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ON SINGULAR INTEGRAL EQUATIONS WITH THE CARLEMAN SHIFTS IN THE CASE OF THE VANISHING COEFFICIENT

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Abstract. Based on the well-known necessary and sufficient condition for the linear-fractional function to be the generator of a cyclic group of n-terms, this paper describes the general form of all linear-fractional functions which are Carleman shifts on the unit circle. Our main result deals with the solvability in a closed form for a class of singular integral equations with Carleman shifts on the unit circle in the case where the coefficient vanishes on the curve.

1. Introduction

The Noetherian theory of singular integral equations of Cauchy's type was started with works of Noether and Carleman in 1921, and then it has been developed by many others (see [1], [2], [3], [6], [8] and references therein). A reason that this theory attracts a lot of attention is that there is an effective relation between Riemann boundary-value problems of the analytic functions and the singular integral equations of Cauchy's type. In [5], the author studied a singular integral equation with the rotation on the unit circle under the assumption that it's coefficient has no zero-points on the curve. The cases of vanishing coefficients in either differential equations or integral equations require new investigations. In [9], one of us has investigated the solvability in a closed form for a class of singular integral equations with the rotation on the unit circle in the case where the coefficient has isolated zero-points on the curve.

In this paper, we study the solvability in a closed form for the class of singular integral equations with the Carleman shifts being the linear-fractional functions in the case where the coefficient has zero-points on the curve.

2. The cyclic group of linear-fractional functions on the complex plane and on the unit circle

It is well-known that the linear-fractional function of the form

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$$\omega(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \tag{2.0.1}$$

caries out a one-to-one and conformal mapping of the extended complex plane $\overline{\mathbb{C}}$ in to itself. If

$$L_1(z) = \frac{\alpha_1 z + \beta_1}{\gamma_1 z + \delta_1} \quad \text{and} \quad L_2(z) = \frac{\alpha_2 z + \beta_2}{\gamma_2 z + \delta_2}$$

are two arbitrary linear-fractional functions then their product, denoted by $L_1 \circ L_2(z)$, is determined as follows:

$$\begin{aligned} L(z) = L_1 \circ L_2(z) &= \frac{\alpha_1 \dfrac{\alpha_2 z + \beta_2}{\gamma_2 z + \delta_2} + \beta_1}{\gamma_1 \dfrac{\alpha_2 z + \beta_2}{\gamma_2 z + \delta_2} + \delta_1} \\ &= \frac{[(\alpha_1\alpha_2 + \beta_1\gamma_2)z + \alpha_1\beta_2 + \beta_1\delta_2)]}{[(\gamma_1\alpha_2 + \delta_1\gamma_2)z + \gamma_1\beta_2 + \delta_1\delta_2)]}. \end{aligned}$$

With the above product, the set of all linear-fractional functions is a group (see [1] or [4]). In the sequel, this group will be denoted by $\mathcal{V}$. It is easy to see that the group $\mathcal{V}$ is infinite. In [1], the author gave a necessary and sufficient condition for a linear-fractional function to be involution of n-order. In Subsection 2.1, we present a brief survey on the results related to the condition given in [1]. In Subsection 2.2, by using the mentioned results we establish a general form of the linear-fractional functions which are the Carleman shifts on the unit circle.

2.1. **The cyclic group of linear-fractional functions on the complex plane.** Two linear-fractional functions

$$L_1(z) = \frac{\alpha_1 z + \beta_1}{\gamma_1 z + \delta_1} \quad \text{and} \quad L_2(z) = \frac{\alpha_2 z + \beta_2}{\gamma_2 z + \delta_2} \tag{2.1.1}$$

will be considered identical in the group $\mathcal{V}$ if and only if $L_1(z) = L_2(z)$ for all values of $z \in \overline{\mathbb{C}}$. For this to be so it is necessary and sufficient that the corresponding coefficients be proportional to each other, i.e $\alpha_2 = \lambda\alpha_1,\ \beta_2 = \lambda\beta_1,\ \gamma_2 = \lambda\gamma_1,\ \delta_2 = \lambda\delta_1,\ \lambda \neq 0$ (see [4] p. 65). So we can assume that

$$\alpha\delta - \beta\gamma = 1 \tag{2.1.2}$$

whenever a linear-fractional function of the form (2.0.1) is dealt with in this paper. On the other hand, we denote by $\mathcal{W}$ the set of all linear-fractional functions of the form (2.0.1) satisfying condition (2.1.2). (Two elements of the form (2.1.1) are identical in $\mathcal{W}$ if and only if $\alpha_1 = \alpha_2,\ \beta_1 = \beta_2,\ \gamma_1 = \gamma_2,\ \delta_1 = \delta_2$). For any $\omega \in \mathcal{W}$, we write

$$A_\omega = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

So A_ω is a 2-order square matrix.

It is easy to prove the following remarks.

Remark 2.1. Let ω_1 and ω_2 be in $\mathcal{W}$. Then the following identity holds

$$A_{\omega_2 \circ \omega_1} = A_{\omega_2}.A_{\omega_1}.$$

Remark 2.2. The function

$$e(z) = \frac{1z+0}{0z+1} \equiv \frac{-1z+0}{0z-1}$$

is the unit element of the group $\mathcal{V}$.

In the sequel, we denote by I the unit element of $\mathcal{V}$.

Remark 2.3. Suppose that $\omega \in \mathcal{W}$. Then $\omega = I$ if and only if either $A_\omega = E$ or $A_\omega = -E$, where E is the unit matrix.

Remark 2.4. Let $\omega(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ be in $\mathcal{W}$. Then for all $n \in \mathbb{N}$ we have

$$A_\omega^n = \lambda_n A_\omega - \lambda_{n-1}E, \tag{2.1.3}$$

where $\lambda_0 = 0,\ \lambda_1 = 1$ and

$$\lambda_k - (\alpha+\delta)\lambda_{k-1} + \lambda_{k-2} = 0, \quad k = 2,3,4,\dots \tag{2.1.4}$$

Proof. We proceed by induction. For $n = 1$, (2.1.3) is trivial. Suppose that (2.1.3) is true for $n = k$. Then for $n = k+1$, we find

$$\begin{aligned} A_\omega^{k+1} = A_\omega^k.A_\omega &= \big(\lambda_k A_\omega - \lambda_{k-1}E\big)A_\omega = \lambda_k A_\omega^2 - \lambda_{k-1}A_\omega \\ &= \lambda_k\big((\alpha+\delta)A_\omega - E\big) - \lambda_{k-1}A_\omega \\ &= \big(\lambda_k(\alpha+\delta) - \lambda_{k-1}\big)A_\omega - \lambda_k E = \lambda_{k+1}A_\omega - \lambda_k E, \end{aligned}$$

where $\lambda_{k+1} = \lambda_k(a+d) - \lambda_{k-1}$. □

Now we determine λ_k from formula (2.1.4). It is easy to see that (2.1.4) is a linear difference equation of 2-order. So we have the following

Remark 2.5. Let $\omega(z) = \dfrac{\alpha z+\beta}{\gamma z+\delta} \in \mathcal{W}$ and let $n \in \mathbb{N}$ be given.

- If $\alpha + \delta = 2$ then
$$\lambda_n = n.$$
- If $\alpha + \delta = -2$ then
$$\lambda_n = (-1)^{n+1}n.$$
- If $\alpha + \delta \neq \pm 2$ then
$$\lambda_n = \frac{1}{\theta_1}(x_1^n - x_2^n) = \frac{1}{\theta_2}(x_2^n - x_1^n),$$
where $\theta_1,\ \theta_2$ are the square roots of the complex number $(\alpha+\delta)^2 - 4$, and
$$x_1 = \frac{\alpha+\delta+\theta_1}{2}, \quad x_2 = \frac{\alpha+\delta+\theta_2}{2}\,.$$

Lemma 2.1. ([1], p. 534.) *Suppose that $\omega \in \mathcal{W}$ is of the form* (2.0.1) *and $\omega \not\equiv I$. Then $\omega^n \equiv I$ if and only if*

$$\alpha + \delta = 2\cos\frac{k\pi}{n}, \quad \textit{for some} \quad k \in \{1, 2, \dots, n-1\}.$$

Theorem 2.1. *Let $\omega \in \mathcal{W}$ be given and let $n \in \mathbb{N}$, $n \geq 2$ be fixed. Then ω satisfies*

$$\begin{cases} \omega^n \equiv I, \\ \omega^m \not\equiv I, m = 1, 2, \dots, n-1 \end{cases} \tag{2.1.5}$$

if and only if

$$\begin{cases} \alpha + \delta = 2\cos\dfrac{k\pi}{n} \quad \textit{for some} \quad k \in \{1, \dots, n-1\}, \ (n, k) = 1, \\ \alpha\delta - \beta\gamma = 1. \end{cases}$$

Proof. By Lemma 2.1 we have $\alpha + \delta = 2\cos\dfrac{k\pi}{n}$ for some $k \in \{1, 2, \dots, n-1\}$. If $(n, k) = l > 1$ then

$$\alpha + \delta = 2\cos\frac{k\pi}{n} = 2\cos\frac{\frac{k}{l}\pi}{\frac{n}{l}} = 2\cos\frac{k_1\pi}{n_1},$$

where $n_1 = \dfrac{n}{l} < n$. By Lemma 2.1, $\omega^{n_1} \equiv I$, which contradicts the assumption $\omega^m \not\equiv I$ for all $m \in \{1, 2, \dots, n-1\}$. Hence $(n, k) = 1$.

Conversely, suppose that $\alpha + \delta = 2\cos\dfrac{k\pi}{n}$ for some $k \in \{1, \dots, n-1\}$ and $(n, k) = 1$. From Lemma 2.1 it follows that $\omega^n \equiv I$. Suppose that there exists $m \in \mathbb{N}$, $2 \leq m < n$, such that $\omega^m = I$. Lemma 2.1 implies that there exists $k_1 \in \{1, 2, \dots, m-1\}$ such that $\alpha + \delta = 2\cos\dfrac{k_1\pi}{m}$. Then we have

$$\cos\frac{k\pi}{n} = \cos\frac{k_1\pi}{m},$$

which implies $\dfrac{k\pi}{n} = \dfrac{k_1\pi}{m}$. The last equality contradicts the condition $(n, k) = 1$. To end the proof, we have to show that $\omega \not\equiv I$. Suppose that $\omega \equiv I$, i.e.

$$\frac{\alpha z + \beta}{\gamma z + \delta} = z, \quad \forall\, z \in \mathbb{C}.$$

By Remark 2.3 we have either $\alpha = \delta = 1$ or $\alpha = \delta = -1$. From the assumption it follows that either $\cos\dfrac{k\pi}{n} = 1$ or $\cos\dfrac{k\pi}{n} = -1$ for some $k \in \{1, \dots, n-1\}$. None of them is possible. □

Remark 2.6. If $\omega \in \mathcal{V}$ satisfies condition (2.1.5) then ω is the generator of the n-terms cyclic group.

2.2. **The cyclic group of linear-fractional functions on unit circle.** Let $\Gamma = \{t \in \mathbb{C} : \ |t| = 1\}$ be the unit circle. In this section, we will determine a general form of the linear-fractional functions $\omega(z)$ satisfying the condition

$$\omega(\Gamma) \subset \Gamma, \quad \omega^n \equiv I, \quad \omega^m \not\equiv I, \ m = 1, 2, \dots, n-1, \tag{2.2.1}$$

where $n \in \mathbb{N}$, $n \geq 2$ is given.

It is well-known that the linear-fractional function $\omega(z)$ maps Γ into Γ if and only if that it is of the form

$$\omega(z) = e^{i\theta} \frac{z - \alpha}{\bar{\alpha} z - 1}\,, \tag{2.2.2}$$

where $\theta \in \mathbb{R}$, α is the zero-point of $\omega(z)$, i.e $\alpha \in \overline{\mathbb{C}}$, $\omega(\alpha) = 0$ (see [4], p. 83).

We now consider two cases

Case 1: $\alpha = \infty$. By (2.2.2) we get $\omega(z) = e^{i\theta} \dfrac{1}{z}$ and $\omega^2 = I$. So we can conclude that

- If $n = 2$ then $\omega(z) = e^{i\theta} \frac{1}{z}$;
- If $n > 2$ then $\omega(z)$ does not exist.

Case 2: $|\alpha| < \infty$. From the condition $\omega(z) \in \Gamma$ for every $z \in \Gamma$ it follows that $|\alpha| \neq 1$. Denote by θ_1 one of the two square roots of $\sqrt{\alpha\bar{\alpha} - 1}$. Dividing both numerator and denominator of (2.2.2) by $e^{\frac{i\theta}{2}} \theta_1$ we get

$$\omega(z) = \frac{\dfrac{e^{\frac{i\theta}{2}}}{\theta_1} z - \dfrac{e^{\frac{i\theta}{2}}}{\theta_1} \alpha}{\dfrac{e^{\frac{-i\theta}{2}}}{\theta_1} \overline{\alpha} z - \dfrac{e^{\frac{-i\theta}{2}}}{\theta_1}}\,.$$

According to Theorem 2.1,

$$\frac{e^{\frac{i\theta}{2}}}{\theta_1} + \frac{-e^{\frac{-i\theta}{2}}}{\theta_1} = 2 \cos \frac{k\pi}{n},$$

for some $k \in \{1, 2, \dots, n-1\}$, $(k, n) = 1$. Therefore

$$\sin \frac{\theta}{2} = \theta_1 . \cos \frac{k\pi}{n}. \tag{2.2.3}$$

- If $n = 2$ then $k = 1$. From (2.2.3) it follows that $\theta = 2m\pi$ for some $m \in \mathbb{Z}$. Then we have

$$\omega(z) = e^{2m\pi i} \frac{z - \alpha}{\overline{\alpha} z - 1} = \frac{z - \alpha}{\overline{\alpha} z - 1}\,.$$

- If $n > 2$, then from the conditions $k \in \{1, 2, \dots, n-1\}$ and $(n, k) = 1$ it follows that $\cos \dfrac{k\pi}{n} \neq 0$. Hence we get
$$\frac{\sin \dfrac{\theta}{2}}{\cos \dfrac{k\pi}{n}} = \theta_1.$$
It implies that θ_1 is a real number; therefore $|\alpha| < 1$. So the condition $|\alpha| < 1$ is necessary. We then have
$$\cos\theta = 1 - 2(1 - \alpha\bar{\alpha}) \cos^2 \frac{k\pi}{n} \cdot$$

So we have proved the following theorem.

Theorem 2.2. *Suppose that $\omega(z)$ is a linear-fractional function satisfying the condition*
$$\omega(\Gamma) \subset \Gamma, \quad \omega^n \equiv I, \quad \omega^m \not\equiv I, \quad m = 1, 2, \dots, n-1,$$
where $n \in \mathbb{N}$, $n \geq 2$ is given.

1) *If $n = 2$ then ω is of the form*
$$\omega(z) = e^{i\theta} \frac{1}{z}, \quad \theta \in \mathbb{R},$$
or
$$\omega(z) = \frac{z - \alpha}{\bar{\alpha} z - 1} \quad |\alpha| \neq 1.$$

2) *If $n > 2$ then ω is of the form*
$$\omega(z) = e^{i\theta} \frac{z - \alpha}{\bar{\alpha} z - 1},$$
where $|\alpha| < 1$, $\cos\theta = 1 - 2(1 - \alpha\overline{\alpha}) \cos^2 \dfrac{k\pi}{n}$, for some $k \in \{1, 2, \dots, n-1\}$, $(k, n) = 1$.

Remark 2.7. Suppose that the linear-fractional function $\omega(z)$ satisfies (2.2.1) and it is the shift in positive orientation of Γ, i.e.

(2.2.4)
$$\begin{cases} \omega(\Gamma) \subset \Gamma, \\ \omega^n \equiv I, \\ \omega^m \not\equiv I, \quad m = 1, 2, \dots, n-1, \\ \omega \text{ positive orientation of } \Gamma, \end{cases}$$

then in the conclusion we have to add the condition $|\alpha| < 1$. In this case, we can conclude that

- If $n = 2$ then ω is of the form
$$\omega(z) = \frac{z - \alpha}{\bar{\alpha} z - 1} \quad (|\alpha| < 1).$$

- If $n > 2$ then ω is of the form
$$\omega(z) = e^{i\theta}\frac{z-\alpha}{\bar{\alpha}z-1},$$
where $|\alpha| < 1$, $\cos\theta = 1 - 2(1-\alpha\overline{\alpha})\cos^2\frac{k\pi}{n}$ for some $k \in \{1, 2, \dots, n-1\}$, $(k, n) = 1$.

3. On a class of singular integral equations with shifts on the unit circle

In this section, we consider the solvability in a closed form of the following equation in $H^\mu(\Gamma)$ $(0 < \mu < 1)$:

$$a(t)\varphi(t) + \frac{b(t)}{n}\sum_{k=0}^{n-1}\varepsilon_\ell^{n-k}\frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau-\omega^k(t)}d\tau = f(t), \tag{3.0.1}$$

where $1 \leq \ell \leq n-1$, $\varepsilon_1 = e^{\frac{2\pi i}{n}}$, $e_\ell = \varepsilon_1^\ell$, $\omega(z) = \dfrac{\alpha z+\beta}{\gamma z+\delta}$, $(\alpha\delta - \beta\gamma = 1)$ is a linear-fractional function satisfying (2.2.4), and $a(t), b(t), f(t)$ are given functions in $H^\mu(\Gamma)$.

Consider the following operators in $X := H^\mu(\Gamma)$:

$$(S\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau-t}d\tau, \tag{3.0.2}$$

$$(W\varphi)(t) = \varphi(\omega(t)), \tag{3.0.3}$$

$$P_k = \frac{1}{n}\sum_{j=1}^{n}\varepsilon_k^{n-j-1}W^{j+1}, \quad k = 1, 2, \dots, n. \tag{3.0.4}$$

In the sequel, we shall need the following identities (see [5], [8]):

$$\begin{cases} W^k = \sum\limits_{j=1}^{n}\varepsilon_j^k P_j, \quad k = 1, 2, \dots, n, \\ P_k P_j = \delta_{kj}P_j, \quad k, j = 1, 2, \dots, n, \\ \sum\limits_{j=1}^{n} P_j = I, \end{cases} \tag{3.0.5}$$

where δ_{kj} is the Kronecker symbol. For every $a \in X$ we write $(K_a\varphi)(t) = a(t)\varphi(t)$.

Lemma 3.1. ([7]) *Suppose that $a \in X$ is fixed. Then for any $k, j \in \{1, 2, \dots, n\}$, there exists an element $b \in X$ such that $K_bX \subset X_k$ and $P_kK_aP_j = K_bP_j$, where $X_k := P_kX$.*

Lemma 3.2. ([9]) *Let $a \in X$ be fixed. Then for any $k, j \in \{1, 2, \dots, n\}$. we have*
$$P_kK_{a_{kj}} = K_{a_{kj}}P_j,$$

where $a_{kj}(t)$ are determined as follows

$$a_{kj}(t)=\frac{1}{n}\sum_{v=1}^{n}\varepsilon_{v+1}^{j-k}a(\omega^{v+1}(t)). \tag{3.0.6}$$

Lemma 3.3. *Let $\varphi\in X$. Then the following identity holds*

$$(SW\varphi)(z)=(WS\varphi)(z)-(S\varphi)\Big(\frac{\alpha}{\gamma}\Big), \tag{3.0.7}$$

where α,γ are the coefficients of $\omega(z)=\dfrac{\alpha z+\beta}{\gamma z+\delta}$.

Proof. We have

$$(SW\varphi)(z)=\frac{1}{\pi i}\int\limits_{\Gamma}\frac{\varphi(\omega(\tau))}{\tau-z}d\tau.$$

Put

$$\tau=\omega^{-1}(x)=\frac{\delta x-\beta}{-\gamma x+\alpha}\quad\text{(see [4])}.$$

Then

$$d\tau=\frac{1}{(-\gamma x+\alpha)^2}dx.$$

Therefore

$$\begin{aligned}
(SW\varphi)(z)&=\frac{1}{\pi i}\int\limits_{\Gamma}\frac{\varphi(x)\dfrac{1}{(\gamma x-\alpha)^2}}{\dfrac{\delta x-\beta}{-\gamma x+\alpha}-z}dx\\
&=\frac{1}{\pi i}\int\limits_{\Gamma}\frac{\varphi(x)}{(-\gamma x+\alpha)(\delta x-\beta-z(-\gamma x+\alpha))}dx\\
&=\frac{1}{\pi i}\int\limits_{\Gamma}\Bigg(\frac{1}{x-\dfrac{\alpha z+\beta}{\gamma z+\delta}}-\frac{1}{x-\dfrac{\alpha}{\gamma}}\Bigg)\varphi(x)dx\\
&=\frac{1}{\pi i}\int\limits_{\Gamma}\frac{\varphi(x)}{x-\omega(z)}-\frac{1}{\pi i}\int\limits_{\Gamma}\frac{\varphi(x)}{x-\dfrac{\alpha}{\gamma}}dx=(WS\varphi)(z)-(S\varphi)\Big(\frac{\alpha}{\gamma}\Big).
\end{aligned}$$

□

Lemma 3.4. *Let $\varphi\in X$. Then*

1) $(SW^k\varphi)(z)=(W^kS\varphi)(z)-(W^{k-1}S\varphi)\Big(\dfrac{\alpha}{\gamma}\Big)$, $k=1,2,\dots,n$, $W^0=I$.
2) $(P_kS\varphi)(z)=(SP_k\varphi)(z)+\dfrac{1}{\varepsilon_k-1}(SP_k\varphi)\Big(\dfrac{\alpha}{\gamma}\Big)$, $k=1,2,\dots,n-1$.

Proof. 1) We prove by induction on k. For $k = 1$, the assertion follows immediately from Lemma 3.3. Suppose 1) is true for $k = m$. For $k = m + 1$ we find

$$\begin{aligned}
&(SW^{m+1}\varphi)(z) = [SW(W^m\varphi)](z) \\
&= [WS(W^m\varphi)](z) - [S(W^m\varphi)]\Big(\frac{\alpha}{\gamma}\Big) = W[(SW^m\varphi)(z)] - (SW^m\varphi)\Big(\frac{\alpha}{\gamma}\Big) \\
&= W[(W^m S\varphi)(z) - (W^{m-1}S\varphi)(\frac{\alpha}{\gamma})] - [(W^m S\varphi)(\frac{\alpha}{\gamma}) - (W^{m-1}S\varphi)(\frac{\alpha}{\gamma})] \\
&= (W^{m+1}S\varphi)(z) - W[(W^{m-1}S\varphi)(\frac{\alpha}{\gamma})] - (W^m S\varphi)(\frac{\alpha}{\gamma}) + (W^{m-1}S\varphi)(\frac{\alpha}{\gamma}).
\end{aligned}$$

Hence $W[(W^{m-1}S\varphi)(\frac{\alpha}{\gamma})] = (W^{m-1}S\varphi)(\frac{\alpha}{\gamma})$, provided that $(W^{m-1}S\varphi)(\frac{\alpha}{\gamma})$ is a constant. Therefore

$$(SW^{m+1}\varphi)(z) = (W^{m+1}S\varphi)(z) - (W^m S\varphi)(\frac{\alpha}{\gamma}).$$

The first part of the lemma is proved.

2) Rewrite the equality in 1) in the form

$$(W^k S\varphi)(z) = (SW^k\varphi)(z) + (W^{k-1}S\varphi)(\frac{\alpha}{\gamma}).$$

We find

$$\begin{aligned}
(P_k S\varphi)(z) &= \frac{1}{n}\sum_{j=1}^{n} \varepsilon_k^{n-j-1}(W^{j+1}S\varphi)(z) \\
&= \frac{1}{n}\sum_{j=1}^{n} \varepsilon_k^{n-j-1}\left[(SW^{j+1}\varphi)(z) + (W^j S\varphi)(\frac{\alpha}{\gamma})\right] \\
&= \left[S\Big(\frac{1}{n}\sum_{j=1}^{n}\varepsilon_k^{n-j-1}W^{j+1}\Big)\varphi\right](z) + \frac{1}{\varepsilon_k}\left[\Big(\frac{1}{n}\sum_{j=1}^{n}\varepsilon_k^{n-j}W^j\Big)S\varphi\right](\frac{\alpha}{\gamma}) \\
&= (SP_k\varphi)(z) + \frac{1}{\varepsilon_k}(P_k S\varphi)(\frac{\alpha}{\gamma}). \qquad (3.0.8)
\end{aligned}$$

Substituting $z = \dfrac{\alpha}{\gamma}$ into formula (3.0.8), we get

$$(1 - \frac{1}{\varepsilon_k})(P_k S\varphi)(\frac{\alpha}{\gamma}) = (SP_k\varphi)(\frac{\alpha}{\gamma}). \qquad (3.0.9)$$

If $k \in \{1, 2, \dots, n-1\}$ then $\varepsilon_k \neq 1$. In this case, substituting (3.0.9) into (3.0.8) we obtain

$$(P_k S\varphi)(z) = (SP_k\varphi)(z) + \frac{1}{\varepsilon_k - 1}(SP_k)\varphi(\frac{\alpha}{\gamma}).$$

$\square$

3.1. **Reducing equation (3.0.1) to a system of singular integral equations.** Now we represent the equation (3.0.1) in the following form

$$a(t)\varphi(t) + b(t)(P_\ell S\varphi)(t) = f(t), \tag{3.1.1}$$

where $a, b, f \in X$ are given and $S, P_\ell,\ (1 \le \ell \le n-1)$ are the operators defined by (3.0.2), (3.0.3), (3.0.4). Suppose that the function $a(t)$ is of the following form

$$a(t) = \prod_{j=1}^{m} (t - \alpha_i)^{r_j} s(t),$$

where $\alpha_j \in \Gamma,\ r_j$ are positive integers $(j = 1, 2, \dots, m)$ and $s(t)$ is a non-vanishing function on Γ. Without loss generality we may assume that $s(t) = 1$. Our assumption means that $a(t)$ has some isolated zero-points with a finite multiplicity on Γ.

Lemma 3.5. *Let $\varphi \in X$. Then φ is a solution of* (3.1.1) *if and only if $\{\varphi_k = P_k\varphi,\ k = 1, 2, \dots, n\}$ is a solution of the following system*

$$a^*(t)\varphi_k(t) + b^*_{k\ell}(t)(S\varphi_\ell)(t) + \frac{b^*_{k\ell}(t)}{\varepsilon_\ell - 1}(S\varphi_\ell)(\frac{\alpha}{\gamma}) = f^*_k(t),\ \ k = 1, 2, \dots, n, \tag{3.1.2}$$

where

$$\begin{aligned} a^*(t) &= \prod_{j=1}^{n} a(\omega^{j+1}(t)), \\ b^*_{k\ell}(t) &= \frac{1}{n}\sum_{j=1}^{n} \varepsilon_{j+1}^{\ell-k} b(\omega^{j+1}(t)) \prod_{\substack{\mu=1\\ \mu\ne j}}^{n} a(\omega^{\mu+1}(t)), \\ f^*_k(t) &= \frac{1}{n}\sum_{j=1}^{n} \varepsilon_k^{n-1-j} f(\omega^{j+1}(t)) \prod_{\substack{\mu=1\\ \mu\ne j}}^{n} a(\omega^{\mu+1}(t)). \end{aligned} \tag{3.1.3}$$

Proof. Since $\varphi \in X$ is the solution of (3.1.1), we have

$$\begin{aligned} &\prod_{\mu=1}^{n} a(\omega^{\mu+1}(t))\varphi(t) + b(t) \prod_{\substack{\mu=1\\ \mu\ne n-1}}^{n} a(\omega^{\mu+1}(t))(P_\ell S\varphi)(t) \\ &\qquad = f(t) \prod_{\substack{\mu=1\\ \mu\ne n-1}}^{n} a(\omega^{\mu+1}(t)). \end{aligned} \tag{3.1.4}$$

Applying the projections $P_k,\ k = 1, 2, \dots , n$ to both sides of (3.1.4) and using Lemma 3.1, we obtain

$$a^*(t)(P_k\varphi)(t) + \Big[\frac{1}{n}\sum_{j=1}^{n} \varepsilon_{j+1}^{\ell-k} b(\omega^{j+1}(t)) \prod_{\substack{\mu=1\\ \mu\neq n-1}}^{n} a(\omega^{\mu+j+2}(t))\Big](P_\ell S\varphi)(t)$$

$$= \frac{1}{n}\sum_{j=1}^{n} \varepsilon_k^{n-j-1} f(\omega^{j+1}(t)) \prod_{\substack{\mu=1\\ \mu\neq n-1}}^{n} a(\omega^{\mu+j+2}(t)). \tag{3.1.5}$$

It is easy to see that

$$\prod_{\substack{\mu=1\\ \mu\neq n-1}}^{n} a(\omega^{\mu+j+2}(t)) \equiv \prod_{\substack{\mu=1\\ \mu\neq j}}^{n} a(\omega^{\mu+1}(t)) \quad \text{for any } j = 1, 2, \dots , n.$$

Hence, (3.1.5) is equivalent to the following system

$$a^*(t)(P_k\varphi)(t) + b_{k\ell}^*(t)(P_\ell S\varphi)(t) = f_k^*(t), \quad k = 1, 2, \dots , n. \tag{3.1.6}$$

By Lemma 3.4, we can rewrite system (3.1.6) in the form

$$a^*(t)(P_k\varphi)(t) + b_{k\ell}^*(t)(SP_\ell\varphi)(t) + \frac{b_{k\ell}^*(t)}{\varepsilon_\ell - 1}(SP_\ell\varphi)(\frac{\alpha}{\gamma}) = f_k^*(t), \quad k = 1, 2, \dots , n.$$

Thus $(P_1\varphi, P_2\varphi, \dots , P_n\varphi)$ is a solution of (3.1.2).

Conversely, suppose that there exists $\varphi \in X$ such that $(P_1\varphi, P_2\varphi, \dots , P_n\varphi)$ is a solution of (3.1.2). Summing by k from 1 to n, we obtain

$$a^*(t)\varphi(t) + \sum_{k=1}^{n} b_{k\ell}^*(t)\Big[(SP_\ell\varphi)(t) + \frac{1}{\varepsilon_\ell - 1}(SP_\ell\varphi)(\frac{\alpha}{\gamma})\Big] = \sum_{k=1}^{n} f_k^*(t). \tag{3.1.7}$$

From (3.1.3), we get

$$\begin{aligned}\sum_{k=1}^{n} b_{k\ell}^*(t) &= \sum_{k=1}^{n}\frac{1}{n}\sum_{j=1}^{n} \varepsilon_{j+1}^{\ell-k} b(\omega^{j+1}(t)) \prod_{\substack{\mu=1\\ \mu\neq j}}^{n} a(\omega^{\mu+1}(t))\\ &= \sum_{j=1}^{n}\Big[\frac{1}{n}\sum_{k=1}^{n} \varepsilon_{j+1}^{\ell-k}\Big] b(\omega^{j+1}(t)) \prod_{\substack{\mu=1\\ \mu\neq j}}^{n} a(\omega^{\mu+1}(t))\\ &= b(t) \prod_{\substack{\mu=1\\ \mu\neq n-1}}^{n} a(\omega^{\mu+1}(t)).\end{aligned}$$

Similarly,

$$\sum_{k=1}^{n} f_k^*(t) = f(t) \prod_{\substack{\mu=1\\ \mu\neq n-1}}^{n} a(\omega^{\mu+1}(t)).$$

Therefore, (3.1.7) is equivalent to the following:

$$a^*(t)\varphi(t)+b(t)\prod_{\substack{\mu=1\\ \mu\neq n-1}}^{n} a(\omega^{\mu+1}(t))\Big[(SP_\ell\varphi)(t)+\frac{1}{\varepsilon_\ell-1}(SP_\ell\varphi)(\frac{\alpha}{\gamma})\Big]$$
$$=f(t)\prod_{\substack{\mu=1\\ \mu\neq n-1}}^{n} a(\omega^{\mu+1}(t)).$$

This implies

$$a(t)\varphi(t)+b(t)(P_\ell S\varphi)(t)=f(t).$$

□

Lemma 3.6. *If* $(\varphi_1,\varphi_2,\dots,\varphi_n)$ *is a solution of the system* (3.1.2) *then* $(P_1\varphi_1,P_2\varphi_2,\dots,P_n\varphi_n)$ *is also its solution.*

Proof. Suppose $(\varphi_1,\varphi_2,\dots,\varphi_n)$ is a solution of the system (3.1.2). Applying the projections P_k to both sides of the k-th equation of (3.1.2) we get

$$a^*(t)(P_k\varphi_k)(t)+P_k\Big[b^*_{k\ell}(t)(S\varphi_\ell)(t)+\frac{b^*_{k\ell}(t)}{\varepsilon_\ell-1}(S\varphi_\ell)(\frac{\alpha}{\gamma})\Big]=P_k(f^*_k(t)). \tag{3.1.8}$$

Then

$$P_k(f^*_k(t))=\Big[\frac{1}{n}\sum_{m=1}^{n}\varepsilon_k^{n-m-1}W^{m+1}\Big]\Big[\frac{1}{n}\sum_{j=1}^{n}\varepsilon_k^{n-j-1}f(\omega^{j+1}(t))\prod_{\substack{\mu=1\\ \mu\neq j}}^{n} a(\omega^{\mu+1}(t))\Big]$$
$$=\frac{1}{n}\sum_{m=1}^{n}\varepsilon_k^{n}\Big[\frac{1}{n}\sum_{j=1}^{n}\varepsilon_k^{n-(m+j+2)}f(\omega^{m+j+2}(t))\prod_{\substack{\mu=1\\ \mu\neq j}}^{n} a(\omega^{m+j+2}(t))\Big]$$
$$=\frac{1}{n}\sum_{m=1}^{n}f^*_k(t)=f^*_k(t). \tag{3.1.9}$$

Moreover, by Lemma 3.2 we get

$$P_k b^*_{k\ell}(t)=b^*_{k\ell}(t)P_\ell. \tag{3.1.10}$$

Substituting (3.1.9), (3.1.10) into (3.1.8) we obtain

$$a^*(t)(P_k\varphi_k)(t)+b^*_{k\ell}(t)(P_\ell S\varphi_\ell)(t)+\frac{b^*_{k\ell}(t)}{\varepsilon_\ell-1}P_\ell\Big((S\varphi_\ell)(\frac{\alpha}{\gamma})\Big)=f^*_k(t). \tag{3.1.11}$$

On the other hand,

$$P_\ell\Big((S\varphi_\ell)(\frac{\alpha}{\gamma})\Big)=\frac{1}{n}\sum_{j=1}^{n}\varepsilon_\ell^{n-j-1}W^{j+1}\Big((S\varphi_\ell)(\frac{\alpha}{\gamma})\Big)$$
$$=\frac{1}{n}(S\varphi_\ell)(\frac{\alpha}{\gamma})\sum_{j=1}^{n}\varepsilon_\ell^{n-j-1}=0.$$

Hence, (3.1.11) is equivalent to the following equation

$$a^*(t)(P_k\varphi_k)(t) + b^*_{k\ell}(t)(SP_\ell\varphi_\ell)(t) + \frac{b^*_{k\ell}(t)}{\varepsilon_\ell - 1}(SP_\ell\varphi_\ell)(\frac{\alpha}{\gamma}) = f^*_k(t),\ k = 1, 2, \dots, n.$$

Thus $(P_1\varphi_1, P_2\varphi_2, \dots, P_n\varphi_n)$ is a solution of (3.1.2). □

3.2. **The solvability of equation (3.0.1).**

Theorem 3.1. *If* $(\varphi_1, \varphi_2, \dots, \varphi_n)$ *is a solution of* (3.1.2)*, then*

$$\varphi = \sum_{i=1}^{n} P_i\varphi_i,$$

is a solution of equation (3.1.1).

Proof. By Lemma 3.6, $(P_1\varphi_1, P_2\varphi_2, \dots, P_n\varphi_n)$ is also a solution of (3.1.2). Put

$$\varphi = \sum_{i=1}^{n} P_i\varphi_i.$$

It is clear that $P_k\varphi = P_k\varphi_k$. This means that $(P_1\varphi, P_2\varphi, \dots, P_n\varphi)$ is a solution of (3.1.2). By Lemma 3.5, φ is a solution of (3.1.1). □

We set

$$\Omega = \Big\{\omega^{-(\mu+1)}(\alpha_i),\ \mu = 1, 2, \dots, n,\ i = 1, 2, \dots, m\Big\},$$

$$\{g(t)\}_{(k,t_0)} = \frac{d^k g(t)}{dt^k}\bigg|_{t=t_0}.$$

Theorem 3.2. *The equation* (3.1.1) *has solutions in* X *if and only if the equation*

$$a^*(t)\varphi(t) + b^*_{\ell\ell}(t)(S\varphi)(t) + \frac{b^*_{\ell\ell}(t)}{\varepsilon_\ell - 1}(S\varphi)(\frac{\alpha}{\gamma}) = f^*_\ell(t), \tag{3.2.1}$$

has a solution $\varphi_0(t)$ *satisfying the following conditions*

$$\Big\{f^*_k(t) - b^*_{k\ell}(t)(S\varphi_0)(t) - \frac{b^*_{k\ell}(t)}{\varepsilon_\ell - 1}(S\varphi_0)(\frac{\alpha}{\gamma})\Big\}_{(j,t_i)} = 0, \quad k = 1, 2, \dots, n, \tag{3.2.2}$$

where $t_i \in \Omega,\ j = 0, 1, \dots, r_i,\ r_i$ *are the multiplicities of zero-points* $\alpha_i,\ i = 1, 2, \dots, m$.

Proof. Since $\varphi \in X$ is a solution of (3.1.1), it follows from Lemma 3.5 that the system (3.1.2) has a solution $(P_1\varphi, P_2\varphi, \dots, P_n\varphi)$. This means that $P_\ell\varphi$ is the solution of the ℓ-th equation of (3.1.2), i.e. the equation (3.2.1). Moreover, for any $k = 1, 2, \dots, n$, φ_k is a solution of the equation

$$a^*(t)(P_k\varphi)(t) = f^*_k(t) - b^*_{k\ell}(t)(SP_\ell\varphi)(t) - \frac{b^*_{k\ell}(t)}{\varepsilon_\ell - 1}(SP_\ell\varphi)(\frac{\alpha}{\gamma}). \tag{3.2.3}$$

The left side of (3.2.3) is the function having zero-point of order r_i at $t_i = \omega^{-(\mu+1)}(\alpha_i) \in \Omega$. Hence condition (3.2.2) is necessary.

Conversely, if $\varphi_0(t)$ is a solution of (3.2.1) satisfying (3.2.2), then it is easy to see that (3.1.2) has a solution $(\varphi_1, \varphi_2, \dots, \varphi_n)$ defined by the formulas

$$\varphi_\ell(t) = \varphi_0(t),$$

$$\varphi_k(t) = \frac{f_k^*(t) - b_{k\ell}^*(t)(S\varphi_\ell)(t) - \dfrac{b_{k\ell}^*(t)}{\varepsilon_\ell - 1}(S\varphi_\ell)(\dfrac{\alpha}{\gamma})}{a^*(t)}, \tag{3.2.4}$$

$$k = 1, 2, \dots, n,\ k \neq \ell.$$

According to Theorem 3.1, $\varphi = \sum\limits_{i=1}^{n} P_i \varphi_j$ is the solution of (3.1.1). □

We set

$$D^+ = \{z \in \mathbb{C} : |z| < 1\}, \qquad D^- = \{z \in \mathbb{C} : |z| > 1\}.$$

Denote by $H^+(D^+)$, $H^-(D^-)$ the sets of the analytic functions in D^+ and D^-, respectively.

Corollary 3.1. *Suppose that the function* $M(t) = \dfrac{b_{\ell\ell}^*(t)}{a^*(t) + b_{\ell\ell}^*(t)}$ *is an analytic continuation on* D^+. *Then the equation* (3.1.1) *is solvable in a closed form.*

Proof. Consider the ℓ-th equation of (3.1.2):

$$a^*(t)\varphi(t) + b_{\ell\ell}^*(t)(S\varphi_\ell)(t) + \frac{b_{\ell\ell}^*(t)}{\varepsilon_\ell - 1}(S\varphi_\ell)(\frac{\alpha}{\gamma}) = f_\ell^*(t). \tag{3.2.5}$$

Put

$$\Phi_\ell(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{\varphi_\ell(\tau)}{\tau - z}d\tau, \quad z \in \mathbb{C} \setminus \Gamma.$$

According to the Sokhotski formula (see [1]), we have

$$\varphi_\ell(t) = \Phi_\ell^+(t) - \Phi_\ell^-(t),$$
$$(S\varphi_\ell)(t) = \Phi_\ell^+(t) + \Phi_\ell^-(t).$$

Moreover, Remark 2.7 implies that $\dfrac{\alpha}{\gamma} \in D^-$. Therefore $(S\varphi_\ell)(\dfrac{\alpha}{\gamma}) = \Phi_\ell^-(\dfrac{\alpha}{\gamma})$. Hence equation (3.2.5) is reduced to the following boundary value problem

$$\Phi_\ell^+(t) + \frac{b_{\ell\ell}^*(t)}{a^*(t) + b_{\ell\ell}^*(t)} \frac{\Phi_\ell^-(\frac{\alpha}{\gamma})}{\varepsilon_\ell - 1} = \frac{a^*(t) - b_{\ell\ell}^*(t)}{a^*(t) + b_{\ell\ell}^*(t)}\Phi_\ell^-(t) + \frac{f_\ell^*(t)}{a^*(t) + b_{\ell\ell}^*(t)}. \tag{3.2.6}$$

From the assumption on $M(t)$ it follows that (3.2.6) is the Riemann boundary value problem for analytic functions. Denote by $\big(\Psi_\ell^+(z), \Psi_\ell^-(z)\big)$ the solution of (3.2.6). We have

$$\Phi_\ell^-(z) = \Psi_\ell^-(z),$$

$$\Phi_\ell^+(z) + \frac{b_{\ell\ell}^*(t)}{a^*(t) + b_{\ell\ell}^*(t)} \frac{\Phi_\ell^-(\frac{a}{c})}{\varepsilon_\ell - 1} = \Psi_\ell^+(z).$$

Hence the solution of (3.2.5) is of the form

$$\varphi_\ell(t) = \Phi_\ell^+(t) - \Phi_\ell^-(t) = \Psi_\ell^+(t) - \frac{b^*_{\ell\ell}(t)}{a^*(t) + b^*_{\ell\ell}(t)} \frac{\Psi_\ell^-(\frac{a}{c})}{\varepsilon_\ell - 1} - \Psi_\ell^-(t). \tag{3.2.7}$$

□

From Theorem 3.1 and 3.2 we conclude that

(i) If neither equation (3.2.6) has solution nor solutions $\varphi_\ell(t)$ of the form (3.2.7) do satisfy condition (3.2.2), then equation (3.1.1) has no solutions.

(ii) If there exists $\varphi_\ell(t)$ of the form (3.2.7) satisfying conditions (3.2.2), then equation (3.1.1) is solvable in a closed form. Solutions of (3.1.1) are given by the following formula

$$\varphi(t) = \sum_{k=1}^{n} (P_k \varphi_k)(t),$$

where $\varphi_\ell(t)$ is defined by (3.2.7) and $\varphi_k(t)$, $1 \le k \ne \ell \le n$ are defined by (3.2.4).

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