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BIFURCATION OF SOLUTIONS FOR AN ELLIPTIC DEGENERATE PROBLEM

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Abstract. We investigate the existence and bifurcation of solutions of a model nonlinear degenerate elliptic differential equation: $-x^r \Delta u = \lambda u + |u|^{p-1} u$ in $(0,1)$; $u(0) = u(1) = 0$. This model is related to a simplified version of the nonlinear Wheeler-De Witt equation as it appears in quantum cosmological models, see [8, 9, 10].

1. Introduction

Bifurcation problems play a very important role in different areas of applied mathematics and have been intensively studied in the literature.

In this work, we shall study the existence and bifurcation of solutions for the elliptic degenerate equation

$$\begin{cases} -x^r \Delta u = \lambda u + |u|^{p-1} u \text{ in } (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where r is a real positive number, $p > 1$, λ is a real parameter and $u = u(x)$ is defined as a continuous function on $[0,1]$.

The equation (1.1) is related to a simplified version of the Wheeler-De Witt equation as it appears in quantum cosmological models. Here nonlinear version looks like

$$\frac{1}{x^2}\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} - \frac{p}{x}\frac{\partial \psi}{\partial x} + x^2\psi - k^2 x^4 \psi + g x^{q-2} |\psi|^s \psi = 0, \tag{1.2}$$

where $y \in \mathbb{R}$ is a scalar field, $x \in (0,R)$, $R > 0$, a scale factor, $p \in \mathbb{R}$, $k^2 > 0$, $g \in \mathbb{R}$, $s > 0$, and $q \geq sp/2$ are given constants (p reflects the factor-ordering ambiguity and k^2 is a cosmological constant). Finally $\psi : (0,R) \times \mathbb{R} \to \mathbb{C}$ is the so-called wave function of the universe for the minisuperspace model (see [8, 9, 10] for more details).

In [6, 7], the above equation was studied as an evolution equation in which y is treated as the evolving time. Stationary solutions of this equation are found in

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[2] by using variational methods. By looking for stationary solutions in the form

$$\psi(x,y) = e^{i\mu y}u(x), \quad \mu \in \mathbb{R},\ u \neq 0, \tag{1.3}$$

and then performing the change of variables $v(x) = x^{p/2}u(x)$, we transform equation (1.2) to the form

$$-x^2\frac{d^2v}{dx^2} + \frac{p}{4}(p-2)v + Vv + gx^{q-sp/2}v = \lambda v, \tag{1.4}$$

where $\lambda = \mu^2 \geq 0$ and $V(x) = x^4 - k^2x^6$. Then equation (1.1) is a simplified version of (1.4).

The case $r = 2$ was studied in [1] by using O.D.E techniques. It was proved that there exists an infinite number of connected branches of solutions which bifurcate from the bottom of the essential spectrum of the corresponding linear operator.

In this paper we shall study (1.1) when $r > 0$. More precisely, we prove that there exists a sequence $(\lambda_n) \subset \mathbb{R}$, $\lambda_n > 2p+4$, $n \in \mathbb{N}$, such that, for any $\overline{\lambda} \in (\lambda_n)$, $(\overline{\lambda}, 0)$ is a bifurcation point of the equation (1.1).

The paper is organized as follows. In Section 2, we study the eigenvalue problem. In Section 3, we reduce the problem to a finite dimensional one, using the Lyapunov-Schmidt procedure and the Banach contraction principle. Finally, in Section 4, using the topological degree theory we prove our main result on the bifurcation of a real sequence of eigenvalues.

2. Eigenvalue problem

Consider $X = H_0^1(0,1) \cap H^2(0,1)$, $Y = L^2(0,1)$ as real Banach spaces and define the mappings $f : [0,1] \to \mathbb{R}$, $T, H, K : X \to Y$, $L : \mathbb{R} \times X \to Y$ by

$$\begin{aligned} &f(x) = -(x^r + 1), \\ &Tu = fu'', \\ &H(u) = |u|^{p-1}u, \\ &L(\lambda, u) = \lambda u, \\ &K(u) = -u'', \end{aligned}$$

for any $x \in [0,1]$, $\lambda \in \mathbb{R}$ and $u \in X$. We can write (1.1) in the form

$$Tu = L(\lambda, u) + H(u) + K(u), \quad (\lambda, u) \in \mathbb{R} \times X. \tag{2.1}$$

By definition, a characteristic value of the pair (T, L) is a number $\overline{\lambda} \in \mathbb{R}$ such that $Tv = L(\overline{\lambda}, v)$ for some $v \in X$, $v \neq 0$. The following proposition is the main result of this section.

Proposition 2.1. *The pair (T, L) has a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of real non null characteristic values tending to $+\infty$, and for any $n \in \mathbb{N}$, we have*

$$\dim\ker(T - L(\lambda_n, .)) = 1.$$

For the proof of the above proposition, we need some preliminary information about the linear operator T. Let us define the mappings

$$b : Y \times Y \to \mathbb{R}$$

by

$$b(u,v) = \int_0^1 \frac{u(x)\,v(x)}{(x^r+1)}dx \tag{2.2}$$

for any $(u,v) \in Y \times Y$, and

$$a : H_0^1(0,1) \times H_0^1(0,1) \to \mathbb{R}$$

by

$$a(u,v) = \int_0^1 u'(x)\,v'(x)\,dx + \int_0^1 \frac{u(x)\,v(x)}{(x^r+1)}dx \tag{2.3}$$

for any $(u,v) \in H_0^1(0,1) \times H_0^1(0,1)$.

Lemma 2.1. (i) *The mapping b is an inner product on Y.*

(ii) *The mapping a is a bilinear symmetric coercive continuous functional on the product space $H_0^1(0,1) \times H_0^1(0,1)$.*

Proof. (i) The bilinearity and the symmetry of a are clear. Let show that a is a coercive continuous functional. For $(u,v) \in H_0^1(0,1) \times H_0^1(0,1)$, we have

$$\begin{aligned} |a(u,v)| &= \left| \int_0^1 u'(x)\,v'(x)\,dx + \int_0^1 \frac{u(x)\,v(x)}{(x^r+1)}dx \right| \\ &\leq \left| \int_0^1 u'(x)\,v'(x)\,dx \right| + \left| \int_0^1 u(x)\,v(x)\,dx \right|. \end{aligned}$$

Hence, by Hölder's inequality, we get

$$|a(u,v)| \leq |u'|_{L^2}\,|v'|_{L^2} + |u|_{L^2}\,|v|_{L^2}\,.$$

It follows that

$$|a(u,v)| \leq \left(|u'|_{L^2} + |u|_{L^2}\right)\left(|v'|_{L^2} + |v|_{L^2}\right).$$

Noting that $|.|_0^1$ is the norm in $H_0^1(0,1)$, we have

$$|a(u,v)| \leq |u|_0^1\,|v|_0^1\,. \tag{2.4}$$

Therefore a is continuous. On the other hand, we have

$$|a(u,u)| = \left| \int_0^1 \left(u'(x)\right)^2 dx + \int_0^1 \frac{(u(x))^2}{(x^r+1)}dx \right|.$$

It follows that

$$|a(u,u)| \geq \left(|u'|^2_{L^2} + \frac{1}{2} |u|^2_{L^2} \right).$$

Thus

$$|a(u,u)| \geq \frac{1}{2} \left(|u|^1_0 \right)^2. \tag{2.5}$$

Hence a is coercive. □

In the sequel, $\langle , \rangle$ denotes the inner product b and X_1 is the space $L^2(0,1)$ equipped with b. Let A be the unbounded operator on X with domain $D(A)$ defined by

$$\begin{aligned} &\text{(i) } \langle Au, v\rangle = a(u,v) \text{ for any } u \in D(A) \text{ and } v \in H^1_0(0,1); \\ &\text{(ii) } D(A) = \{u \in H^1_0(0,1) \,|\, v \longmapsto a(u,v) \\ &\qquad \text{is continuous on } H^1_0(0,1) \text{ for the } X_1 \text{ topology}\}. \end{aligned} \tag{2.6}$$

Finally, let B be the operator defined by

$$\begin{aligned} &Bu = fu'' + u \quad \text{for any } u \in D(B) \\ &D(B) = H^1_0(0,1). \end{aligned} \tag{2.7}$$

Lemma 2.2. *The following assertions hold:*

(i) *The injection of $H^1_0(0,1)$ in X_1 is compact.*

(ii) *$H^1_0(0,1)$ is dense in X_1.*

(iii) *$Au = Bu$ for any $u \in D(A)$.*

(iv) *A is a self-adjoint operator.*

Proof. We have

$$|u|^2_{X_1} = \int_0^1 \frac{u^2(x)}{(x^r+1)} dx.$$

Hence

$$\frac{1}{2} \int_0^1 u^2(x)\, dx \leq |u|^2_{X_1} \leq \int_0^1 u^2(x)\, dx,$$

and then

$$\frac{\sqrt{2}}{2} |u|_{L^2} \leq |u|_{X_1} \leq |u|_{L^2}. \tag{2.8}$$

It follows that the norms of X_1 and $L^2(0,1)$ are equivalent. Since I=$[0,1]$ is bounded, we know that $H^1_0(0,1)$ is dense in $L^2(0,1)$ and the injection of $H^1_0(0,1)$

in $L^2(0,1)$ is compact. Thus (i) and (ii) hold by (2.8). For any $u \in D(A)$ and $v \in H_0^1(0,1)$ we have

$$\langle Au, v\rangle = a(u,v) = \int_0^1 u'(x)\, v'(x)\, dx + \int_0^1 \frac{u(x)\, v(x)}{(x^r+1)} dx \tag{2.9}$$

On the other hand, we have

$$\begin{aligned} \langle Bu, v\rangle &= \left(fu^{"} + u, v\right) \\ &= \int_0^1 \frac{-(x^r+1)\, u^{"}(x) + u(x)}{x^r+1} v(x)\, dx \\ &= \int_0^1 \left(-u^{"}(x)\, v(x) + \frac{u(x)\, v(x)}{x^r+1}\right) dx. \end{aligned}$$

Thus, using an integration by parts argument, we have

$$\langle Bu, v\rangle = -\left[u'(x)\, v(x)\right]_0^1 + \int_0^1 u'(x)\, v'(x)\, dx + \int_0^1 \frac{u(x)\, v(x)}{x^r+1} dx.$$

Hence

$$\langle Bu, v\rangle = \int_0^1 u'(x)\, v'(x)\, dx + \int_0^1 \frac{u(x)\, v(x)}{x^r+1} dx. \tag{2.10}$$

Then by (2.9) and (2.10) we infer that

$$\langle Bu, v\rangle = \langle Au, v\rangle \text{ for any } v \in H_0^1(0,1).$$

Since $H_0^1(0,1)$ is dense in X_1, it follows that

$$\langle Bu, v\rangle = \langle Au, v\rangle \text{ for any } v \in X_1.$$

Therefore

$$Bu = Au \text{ for any } u \in D(A).$$

Finally, (iv) is immediate since $a(.,.)$ is symmetric. This ends the proof. □

Lemma 2.3. [4] *Let $V \subset H$ be two Hilbert spaces, such that, the injection of V in H is compact, V is dense in H and $a(u,v)$ is a sesquilinear hermitian continuous form on $V \times V$ satisfying*

$$a(u,u) \geq \alpha\, |u|_V^2 \quad \text{for any } u \in V$$

for some $\alpha > 0$. Let A be the operator defined by (2.6). Then, its spectrum $\sigma(A)$ is punctual,

$$\sigma(A) = \sigma_p(A) = \{\beta_n\}_{n\in\mathbb{N}}$$

and

$$0 < \alpha \leq \beta_n \to +\infty \qquad as\ n \to +\infty.$$

Proof of Proposition 2.1.

By (2.5), Lemmas 2.2 and 2.3, the operator A possesses a sequence $(\beta_n)_n$ of eigenvalues such that

$$\left\{ \begin{array}{l} \beta_n \geq \dfrac{1}{2} \\ \lim\limits_{n\to\infty} \beta_n = +\infty. \end{array} \right. \tag{2.11}$$

If we denote the eigenvectors of A in $H_0^1(0,1)$ by w_n, then

$$Aw_n = \beta_n w_n.$$

Hence

$$fw_n'' + w_n = \beta_n w_n.$$

So

$$w_n \in \mathsf{C}^{\infty}([0,1], \mathbb{R}). \tag{2.12}$$

Therefore $w_n \in X$. and

$$fw_n'' = (\beta_n - 1) w_n.$$

Consequently,

$$Tw_n = (\beta_n - 1) w_n.$$

Put $\lambda_n = \beta_n - 1$. Since $L(\lambda, u) = \lambda u$, we have

$$[T - L(\lambda, .)] w_n = 0.$$

Therefore the pair (T, L) possesses a sequence $\lambda_n \geq -\dfrac{1}{2}$ of real characteristics values tending to $+\infty$.

Let us show that $\lambda_n \neq 0$ for any $n \in \mathbb{N}$. Suppose that there exists $n \in \mathbb{N}$ such that $\lambda_n = 0$. Then there exists a non zero $v \in X$ such that

$$-(x^r + 1) v" = 0.$$

Since by (2.12), $v \in \mathsf{C}^{\infty}([0,1], \mathbb{R})$, there exists $(a, b) \in \mathbb{R}^2$ such that $v(x) = ax + b$ for any $x \in [0,1]$. As $v(0) = v(1) = 0$, we have $v = 0$. Hence $\lambda_n \neq 0$.

Finally, let us show that $\dim\ker(T - L(\lambda_n, .)) = 1$. If u and v be two eigenfunctions associated to the characteristic value $\lambda = \lambda_n$, then

$$-(x^r + 1)\lambda v u'' = -(x^r + 1)\lambda u v",$$

but $\lambda \neq 0$, hence $vu'' = uv"$. This implies that for any $x \in [0,1]$,

$$\int_0^x \left(u''v - v"u\right)(y) dy = 0.$$

Thus

$$\left[u'(y)v(y)\right]_0^x - \left[u(y)v'(y)\right]_0^x = 0;$$

but $u(0) = u(1) = 0$, so $u'(x)\, v(x) - v'(x)\, u(x) = 0$.

Hence $\dfrac{u'(x)\, v(x) - v'(x)\, u(x)}{v^2(x)} = 0$, and then $\left(\dfrac{u}{v}\right)'(x) = 0$. Therefore we can find $c \in \mathbb{R}$ such that $u = cv$; so $\dim \ker (T - L(\lambda_n, .)) = 1$. $\square$

3. Reduction of the problem to a finite dimensional one

In this section, using the Lyapunov-Schmidt procedure and the Banach contraction principle, we shall reduce the problem of solving (1.1) to that of solving a system in a finite dimension.

The following lemmas play an important role in the sequel.

Lemma 3.1. *There exists a constant $\rho > 0$ such that*

$$|H(u) - H(v)|_Y \leq p\,|u - v|_X \quad \text{for all } u, v \in B_X(0, \rho). \tag{3.1}$$

where $B_X(0, \rho)$ denotes the open ball with the center at the origin in X and the radius $\rho > 0$.

Proof. There exists a constant $c > 0$ such that

$$|u|_{L^\infty(0,1)} \leq c\,|u|_{H^1(0,1)} \quad \text{for all } \ u \in H^1(0,1). \tag{3.2}$$

Let ρ be a constant in the interval $\left(0, \dfrac{1}{c}\right)$. Then for any $u \in B_X(0, \rho)$ we have

$$|u|_{L^\infty(0,1)} \leq c\,|u|_{H^1(0,1)} \leq c\,|u|_X < 1.$$

Hence

$$|u(x)| < 1 \tag{3.3}$$

for any $x \in [0, 1]$ and for any $u \in B_X(0, \rho)$. Now let $u, v \in B_X(0, \rho)$, $x \in (0, 1)$. Suppose for instance that $u(x) \leq v(x)$. Then by (3.3) and the fact that the function $x \mapsto px - |x|^{p-1}x$ is increasing on the interval $[-1, 1]$, we have

$$pu(x) - |u(x)|^{p-1} u(x) \leq pv(x) - |v(x)|^{p-1} v(x).$$

Hence

$$|v(x)|^{p-1} v(x) - |u(x)|^{p-1} u(x) \leq p\,(v(x) - u(x)).$$

As $u(x) \leq v(x)$, we obtain

$$\left| |v(x)|^{p-1} v(x) - |u(x)|^{p-1} u(x) \right| \leq p\,|(v(x) - u(x))|.$$

We have the same formula if $v(x) \leq u(x)$. It follows that

$$\left| |v(x)|^{p-1} v(x) - |u(x)|^{p-1} u(x) \right|^2 \leq p^2\,|(v(x) - u(x))|^2 \quad \text{for any } \ x \in (0, 1),$$

and then

$$\int_0^1 \left| |v(x)|^{p-1} v(x) - |u(x)|^{p-1} u(x) \right|^2 dx \leq p^2 \int_0^1 |(v(x) - u(x))|^2 \, dx.$$

Therefore

$$\left| |v|^{p-1} v - |u|^{p-1} u \right|_Y \leq p \, |v-u|_Y \leq p \, |v-u|_X \,,$$

which proves (ii). □

Let us regard $\overline{\lambda} = \lambda_n$ as a fixed characteristic value of the pair (T, L) and we set

$$\begin{aligned} X_0 &= \ker \left(T - L\left(\overline{\lambda}, .\right)\right) = [v] \\ X_1 &= \{x \in X | \, \langle x, v \rangle = 0\} \\ Y_1 &= \{y \in Y | \, \langle y, v \rangle = 0\} \, . \end{aligned} \tag{3.4}$$

It can be seen that $X = X_0 \bigoplus X_1$, $Y = X_0 \bigoplus Y_1$ and the restriction of the mapping $T - L\left(\overline{\lambda}, .\right)$ is an one-to-one linear continuous mapping from X_1 onto Y_1. In the sequel, N will denote the map $T - L\left(\overline{\lambda}, .\right)$.

Lemma 3.2. *The mapping N is Fredholm with nullity zero and index zero.*

Proof. Since the injection of X in Y is compact, the operator $L\left(\overline{\lambda}, .\right)$ is compact from X into Y. Then it suffices to prove the lemma for the operator T (see for instance [3]). Let $u \in \ker T$. Then $-(x^r + 1) \, u''(x) = 0$ for any $x \in (0, 1)$. Hence $u'' = 0$. Since $u(0) = u(1) = 0$, we have $u = 0$. Therefore $\ker T = \{0\}$. Let $v \in Y$ and $u \in X$ be such that $Tu = v$. Then $fu'' = v$. Hence $u'' = \frac{v}{f}$, and then $u'(x) - u'(0) = \int\limits_0^x \frac{v(y)}{y^r + 1} dy$ for any x in $[0, 1]$. It follows that

$$u(x) - u'(0) \, x = -\int_0^x \left[\int_0^\tau \frac{v(y)}{y^r + 1} dy \right] d\tau.$$

Then it suffices to take

$$u(x) = \alpha x - \int_0^x \left[\int_0^\tau \frac{v(y)}{y^r + 1} dy \right] d\tau,$$

where

$$\alpha = \int_0^1 \left[\int_0^\tau \frac{v(y)}{y^r + 1} dy \right] d\tau.$$

Therefore T is an one-to-one on mapping from X onto Y and the lemma follows. □

We denote by S the inverse of the operator N and by $\|S\|$ the norm of the map S. Then we have

Lemma 3.3. *If* $|\lambda| > 2$, *then*

$$\|S\| < \frac{1}{|\lambda| - 2}.$$

Proof. Let $u \in X$. Then

$$\|T\|_Y = \left|-(x^r + 1)\, u''\right|_Y \leq 2\left|u''\right|_Y \leq 2\,|u|_X.$$

It follows that

$$\|T\| \leq 2. \tag{3.5}$$

In the other hand, we have

$$\begin{aligned} S &= (T - L(\lambda, .))^{-1} \\ &= -\frac{1}{\lambda}\left(I - \frac{1}{\lambda}T\right)^{-1} \\ &= -\frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{1}{\lambda^n}T^n. \end{aligned}$$

It follows that

$$\|S\| < \frac{1}{|\lambda| - 2}. \tag{3.6}$$

□

Next, we put $M(u) = H(u) + K(u)$ for any $u \in X$ and we define the mappings $P_X : X \to X_0$, $Q_X : X \to X_1$, $P_Y : Y \to X_0$ and $Q_Y : Y \to Y_1$ by

$$\begin{aligned} P_X(x) &= \langle x, v\rangle\, v, \quad Q_X(x) = x - P_X(x) \quad x \in X, \\ P_Y(y) &= \langle y, v\rangle\, v, \quad Q_Y(y) = y - P_Y(y) \quad y \in Y. \end{aligned}$$

Evidently, P_X, P_Y, Q_X and Q_Y are projectors of X to X_0, X to X_1, Y to X_0, and Y to Y_1, respectively. Now, we observe that the totality of solutions of equation (1.1) can be obtained by solving the two following equations

$$\begin{aligned} Q_Y(N(\lambda, u) - M(u)) &= 0, \\ \langle N(\lambda, u) - M(u), v\rangle &= 0. \end{aligned} \tag{3.7}$$

Since any $u \in X$ can be written as $u = \varepsilon v + \omega$ for some $\omega \in X_1$ and $\varepsilon \in \mathbb{R}$, we conclude that the system of equation (3.7) is equivalent to the system

$$\begin{cases} Q_Y(T(\varepsilon v + \omega) - L(\lambda, \varepsilon v + \omega) - M(\varepsilon v + \omega)) = 0 \\ \langle T(\varepsilon v + \omega) - L(\lambda, \varepsilon v + \omega) - M(\varepsilon v + \omega), v\rangle = 0, \end{cases} \tag{3.8}$$

where $\lambda \in \Lambda$, $\varepsilon \in \mathbb{R}$, $\omega \in X_1$ are the unknown.

Let $I_1 = [-1, 1]$, $D = B_X(0, \rho)$ where the radius $\rho > 0$ is given by (3.1). Set $D_1 = Q_X(D)$. Choosing D smaller if necessary, we may assume that $D_1 = D_1(0, r_1)$ is the open ball with center at the origin in X_1, and radius $r_1 > 0$. Let

$U_1 =]-r, r[$ be an open interval, where $r \in \mathbb{R}^+$, $r < \frac{r_1}{2}$, such that $\varepsilon v \in P_X(D)$ for all $\varepsilon \in U_1$. We define the mapping $G_\pm : I_1 \times U_1 \times \overline{D_1} \to X_1$ by

$$G(\alpha, \varepsilon, \omega) = -SQ_Y \left(\pm |\alpha| T(\varepsilon v + \omega) - (1 \pm |\alpha|) M(\varepsilon v + \omega) \right). \tag{3.9}$$

In the sequel, we suppose that

$$\overline{\lambda} = \lambda_n > 2p + 6. \tag{3.10}$$

Proposition 3.1. *Let I_1, U_1, D_1 be as above. Then there exist neighborhoods I_2 of zero in $\mathbb{R}$, $I_2 \subset I_1$, D_2 of the origin in X_1, $D_2 \subset D_1$, such that for any $\left(|\alpha|^{p-1}, |\alpha| x\right) \in I_2 \times U_1$ one can find a point $\omega\left(|\alpha|^{p-1}, |\alpha| x\right) \in D_2$ satisfying:*

1) $G\left(|\alpha|^{p-1}, |\alpha| x, \omega\left(|\alpha|^{p-1}, |\alpha| x\right)\right) = \omega\left(|\alpha|^{p-1}, |\alpha| x\right)$.

2) *There exists a constant $c > 0$ such that for any $|\alpha|^{p-1} \in I_2$, $|\alpha| x$, $|\alpha| y \in U_2$ we have*

$$\left|\omega\left(|\alpha|^{p-1}, |\alpha| x\right) - \omega\left(|\alpha|^{p-1}, |\alpha| y\right)\right| \leq c |\alpha| |x - y|,$$

(consequently, for any fixed $\alpha \in I_2$, $\omega\left(|\alpha|^{p-1}, |\alpha| .\right)$ is a continuous mapping with respect to $x \in U_1$).

3) $\omega\left(|\alpha|^{p-1}, 0\right) = 0$ *for any fixed $\alpha \in I_2$.*

Proof. 1) Set

$$\begin{aligned}
c_0 &= \frac{r_1 (1 - 2\|S\|)}{(r_1 + r)\|S\| (\|T\| + 2p)}, \\
t_0 &= \left[\inf\left(1, \frac{1}{2} c_0\right)\right]^{1/(p-1)}, \\
D_2 &= t_0 D_1, \\
I_2 &= t_0 I_1.
\end{aligned}$$

We claim that $G\left(|\alpha|^{p-1}, |\alpha| x, .\right)$ is a contraction mapping, and it maps $\overline{D_2}$ into itself. To prove this, let ω^1, $\omega^2 \in D_2$, $\alpha \in I_2$ and $\omega^1 = t_0 \omega'^1$, $\omega^2 = t_0 \omega'^2$, $\alpha = t_0 \alpha'$, $\left(\omega'^1, \omega'^2, \alpha'\right) \in D_1 \times D_1 \times I_1$. We have

$$\begin{aligned}
&\left|G\left(|\alpha|^{p-1}, |\alpha| x, \omega^1\right) - G\left(|\alpha|^{p-1}, |\alpha| x, \omega^2\right)\right|_X \\
&= | - SQ_Y\left(\pm |\alpha|^{p-1} T\left(|\alpha| xv + \omega^1\right) - \left(1 \pm |\alpha|^{p-1}\right) M\left(|\alpha| xv + \omega^1\right)\right)_X + \\
&\quad SQ_Y\left(\pm |\alpha|^{p-1} T\left(|\alpha| xv + \omega^2\right) - \left(1 \pm |\alpha|^{p-1}\right) M\left(|\alpha| xv + \omega^2\right)\right)_X | \\
&\leq \|S\| |\alpha|^{p-1} \|T\| \left|\omega^1 - \omega^2\right|_X + \\
&\quad \|S\| \left(1 + |\alpha|^{p-1}\right) \left|Q_Y\left[H\left(t_0\left(\left|\alpha'\right| xv + \omega'^1\right)\right) - H\left(t_0\left(\left|\alpha'\right| xv + \omega'^2\right)\right)\right]\right|_Y + \\
&\quad \|S\| \left(1 + |\alpha|^{p-1}\right) \left|Q_Y K\left(\omega^1 - \omega^2\right)\right|_Y.
\end{aligned}$$

Hence by Lemma 3.1 we have

$$\begin{aligned}
&\left|G\left(|\alpha|^{p-1},|\alpha|\,x,\omega^1\right)-G\left(|\alpha|^{p-1},|\alpha|\,x,\omega^2\right)\right|_x \\
\leq\ & \|S\|\,|\alpha|^{p-1}\,\|T\|\left|\omega^1-\omega^2\right|_X+\|S\|\left(1+|\alpha|^{p-1}\right)t_0^p p\left|\left|\omega'^1-\omega'^2\right|_X\right| \\
&+2\,\|S\|\left|\omega^1-\omega^2\right|_X \\
\leq\ & \left\{t_0^{p-1}\,\|S\|\,(\|T\|+2p)+2\,\|S\|\right\}\left|\omega^1-\omega^2\right|_X.
\end{aligned}$$

If we set $G_1\left(t_0\right)=t_0^{p-1}\,\|S\|\,(\|T\|+2p)+2\,\|S\|$, then

$$\begin{aligned}
G_1\left(t_0\right) &\leq \frac{1}{2}c_0\,\|S\|\,(\|T\|+2p)+2\,\|S\| \\
&\leq \frac{r_1\left(1-2\,\|S\|\right)}{2\left(r_1+r\right)}+2\,\|S\| \\
&< 1.
\end{aligned}$$

Therefore

$$\left|G\left(|\alpha|^{p-1},|\alpha|\,x,\omega^1\right)-G\left(|\alpha|^{p-1},|\alpha|\,x,\omega^2\right)\right|_X\leq G_1\left(t_0\right)\left|\omega^1-\omega^2\right|_X, \tag{3.11}$$

with $G_1\left(t_0\right)<1$. On the other hand, we have

$$\begin{aligned}
&\left|G\left(|\alpha|^{p-1},|\alpha|\,x,\omega^1\right)\right|_X \\
=\ & |-SQ_Y\left(+|\alpha|^{p-1}\,T\left(|\alpha|\,xv+\omega^1\right)-\left(1+|\alpha|^{p-1}\right)H\left(|\alpha|\,xv+\omega^1\right)\right)+ \\
& SQ_Y\left(\left(1+|\alpha|^{p-1}\right)K\left(|\alpha|\,xv+\omega^2\right)\right)|_X \\
\leq\ & t_0^p\left(\|S\|\,\|T\|+2\,\|S\|\,p\right)\left(r_1+r\right)+2\,\|S\|\,t_0\left(r_1+r\right) \\
\leq\ & t_0\left(\frac{1}{2}+2\,\|S\|\right)r_1.
\end{aligned}$$

Then, by Lemma 3.3 we have

$$\left|G\left(|\alpha|^{p-1},|\alpha|\,x,\omega^1\right)\right|_X\leq t_0\left(\frac{1}{2}+\frac{2}{|\lambda|-2}\right)r_1.$$

Therefore from the hypothesis (3.10) we deduce that

$$\left|G\left(|\alpha|^{p-1},|\alpha|\,x,\omega^1\right)\right|_X\leq t_0 r_1. \tag{3.12}$$

Applying the Banach contraction principle, by (3.11) and (3.12) we conclude that $G\left(|\alpha|^{p-1},|\alpha|\,x,.\right)$ possesses a fixed point $\omega\left(|\alpha|^{p-1},|\alpha|\,x\right)$ in D_2,

2) We have

$$
\begin{aligned}
&\left|\omega\left(|\alpha|^{p-1},|\alpha|\,x\right)-\omega\left(|\alpha|^{p-1},|\alpha|\,y\right)\right|_X\\
=\ &\left|G\left(|\alpha|^{p-1},|\alpha|\,x,\omega\left(|\alpha|^{p-1},|\alpha|\,x\right)\right)-G\left(|\alpha|^{p-1},|\alpha|\,x,\omega\left(|\alpha|^{p-1},|\alpha|\,y\right)\right)\right|_X\\
\leq\ &\|S\|\left\{\|T\|\left(|\alpha|^p\,|x-y|+|\alpha|^{p-1}\left|\omega\left(|\alpha|^{p-1},|\alpha|\,x\right)-\omega\left(|\alpha|^{p-1},|\alpha|\,y\right)\right|_X\right)+\right.\\
&\left.\left(1+|\alpha|^{p-1}\right)(p+1)\left(|\alpha|\,|x-y|+\left|\omega\left(|\alpha|^{p-1},|\alpha|\,x\right)-\omega\left(|\alpha|^{p-1},|\alpha|\,y\right)\right|_X\right)\right\}\\
\leq\ &\|S\|\left\{\left(|\alpha|^p\,\|T\|+\left(1+|\alpha|^{p-1}\right)(p+1)\,|\alpha|\right)|x-y|+\right.\\
&\left.\left(|\alpha|^{p-1}\,\|T\|+\left(1+|\alpha|^{p-1}\right)(p+1)\right)\left(\left|\omega\left(|\alpha|^{p-1},|\alpha|\,x\right)-\omega\left(|\alpha|^{p-1},|\alpha|\,y\right)\right|_X\right)\right\}.
\end{aligned}
$$

But by (3.6) we have

$$\|S\|\,(p+1)\leq\frac{p+1}{\frac{|\lambda|}{2}-1},$$

and by hypotheses we have $2p+6<\lambda$. Then

$$\|S\|\,(p+1)<1. \tag{3.13}$$

Choosing I_2 smaller if necessary, we may assume that

$$1-\|S\|\left(|\alpha|^{p-1}\,\|T\|+\left(1+|\alpha|^{p-1}\right)(p+1)\right)>0.$$

Then

$$c=\frac{\|S\|\left(|\alpha|^{p-1}\,\|T\|+\left(1+|\alpha|^{p-1}\right)(p+1)\right)}{1-\|S\|\left(|\alpha|^{p-1}\,\|T\|+\left(1+|\alpha|^{p-1}\right)(p+1)\right)}>0 \tag{3.14}$$

and then, by (3.13) we obtain conclusion 2) of the proposition.

3) We have

$$
\begin{aligned}
&\left|\omega\left(|\alpha|^{p-1},0\right)\right|_X\\
=\ &\left|G\left(|\alpha|^{p-1},0,\omega\left(|\alpha|^{p-1},0\right)\right)\right|_X\\
=\ &\left|-SQ_Y\left(|\alpha|^{p-1}\,T\left(\omega\left(|\alpha|^{p-1},0\right)\right)-(1+|\alpha|)\,M\left(\omega\left(|\alpha|^{p-1},0\right)\right)\right)\right|\\
\leq\ &\|S\|\left(2\left|\omega\left(|\alpha|^{p-1},0\right)\right|_X+2\,(p+1)\left|\omega\left(|\alpha|^{p-1},0\right)\right|_X\right)\\
\leq\ &\frac{2(p+2)}{|\lambda|-2}\left|\omega\left(|\alpha|^{p-1},0\right)\right|_X.
\end{aligned}
$$

Then, by (3.10), $\omega\left(|\alpha|^{p-1},0\right)=0$. □

Remark 3.1. Equation (1.1) is equivalent to the system of equation (3.7), which is equivalent to finding $\lambda\in\Lambda$, $\varepsilon\in\mathbb{R}$, $\omega\in X_1$ satisfying (3.8). In the sequel, we can see that ω is given by the Proposition 3.1, and it remains to find $\lambda \in \Lambda$, $\varepsilon \in \mathbb{R}$. such that

$$\langle T(\varepsilon v+\omega)-L(\lambda,\varepsilon v+\omega)-M(\varepsilon v+\omega),v\rangle=0.$$

Therefore problem (1.1) is reduced to a finite dimensional one.

4. The main result

Let $\overline{\lambda}$ be a fixed characteristic value of the pair (T,L). By (2.7) and Lemma 2.2, $T-L\left(\overline{\lambda},.\right)$ is a self-adjoint operator. Hence by (3.4) we have

$$\ker\left(T-L\left(\overline{\lambda},.\right)\right)^{*}=[v],\tag{4.1}$$

where $\left(T-L\left(\overline{\lambda},.\right)\right)^{*}$ denotes the adjoint mapping of the map $T-L\left(\overline{\lambda},.\right)$.

The main result of this paper is stated as follows

Theorem 4.1. *There exists a sequence $(\lambda_n)\subset\mathbb{R}$, $\lambda_n>2p+6$, $n\in\mathbb{N}$, such that, for any $\overline{\lambda}\in(\lambda_n)$, $\left(\overline{\lambda},0\right)$ is a bifurcation point of equation* (1.1).

More precisely, there exists a sequence $(\lambda_n)\subset\mathbb{R}$, $\lambda_n>2p+6$, $n\in\mathbb{N}$, such that, for any $\overline{\lambda}\in(\lambda_n)$, there exist a function $v\in H_0^1([0,1])\cap H^2([0,1])$, an open $U^*\subset\mathbb{R}$ not containing the origin, such that, to any given $\delta>0$ there exists a neighborhood J of zero in $\mathbb{R}$ satisfying:

For each $\alpha\in J$, $\alpha\neq0$, we can find $x(\alpha)\in U^*$, and a non trivial solution $(\lambda^+(\alpha),v(\alpha))$ of (1.1), *such that*

$$\begin{cases}\lambda(\alpha)=\dfrac{\overline{\lambda}}{\left(1+|\alpha|^{p-1}\right)}\\ \left|\lambda(\alpha)-\overline{\lambda}\right|<\delta,\end{cases}$$

and $v(\alpha)$ is of the form

$$\begin{cases}v(\alpha)=|\alpha|\,x(\alpha)\,v+\omega\left(|\alpha|^{p-1},|\alpha|\,x(\alpha)\right)\\ 0<|v(\alpha)|<\delta,\end{cases}$$

where ω is defined in Proposition 3.1.

We define the mapping $\mathcal{A}:\mathbb{R}\to\mathbb{R}$ by

$$\mathcal{A}(x)=\langle(T-H)(xv),v\rangle,\qquad x\in\mathbb{R}.\tag{4.2}$$

To prove the theorem,we need the following lemma.

Lemma 4.1. *If $\overline{\lambda}>0$, then there is a point $\overline{x}\in\mathbb{R}$ and an open neighborhood U^* of $\overline{x}$ not containing the origin in $\mathbb{R}$, such that the topological degree* $\deg(\mathcal{A},U^*,0)$ *of the mapping $\mathcal{A}$ with respect to U^* and the origin is defined and different from zero.*

Proof. The main idea of the proof is to take

$$\overline{x} = \left[\frac{\overline{\lambda}}{\int\limits_0^1 \frac{|v(y)|^{p+1}}{y^r+1} dy} \right]^{\frac{1}{p-1}} \tag{4.3}$$

and

$$U^* = \left] \frac{\overline{x}}{2}, \frac{3\overline{x}}{2} \right[. \tag{4.4}$$

For any $x \in U^*$, we have

$$\begin{aligned} \mathcal{A}(x) &= \langle (T - H)(xv), v \rangle \\ &= \langle xTv - H(xv), v \rangle \\ &= \left\langle xTv - |xv|^{p-1} xv, v \right\rangle . \end{aligned}$$

Hence

$$\mathcal{A}(x) = \overline{\lambda} x - x^p \int\limits_0^1 \frac{|v(y)|^{p+1}}{y^r+1} dy \tag{4.5}$$

and

$$\mathcal{A}'(x) = \overline{\lambda} - p x^{p-1} \int\limits_0^1 \frac{|v(y)|^{p+1}}{y^r+1} dy, \tag{4.6}$$

where $\mathcal{A}'$ denotes the derivative of the mapping $\mathcal{A}$. Then

$$\mathcal{A}(\overline{x}) = 0 \tag{4.7}$$

and $\mathcal{A}'(\overline{x}) = (1-p)\overline{\lambda}$. But $\overline{\lambda} \neq 0$ and $p \neq 1$, so

$$\mathcal{A}'(\overline{x}) \neq 0. \tag{4.8}$$

On the other hand, since

$$\begin{aligned} \mathcal{A}\left(\frac{\overline{x}}{2}\right) &= \overline{\lambda}\frac{\overline{x}}{2} - \left(\frac{\overline{x}}{2}\right)^p \int\limits_0^1 \frac{|v(y)|^{p+1}}{y^r+1} dy \\ &= \frac{\overline{x}}{2} \left[\overline{\lambda} - \frac{\overline{\lambda}}{2^{p-1} \int\limits_0^1 \frac{|v(y)|^{p+1}}{y^r+1} dy} \int\limits_0^1 \frac{|v(y)|^{p+1}}{y^r+1} dy \right] \\ &= \frac{1}{2}\overline{\lambda}\overline{x} \left[1 - \frac{1}{2^{p-1}} \right], \end{aligned}$$

$\mathcal{A}\left(\frac{\overline{x}}{2}\right) \neq 0$. By the same manner, we have

$$\mathcal{A}\left(\frac{3\overline{x}}{2}\right) = \frac{3}{2}\overline{\lambda}\,\overline{x}\left[1 - \left(\frac{3}{2}\right)^{p-1}\right],$$

and then $\mathcal{A}\left(\frac{3\overline{x}}{2}\right) \neq 0$. Therefore

$$0 \notin \mathcal{A}\left(\partial U^*\right). \tag{4.9}$$

From (4.7), (4.8) and (4.9) we deduce the proposition (see for instance [5]). □

Proof of Theorem 4.1. Let (λ_n) be the sequence given by Proposition 2.1; I_1, U_1, D_1 be as above and let $\delta > 0$ be given. Using Proposition 3.1, we conclude that for any $\overline{\lambda} \in (\lambda_n)$, $\overline{\lambda} > 2p + 6$, there is a neighborhood I_2 of zero in $\mathbb{R}$, $I_2 \subset I_1$ such that for any $\alpha \in I_2$, $\alpha \neq 0$, $|\alpha|\, x \in U_1$, we can find a fixed point $\omega\left(|\alpha|^{p-1}, |\alpha|\, x\,(\alpha)\right)$ of the mapping $G\left(|\alpha|^{p-1}, |\alpha|\, x\,(\alpha), .\right)$. It follows that

$$\begin{aligned}
&\omega\left(|\alpha|^{p-1}, |\alpha|\, x\,(\alpha)\right)\\
&= - SQ_Y\left(+|\alpha|^{p-1}\, T\left(|\alpha|\, xv + \omega\left(|\alpha|^{p-1}, |\alpha|\, x\,(\alpha)\right)\right)\right)\\
&\quad + SQ_Y\left(\left(1 + |\alpha|^{p-1}\right) M\left(|\alpha|\, xv + \omega\left(|\alpha|^{p-1}, |\alpha|\, x\,(\alpha)\right)\right)\right),
\end{aligned}$$

but

$$S^{-1}(\omega) = \left(T - L\left(\overline{\lambda}, .\right)\right)(\omega) \in Y_1,$$

where

$$\omega = \omega\left(|\alpha|^{p-1}, |\alpha|\, x\,(\alpha)\right).$$

Therefore

$$\begin{aligned}
Q_Y\Big\{T(\omega) - L\left(\overline{\lambda}, \omega\right) + |\alpha|^{p-1}\, T\left(|\alpha|\, xv + \omega\right) -&\\
- \left(1 + |\alpha|^{p-1}\right) M\left(|\alpha|\, xv + \omega\right)\Big\} &= 0.
\end{aligned}$$

Since $T\left(|\alpha|\, xv\right) = L\left(\overline{\lambda}, |\alpha|\, xv.\right)$,

$$Q_Y\left\{T\left(|\alpha|\, xv + \omega\right) - L\left(\frac{\overline{\lambda}}{1 + |\alpha|^{p-1}}, |\alpha|\, xv + \omega\right) - M\left(|\alpha|\, xv + \omega\right)\right\} = 0. \tag{4.10}$$

Furthermore, by choosing $I_2' \subset I_2$ if necessary we may assume that

$$\left|\frac{\overline{\lambda}}{1 + |\alpha|^{p-1}} - \overline{\lambda}\right| < \delta$$

and $\alpha U^* \subset U_1$ for all $\alpha \in I_2$, where U^* is from Lemma 4.1. For any $(t, \alpha, x) \in [0,1] \times I_2 \times U_2$, $\alpha \neq 0$ we put

$$g_1(t,\alpha,x) = \left\langle T\left(xv + \frac{\omega\left(\|T\|^{p-1}, \|T\| x\right)}{|\alpha|}\right), v\right\rangle,$$

$$g_2(t,\alpha,x) = -\left\langle \left(1 + t|\alpha|^{p-1}\right) H\left(xv + \frac{\omega\left(\|T\|^{p-1}, \|T\| x\right)}{|\alpha|}, v\right)\right\rangle,$$

$$g_3(t,\alpha,x) = -\left\langle \left(1 + t|\alpha|^{p-1}\right)|\alpha|^{-p} K\left(t|\alpha| xv + \omega\left(\|T\|^{p-1}, \|T\| x\right)\right), v\right\rangle,$$

and define the function $\mathcal{A}_\alpha$ for $\alpha \in I$, $\alpha \neq 0$ by

$$\mathcal{A}_\alpha : [0,1] \times \overline{U_2} \to \mathbb{R},$$

$$\mathcal{A}_\alpha(t,x) = \sum_{m=1}^{3} g_m(t,\alpha,x),$$

where $\mathcal{A}$ is given by (4.2). By Proposition 3.1, the mapping $\mathcal{A}_\alpha$ is continuous and there is a neighborhood J of zero, $J \subset I_2$, such that $\mathcal{A}_\alpha(t,x) \neq 0$ for all $\alpha \in J, \alpha \neq 0$, $t \in (0,1)$, $x \in \partial U^*$). It follows that for any fixed $\alpha \in J, \alpha \neq 0$, the mapping $\mathcal{A}_\alpha(1,.)$ is homotopic to $\mathcal{A}_\alpha(0,.) = \mathcal{A}$ on U^*. Therefore, by the basic theorem on the topological degree of continuous mappings in a finite dimensional space, we deduce from Lemma 4.1 that

$$\deg\left(\mathcal{A}_\alpha(1,.), U^*, 0\right) = \deg\left(\mathcal{A}, U^*, 0\right) \neq 0.$$

Then we conclude that for each $\alpha \in I$, $\alpha \neq 0$, there is a point $x(\alpha) \in U^*$ such that $\mathcal{A}_\alpha(1, x(\alpha)) = 0$. By the definition of $\mathcal{A}_\alpha(1,.)$ we obtain

$$\left\langle T\left(\frac{v(\alpha)}{|\alpha|}\right) - \left(1 + |\alpha|^{p-1}\right)|\alpha|^{-p} M(v(\alpha)), v\right\rangle = 0, \tag{4.11}$$

where

$$v(\alpha) = |\alpha| x(\alpha) v + \omega\left(|\alpha|^{p-1}, |\alpha| x(\alpha)\right). \tag{4.12}$$

Multiplying both sides of (4.11) by $|\alpha|^p$, we get

$$\left\langle |\alpha|^{p-1} T(v(\alpha)) - \left(1 + |\alpha|^{p-1}\right) M(v(\alpha)), v\right\rangle = 0.$$

But

$$\left\langle T(v(\alpha)) - L\left(\overline{\lambda}, v(\alpha)\right), v\right\rangle = \left\langle v(\alpha), \left(T - L\left(\overline{\lambda},\right)\right)^* v\right\rangle = 0,$$

then

$$\left\langle \left(1 + |\alpha|^{p-1}\right) T(v(\alpha)) - L\left(\overline{\lambda}, v(\alpha)\right) - M(v(\alpha)), v\right\rangle = 0.$$

Therefore

$$\left\langle T(v(\alpha)) - L\left(\frac{\overline{\lambda}}{1 + |\alpha|^{p-1}}, v(\alpha)\right) - M(v(\alpha)), v\right\rangle = 0. \tag{4.13}$$

Combining (4.10) and (4.13) gives

$$T\left(v\left(\alpha\right)\right)-L\left(\lambda\left(\alpha\right),v\left(\alpha\right)\right)-M\left(v\left(\alpha\right)\right)=0,$$

where $v\left(\alpha\right)$ is given by (4.12) and

$$\lambda\left(\alpha\right)=\frac{\overline{\lambda}}{1+\left|\alpha\right|^{p-1}}\,.$$

This ends the proof. □

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