int_{r_0}^{r_0+\bar{r}} H(r)dr.$$

Since $H(r) \ge 0$, the obtained inequality yields

$$r(y) \le r_0 + \bar{r}. \tag{1}$$

Applying Theorem 2 for $\varepsilon = H(r_0 + \bar{r})$ and $f\big|_{E(x_0)}$ one finds $z^* \in E(x_0)$ such that

$$f(z^*) - f(x_0) + H(r_0 + \bar{r})d(x_0, z^*) \le 0, \tag{2}$$

$$f(x) - f(z^*) + H(r_0 + \bar{r})d(z^*, x) > 0, \quad \forall x \in E(x_0) \setminus \{z^*\}. \tag{3}$$

Again, using Theorem 2 for $f$ (on $X$), $H(r(z^*))$ in the place of $\varepsilon$, and $z^*$ as the starting point $x_0$, one obtains $x^* \in X$ satisfying

$$f(x^*) - f(z^*) + H(r(z^*))d(z^*, x^*) \le 0, \tag{4}$$

$$f(x) - f(x^*) + H(r(z^*))d(x^*, x) > 0, \quad \forall x \ne x^*. \tag{5}$$

Adding (2) and (4) gives

$$f(x^*) - f(x_0) + H(r_0 + \bar{r})d(x_0, z^*) + H(r(z^*))d(z^*, x^*) \le 0,$$

which indicates that $f(x^*) \le f(x_0)$. Thus assertion (a) holds.

If $x^* = z^*$, then (5) implies (c) and (1) implies (b). Hence $x^* = z^*$ is the required point. If $x^* \ne z^*$, then we have three possibilities. First, if $x^* \in E(x_0)$, then (3) and (1) imply that

$$f(x^*) - f(z^*) + H(r(z^*))d(z^*, x^*) > 0,$$

contradicting (4). Next, if $x^* \notin E(x_0)$ (then $x^* \notin E(z^*)$) and if $r(z^*) \le r(x^*)$, then by the definition of $E(z^*)$ one gets

$$f(x^*) - f(z^*) + H(r(z^*))d(z^*, x^*) \ge f(x^*) - f(z^*) + \int\limits_{r(z^*)}^{r(x^*)} H(r)dr > 0,$$

again contradicting (4). So it remains to consider the case where $x^* \notin E(x_0)$ and $r(x^*) < r(z^*)$. In this case we have $d(y_0, x^*) = r(x_*) < r(z^*) \le r_0 + \bar{r}$ by (1). Hence (b) holds. Besides, $r(x^*) < r(z^*)$ implies that $H(r(x^*)) \ge H(r(z^*))$, and hence (c) follows from (5). □

**Remark 1**. If one chooses, in the statement of the Theorem 3, $y_0 = x_0$, $\bar{r} = \lambda$ and $H(r) \equiv \dfrac{e}{\lambda}$, then one gets Theorem 1 from Theorem 3. So, Theorems 1, 2 and 3 are equivalent.

Theorem 3 implies the following main result of [11].

**Theorem 4.** [11]. *Let $(X, d)$, $f$ and $y_0$ be as in Theorem 3. Let $\lambda > 0$, $\varepsilon > 0$ and $x_0 \in X$ such that $f(x_0) < \inf\limits_{x \in X} f(x) + \varepsilon$. Let $h : [0, +\infty) \to [0, +\infty)$ be a continuous nondecreasing function such that $\int\limits_0^{+\infty} \dfrac{dr}{1 + h(r)} = +\infty$. Let $r_0 = d(y_0, x_0)$ and $\bar{r}$ be such that*

$$\int\limits_{r_0}^{r_0+\bar{r}} \frac{dr}{1 + h(r)} \ge \lambda.$$

*Then, there exists $x^* \in X$ such that assertions* (a), (b) *of Theorem 3 and*

(c′) $f(x) - f(x^*) + \dfrac{\varepsilon}{\lambda(1 + h(d(y_0, x^*)))} d(x, x^*) \ge 0, \quad \forall\, x \in X,$

*hold.*

*Proof.* For $H(r) = \dfrac{\varepsilon}{\lambda(1 + h(r))}$ we see that all the assumptions of Theorem 3 are satisfied, and (c) implies (c′). □

Observe that the assumed continuity of $h(.)$, which is inevitable in the proof in [11], is more restrictive than the continuity of $\int\limits_0^s H(r)dr$ needed for Theorem 3. Moreover, (c) implies even the strict inequality in (c′) for all $x \in X \setminus \{x^*\}$.

We now restate and demonstrate the following Theorem 5, which is the main result of [12]. Let $(X, d)$, $x_0$, $y_0$ and $h(.)$ be as in Theorem 4. Let $F : X \times X \to R \cup \{-\infty, +\infty\}$ be lower semicontinuous in the second variable and satisfy:

(i) $F(x, x) = 0, \ \forall x \in X;$

(ii) $F(x, z) \le F(x, y) + F(y, z), \ \forall\, x, y, z \in X;$

(iii) $-\lambda := \inf_{x\in X} F(x_0, x) > -\infty$.

Let $\bar{r}$ satisfy $\int\limits_{r_0}^{r_0+\bar{r}} \frac{dr}{1+h(r)} > \lambda$.

**Theorem 5.** [12]. *Under the just mentioned assumptions, there exists $x^* \in X$ such that*

(a$_1$) $F(x_0, x^*) \le 0$,

$$d(x_0, x^*) \le -(1 + h(r_0 + \bar{r}))F(x_0, x^*) \le \lambda(1 + h(r_0 + \bar{r}));$$

(b$_1$) $d(y_0, x^*) \le r_0 + \bar{r}$;

(c$_1$) $F(x_*, x) + \frac{1}{1 + h(d(y_0, x^*))} d(x^*, x) > 0, \ \ \forall x \ne x^*$.

*Proof.* We will use Theorems 2 and 3 and also some arguments in the proof of Theorem 3. Set $H(.) = (1 + h(.))^{-1}$ and $f(.) = F(x_0, .)$. Then, it is easy to see that all the assumptions of Theorem 3 are satisfied. Hence, so are the assumptions of Theorem 2. Let us choose $z^*$ and $x^*$ by the same method as in the proof of Theorem 3. Then $z^*$ satisfies (2) and $x^*$ satisfies (4). Besides, by (1), $H(r(z^*)) \ge H(r_0 + \bar{r})$. From (4) it follows that

$$f(x^*) - f(z^*) + H(r_0 + \bar{r})d(z^*, x^*) \le 0. \tag{6}$$

Adding (2) and (6) gives

$$f(x^*) - f(x_0) + H(r_0 + \bar{r})d(x_0, x^*) \le 0. \tag{7}$$

From this and the definitions of $H(.)$ and $f(.)$ we have

$$d(x_0, x^*) \le -(1 + h(r_0 + \bar{r}))F(x_0, x^*) \le \lambda(1 + h(r_0 + \bar{r})),$$

which together with (7) yields (a$_1$). Finally, since $x^*$ satisfies the conclusions of Theorem 3 and since (b$_1$) and (c$_1$), respectively, are nothing else than (b) and (c) of Theorem 3, we obtain the conclusion of Theorem 5. □

Note that the results of [7] are contained in the corresponding results of [12]. So they can be also derived from Theorem 3 by setting $f(.) := F(x_0, .)$. For $F(x, y) := f(y) - f(x)$ and $h(.) := H(.)^{-1} - 1$, the conclusion of Theorem 5 clearly implies that of Theorem 3. (However, the assumed continuity of $h(.)$ is more restrictive than the continuity of $\int_0^s H(r)dr$ assumed in Theorem 3.)

**Remark 2**. Several authors used the term "generalization" to mean also an extension to the case where $f$ is a vector-valued function. We would like to observe that, using scalarization one can derive Ekeland's variational principle for the vector-valued case from the corresponding result for the scalar case (see [1, 5, 8]). So these principles for the vector-valued case can be viewed as equivalent to Theorems 2 and 3. However, Ekeland's variational principles for vector-valued case in [4, 6, 9, 10], using vector metrics, are not equivalent to these theorems, since they imply Theorem 2 when $f$ and the mentioned metric are scalar, but Theorem 2 does not imply them.

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