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LOCALLY BOUNDED HOLOMORPHIC FUNCTIONS AND THE MIXED HARTOGS THEOREM

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ABSTRACT. This paper deals with the conditions which allow the local boundedness of every holomorphic function taking value in Frechet space. These results are also applied to get different versions of the Hartogs theorem of mixed type.

0. INTRODUCTION

Let E, F be Frechet spaces and D be an open subset of E. By $\mathcal{H}(D, F)$ we denote the vector space of holomorphic functions defined on D and taking values in F. As in [6], a function $f \in \mathcal{H}(E, F)$ is called of uniformly bounded type if there exists a neighbourhood U of $0 \in E$ such that $f(rU)$ is bounded for all $r > 0$. The uniform boundedness of scalar holomorphic functions on a nuclear Frechet space was investigated by several authors, in particular by Meise and Vogt [6]. Later L. M. Hai [2] has extended some results of Meise and Vogt to holomorphic functions taking value in a Frechet space. Recently, the second named author of this paper also proved that every $f \in \mathcal{H}(E, F)$ is of uniformly bounded type if E is a nuclear Frechet space with $(\overline{\overline{\Omega}})$ and F a Frechet space having the property $(\underline{DN})$ [5]. Unlike the previous studies, in the present paper we consider the local boundedness of Frechet-valued holomorphic functions defined on open subsets of a Frechet space.

Let E, F and D be as above, we call a holomorphic function $f : D \longrightarrow F$ locally bounded if for every $z \in D$ there exists a neighbourhood U of z in D such that $f(U)$ is bounded. We set

$$\mathcal{H}_{Lb}(D, F) = \{f \in \mathcal{H}(D, F) \ : f \text{ is locally bounded on } D\}.$$

We now describe briefly the content or our paper. In Section 2 we give sufficient conditions for E and F such that the relation

$$(Lb) : \mathcal{H}_{Lb}(D, F) = \mathcal{H}(D, F)$$

holds for every open set D of E.

These results are then applied in Section 3 to study holomorphicity of separately holomorphic functions in the mixed case. Finally, in Section 4, some

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examples on the existence of non locally bounded holomorphic function with values in Frechet spaces are given.

1. Preliminaries

In the sequel, we shall use standard notations from the theory of locally convex spaces as presented in the book of Pietsch [9] or of Schaefer [10]. All convex spaces are assumed to be complex and Hausdorff.

1.1. **Linear Topological Invariants.** Let E be a Frechet space with fundamental system of semi-norms $\{\|.\|_k\}$. For a subset B of E, we define the generalized semi-norm $\|.\|_B^* : E' \to [0, +\infty]$, where E' is the dual space of E, by $\|u\|_B^* = \sup\{|u(x)| : x \in B\}$. Write $\|.\|_k^*$ for $B = U_k = \{x \in E : \|x\|_k \ \leq 1\}$.

We say that E has the property

$$
\begin{aligned}
(\tilde{\Omega}) &\iff \forall\, p \exists\, q, d\, \forall\, k \exists\, C > 0 : \|.\|_q^{*(1+d)} \leq C \|.\|_k^* \|.\|_p^{*d},\\
(DN) &\iff \exists\, p\, \forall\, q, d > 0 \exists\, k, C > 0 : \|.\|_q^{1+d} \leq C \|.\|_k \|.\|_p^d,\\
(\underline{DN}) &\iff \exists\, p\, \forall\, q \exists\, k, d, C > 0 : \|.\|_q^{1+d} \leq C \|.\|_k \|.\|_p^d,\\
(LB_\infty) &\iff \forall\, \rho_k \ > 0, \rho_k \uparrow\ \exists\, p\, \forall\, q \exists\, k_q \geq q, C > 0\ \forall x \in E, \exists\, q \ \leq \ k \leq k_q :\\
&\qquad \|x\|_q^{1+\rho_k} \leq C \|x\|_k \|x\|_p^{\rho_k}.
\end{aligned}
$$

The above properties were introduced and investigated by Vogt ([11], [12]).

1.2. **Holomorphic functions.** Let E, F be locally convex space and D an open set in E. A function $f : D \to F$ is called holomorphic if f is continuous and $u \circ f$ is Gateaux holomorphic for all $u \in F'$. For more details concerning holomorphic functions on locally convex spaces we refer the readers to the book of Dineen [1] or of Noverraz [8].

2. Locally bounded holomorphic functions

First we prove the following theorem.

Theorem 2.1. *Let E and F be Frechet space with $E \in (\tilde{\Omega})$ and $F \in (DN)$. Assume that E is nuclear. Then (Lb) holds for every open subset D of E.*

Proof. Let $\{\|.\|\}_\alpha$ and $\{\|.\|\}_k$ be increasing sequences of semi-norms defining the topology of E and F respectively.

Given $f \in \mathcal{H}(D, F)$ and $z_0 \in D$, we may assume that D is balanced and $z_0 = 0$.

(i) Let $\alpha \geq 1$ be such that

$$U_\alpha = \{z \in E : \|z\|_\alpha < 1\} \subset D$$

and

$$M(\alpha, p) = \sup\{\|f(z)\|_p : \|z\|_\alpha < 1\} < \infty,$$

where $p \geq 1$ is chosen such that (DN) holds.

Consider $E/\mathrm{Ker}\|.\|_\alpha$ equipped with the quotient topology and the canonical map $\mathcal{R} : E \longrightarrow E/\mathrm{Ker}\|.\|_\alpha$. By [1, Proposition 2.23], $\omega_p f$ can be written in the form $\omega_p f = g\mathcal{R}$, where $g : \mathcal{R}(U_\alpha) \longrightarrow F$ is a holomorphic function and $\omega_p : F \longrightarrow F_p$ the canonical map from F into the Banach space F_p associated to the semi-norm $\|.\|_p$. Since ω_p is injective, $g = \omega_p \tilde{f}$, where $\tilde{f}$ is a function on $\mathcal{R}(U_\alpha)$ with values in F. From the openness of $\mathcal{R}$, it follows that $\tilde{f}$ is holomorphic. On the orther hand, since $E \in (\overline{\overline{\Omega}})$, we have $E/\mathrm{Ker}\|.\|_\alpha \in (\overline{\overline{\Omega}})$.

Thus replacing f by $\tilde{f}$, without loss of generality we may assume that $\|.\|_\alpha$ is a norm on E.

Choose a balanced convex compact set K in $E, \beta \geq \alpha$ and $C, d > 0$ such that

$$(\tilde{\Omega}_K) : \|.\|_\beta^{*(1+d)} \leq C\|.\|_K^*\|.\|_\alpha^{*d} \tag{1}$$

(see [11] for the existence of this set).

By scaling K we may assume further that $C = 1$.

Let $\omega_\alpha : E \to E_\alpha$ be the canonical map from E into the Banach space E_α associated to $\|.\|_\alpha$ and $A = \omega_{\alpha}|_{E(K)}$, where $E(K)$ is the Banach space spanned by K. Since E is nuclear, we may assume that $E(K)$ and E_α are Hilbert spaces. Thus A can be written in the form

$$Ax = \sum_{j=1}^{\infty} \lambda_j \langle x, y_j \rangle z_j,$$

where $\lambda = (\lambda_j)_{j\in\mathbf{N}} \in s$, s is the space of rapidly decreasing sequences, $\lambda_j > 0$ $\forall\ j \geq 1$, $\{y_j\}_{j\in\mathbf{N}}$ is a complete orthonormal system in $E(K)$ and $\{z_j\}_{j\in\mathbf{N}}$ an othonormal system in E_α.

Since $A\Big(\dfrac{y_j}{\lambda_j}\Big) = z_j \in \omega_\alpha(U_\alpha)\ \forall\ j \geq 1$ we have $\dfrac{y_j}{\lambda_j} \in U_\alpha\ \forall\ j \geq 1$. It follows that $\sum\limits_{j=1}^{m} \Big(\dfrac{\tilde{\mu}_j}{\lambda_j}\Big) y_j \in U_\alpha\ \forall\ m \geq 1$, where $\tilde{\mu}_j = \dfrac{\delta}{j}$ and $\delta > 0$ are chosen such that

$$\sum_{j=1}^{\infty} \tilde{\mu}_j^2 \leq 1$$

and the set $\Big\{u \in E_\alpha :\ u = \sum\limits_{j=1}^{\infty} \xi_j z_j \text{ with } |\xi_j| \leq \tilde{\mu}_j\ \forall\ j \geq 1\Big\}$ is contained in $\omega_\alpha(U_\alpha)$. Take $0 < \epsilon < 1$ such that if $\mu = \epsilon\tilde{\mu}$, then

$$\lambda_j \mu_j \leq \tilde{\mu}_j\ \forall\ j \geq 1.$$

We let $\chi_k \in E'_\alpha$ be the functional on E'_α defined by $z \mapsto \langle z, z_k \rangle_\alpha$, the scalar product in E_α.
Then

$$\|\chi_k\| = 1 \quad \forall\ k \geq 1 \tag{2}$$

and

$$\|A^*\chi_k\|_K^* = \sup_{\|x\|\le 1} |\chi_k A(x)| = \sup_{\|x\|\le 1} |\lambda_k\langle x, y_k\rangle| \le \lambda_k.$$

Next we put

$$\varphi_k = \omega_\alpha^*(\chi_k). \tag{3}$$

From (1), (2), (3) we deduce

$$\|\varphi_k\|_\beta^{*(1+d)} = \|\omega_\alpha^*\chi_k\|_\beta^{*(1+d)} \le \|A^*\chi_k\|_K^*\|\chi_k\|_\alpha^{*(1+d)}, \quad \forall k \ge 1.$$

It follows that

$$\|\varphi_k\|_\beta^* \le (\lambda_k)^{\frac{1}{1+d}} \quad \forall\ k \ge 1.$$

(ii) Let $g = \omega_p f$. We put

$$\mathbf{M} = \ \{m = (m_j) \in \mathbf{N}^{\mathbf{N}} : m_j \ne 0 \text{ only for finitely many } j\}.$$

For each $m = (m_1, m_2, \dots, m_n, 0, 0, \dots) \in \mathbf{M}$ we define

$$a_m = \Big(\frac{1}{2\pi i}\Big)^n \int\limits_{|\rho_1|=\tilde{\mu}_1} \int\limits_{|\rho_2|=\tilde{\mu}_2} \cdots \int\limits_{|\rho_n|=\tilde{\mu}_n} \frac{g(\rho_1 z_1 + \rho_2 z_2 + \dots + \rho_n z_n)}{\rho^{m+1}} d\rho,$$

where

$$\rho^{m+1} := \rho_1^{m_1+1}\rho_2^{m_2+1}\dots\rho_n^{m_n+1},$$
$$d\rho := d\rho_1 d\rho_2 \dots d\rho_n.$$

Then

$$\|a_m\|_p \le \frac{M(\alpha, p)}{\tilde{\mu}^m} \le \frac{M(\alpha, p)}{\mu^m},$$

where $\tilde{\mu}^m := \tilde{\mu}_1^{m_1}\tilde{\mu}_2^{m_2}\dots\tilde{\mu}_n^{m_n}$ and $\mu^m = \mu_1^{m_1}\cdots\mu_n^{m_n}, \forall m \in \mathbf{M}$.

Since

$$\sum_{j=1}^m \Big(\frac{\rho_j}{\lambda_j}\Big) y_j \in U_\alpha \subset D \quad \forall\ m \ge 1,$$

we obtain

$$g\Big(\sum_{j\ge 1}\rho_j z_j\Big) = gA\Big(\sum_{j=1}^\infty \frac{\rho_j}{\lambda_j} y_j\Big) = \omega_p f\Big(\sum_j \frac{\rho_j}{\lambda_j} y_j\Big).$$

On the other hand, by applying the Cauchy integral formula we get

$$\begin{aligned} a_m &= \Big(\frac{1}{2\pi i}\Big)^n \int\limits_{|\rho_1|=\lambda_1\mu_1} \cdots \int\limits_{|\rho_n|=\lambda_n\mu_n} \frac{\omega_\rho f(\frac{\rho_1}{\lambda_1} y_1 + \frac{\rho_2}{\lambda_2} y_2 + \dots + \frac{\rho_n}{\lambda_n} y_n)}{\lambda^{m+1}(\frac{\rho}{\lambda})^{m+1}} d\rho \\ &= \omega_p \underbrace{\Big(\frac{1}{\lambda^m}\Big(\frac{1}{2\pi i}\Big)^n \int\limits_{|\theta_1|=\mu_1} \cdots \int\limits_{|\theta_n|=\mu_n} \frac{f(\theta_1 y_1 + \theta_2 y_2 + \dots \theta_n y_n)}{\theta^{m+1}} d\theta\Big)}_{b_m} \end{aligned}$$

where $\theta_j = \dfrac{\rho_j}{\lambda_j}$ for all $j \geq 1$.

From the definition of b_m we have

$$\|b_m\|_q \leq \frac{N(q)}{\lambda^m \mu^m}, \quad \forall\, m \in M,\ \forall\, q \geq p,$$

where

$$N(q) = \sup\Big\{\|f(x)\|_q :\ x = \sum_{j=1}^{\infty} \xi_j y_j \text{ and } |\xi_j| \leq \tilde{\mu_j}\Big\}$$

Note that $N(q) < \infty$ since

$$\Big\{x :\ x = \sum_{j=1}^{\infty} \xi_j y_i : |\xi_j| \leq \tilde{\mu_j}\Big\}$$

is compact in $E(K)$.

Since $F \in (DN)$, for $q \geq p$ and $\bar{d} = \dfrac{d}{\delta}$ there exists $k \geq q$ and $C > 0$ such that

$$\|.\|_q^{1+\bar{d}} \leq C\|.\|_k \|.\|_p^{\bar{d}},$$

where $0 < \delta < 1$ is chosen so that $\epsilon \;=\; \gamma - \dfrac{1-\gamma}{1+\bar{d}} > 0$ with $\gamma = \dfrac{1}{2(1+d)}\cdot$

Again, by choosing k sufficiently large, without loss of generality we may assume that $C = 1$.

We have

$$\begin{aligned}
&\sum_{m\in M} r^m \|b_m\|_q \prod_{j=1}^{\infty} \|\varphi_j\|_{\beta}^{*m_j} \leq \sum_{m\in M} r^m \|b_m\|_q \prod_{j=1}^{\infty} (\lambda_j)^{\frac{m_j}{1+d}} \\
&= \sum_{m\in M} r^m (\lambda)^{2\gamma m} \|b_m\|_q = \sum_{m\in M} r^m (\lambda^m \|b_m\|_q)^{\gamma} (\lambda)^{\gamma m} \|b_m\|_q^{1-\gamma} \\
&\leq N(q)^{\gamma} N(k)^{\frac{1-\gamma}{1+\bar{d}}} M(\alpha,p)^{\frac{(1-\gamma)\bar{d}}{1+\bar{d}}} \sum_{m\in M} r^m \Big[\frac{\lambda^{m(\gamma - \frac{1-\gamma}{1+\bar{d}})}}{\mu^{m(\gamma + \frac{1-\gamma}{1+\bar{d}} + \frac{(1-\gamma)\bar{d}}{1+\bar{d}})}}\Big] \\
&= N(q)^{\gamma} N(k)^{\frac{1-\gamma}{1+\bar{d}}} M(\alpha,p)^{\frac{(1-\gamma)\bar{d}}{1+\bar{d}}} \prod_{j=1}^{\infty} \sum_{m_j=0}^{\infty} \Big(\frac{r\lambda_j^{\epsilon}}{\mu_j}\Big)^{m_j} \\
&= N(q)^{\gamma} N(k)^{\frac{1-\gamma}{1+\bar{d}}} M(\alpha,p)^{\frac{(1-\gamma)\bar{d}}{1+\bar{d}}} \prod_{j=1}^{\infty} \Big(1 \;-\; \frac{r\lambda_j^{\epsilon}}{\mu_j}\Big)^{-1}
\end{aligned}$$

(see [6], p. 155).

The convergence of the right hand side for $r > 0$ small enough follows from the following observation. If $\lambda = (\lambda_j) \in l_1$, $\lambda_j \geq 0$, $\forall j \geq 1$ and $\sup_{j\geq 1} \lambda_j < 1$, then

$\sum\limits_{\mathbf{M}} \lambda^m$ is convergent. To check this statement, we write

$$\begin{aligned}\sum_{\mathbf{M}} \lambda^m &= \lim_{n\to\infty} \sum_{\mathbf{M}_n} \lambda^m = \lim_{n\to\infty} \prod_{j=1}^{n} \sum_{m_j=0}^{\infty} (\lambda_j)^{m_j} \\ &= \lim_{n\to\infty} \prod_{j=1}^{n} (1-\lambda_j)^{-1} = \prod_{j=1}^{\infty} (1-\lambda_j)^{-1},\end{aligned}$$

where $\mathbf{M}_n = \{(m_1, \dots, m_n, 0, \dots, 0, \dots) : (m_1, \dots, m_n) \in \mathbf{N}^m\} \subset \mathbf{M}$.

The convergent of the last product follows from the following estimate

$$0 < \sum_{j=1}^{\infty} \big(\frac{1}{1-\lambda_j} - 1\big) = \sum_{j=1}^{\infty} \frac{\lambda_j}{1-\lambda_j} < \frac{1}{c} \sum_{j=1}^{\infty} \lambda_j < \infty,$$

where

$$c = \inf_{j\geq 1} (1-\lambda_j) > 0.$$

Since $\lambda = (\lambda_j)$ is in s, we have $\Big(\dfrac{\lambda_j^{\epsilon}}{\mu_j}\Big)$ is in l^1, and hence, for $R = \sum\limits_{j=1}^{\infty} \dfrac{\lambda_j^{\alpha}}{\mu_j}$ we obtain $R \geq \dfrac{\lambda_j^{\epsilon}}{\mu_j}$ for $j \geq 1$.

This implies $0 < \sup\Big\{\dfrac{\lambda_j^{\epsilon}}{2R\mu_j} : j \geq 1\Big\} < \dfrac{1}{2}$. Now we have

$$\sum_{M} r^m \|b_m\|_q \prod_{j=1}^{\infty} \|\varphi_j\|_{\beta}^{*m_j} \leq N(q)^{\gamma} N(k)^{\frac{1-\gamma}{1+d}} M(\alpha, p)^{\frac{(1-\gamma)\bar{d}}{1+d}} \prod_{j=1}^{\infty} \Big(1 - \frac{r\lambda_j^{\epsilon}}{\mu_j}\Big)^{-1}$$

Hence

$$h(x) = \sum_{M} b_m \prod_{j=1}^{\infty} (\varphi_j(x))^{m_j}$$

defines a locally bounded holomorphic function on $\dfrac{1}{3R}U_{\beta}$. The function h is equal to f on $\dfrac{1}{3R}U_{\beta}$, because $h(x) = f(x)$ for $x = \sum\limits_{j=1}^{m} \xi_j y_j \in \dfrac{1}{3R}U_{\beta} \cap E(K)$ and $E(K)$ is dense in E. $\square$

Theorem 2.2. *Let E, F be Frechet spaces, $E \in (\tilde{\Omega})$ and $F \in (LB_{\infty})$. Assume that B is a Banach space and E a nuclear space. Then, (Lb) holds for every open set D in $B \times E$.*

Proof. Let D be an open set in $B \times E$, $f \in \mathcal{H}(D, F)$ and $(x_0, y_0) \in D$. It suffices to consider the case where D is balanced and $(x_0, y_0) = 0$. Without loss of generality we may assume that B has an absolute basis $(e_i)_{i\in\mathbf{I}}$ (not-necessarily countable), $\|e_i\| = 1$, $\forall\ i \in \mathbf{I}$, since B is a quotient of such a space. Choose two sequences $1 \geq \epsilon_k \searrow 0$ and $\alpha_k \nearrow \infty$ such that

$$M_k = \sup\Big\{\|f(x,y)\|_k : \|x\| < \epsilon_k, \|y\|_{\alpha_k} < 1\Big\} < \infty \quad \text{for } k \geq 1.$$

Put $d_k = \dfrac{1}{\epsilon_k}$. Since $\lim\limits_{k\to\infty} \epsilon_k^{\epsilon_k} = 1$,

$$\lim_{k\to\infty} \frac{\delta^{\bar{d}_k}}{\epsilon_k} = \lim_{k\to\infty} \Big(\frac{\delta}{\epsilon_k^{\epsilon_k}}\Big)^{\frac{1}{\epsilon_k}} = 0 \quad \text{for } 0 < \delta < 1. \tag{4}$$

By applying the hypothesis $F \in (LB_\infty)$ to the sequence $\{d_k\} \uparrow \infty$ we have

$$\begin{gathered} \exists\ p\ \forall\ q \geq p\ \exists k_q, C_q > 0\ \forall\ \omega \in F\ \exists q \leq k \leq k_q : \\ \|\omega\|_q^{1+d_k} \leq C_q \|\omega\|_k \|\omega\|_p^{d_k}. \end{gathered} \tag{5}$$

By letting $\delta = \dfrac{\epsilon_{p+1}}{\epsilon_p}$ in (4) we get

$$\exists\ k_0\ \forall\ k \geq k_0 : \epsilon_{p+1}^{1+d_k} \leq \epsilon_k \epsilon_p^{d_k}. \tag{6}$$

We split the rest of the proof into two steps.

Step 1. We will prove that (Lb) holds in the case of $E = \{0\}$. For this, it is enough to check that f is bounded on $\{x : \ \|x\| < r\epsilon_{p+1}\}$, where $r > 0$ chosen so small that

$$\sum_{n\geq 0} \frac{r^n n^n}{n!} < \infty. \tag{7}$$

For $\|x\| < \epsilon_{p+1}$ we write

$$f(rx) \ = \ \sum_{n\geq 0} r^n \sum_{i_1,i_2,\ldots,i_n \in I} e_{i_1}^*(x)\ldots e_{i_n}^*(x) P_n f(e_1,\ldots,e_n).$$

For each $k \in [q, k_q]$ we put

$$F_k = \Big\{\omega \in F : \|\omega\|_q^{1+d_k} \leq C_q \|\omega\|_k \|\omega\|_p^{d_k}\Big\}.$$

Without loss of generality we may assume that F_k are disjoint. This implies that the sets J_k^n given by

$$J_k^n = \Big\{(i_1, i_2, \ldots, i_n) : P_n f(e_1, e_2, \ldots, e_n) \in F_k\Big\}.$$

are also disjoint. Then, from (5), (6) and (7) it follows that

$$\begin{aligned}
&\sum_{n\geq 0} r^n \sum_{i_1,i_2,\dots,i_n\in I} |e^*_{i_1}(x)|\dots|e^*_{i_n}(x)|\|P_n f(e_1,\dots,e_n)\|_q = \\
&= \sum_{n\geq 0}\sum_{q<k<k_q}\sum_{J^n_k} |e^*_{i_1}(x)|\dots|e^*_{i_n}(x)|\|P_n f(e_1,\dots,e_n)\|_q \\
&\leq \sum_{n\geq 0}\sum_{q<k<k_q}\sum_{J^n_k} |e^*_{i_1}(x)|\dots|e^*_{i_n}(x)|\; C_q^{\frac{1}{1+d_k}} \epsilon_k^{\frac{-n}{1+d_k}} \epsilon_p^{\frac{-nd_k}{1+d_k}} \times \\
&\quad \times \|P_n f(\epsilon_k e_{i_1},\dots,\epsilon_k e_{i_n}\|_k^{\frac{1}{1+d_k}} \times \|P_n f(\epsilon_p e_{i_1},\dots,\epsilon_p e_{i_n})\|_p^{\frac{d_k}{1+d_k}} \\
&\leq \sum_{n\geq 0} r^n \sum_{q\leq k\leq k_q} C_q^{\frac{1}{1+d_k}} \epsilon_{p+1}^{-n} M_k^{\frac{1}{1+d_k}} M_p^{\frac{d_k}{1+d_k}} \Big(\frac{n^n}{n!}\Big)\Big(\sum_{i\in I}|e^*_i(x)|\Big)^n \\
&\leq \sum_{q\leq k\leq k_q} C_q^{\frac{1}{1+d_k}} M_k^{\frac{1}{1+d_k}} M_p^{\frac{d_k}{1+d_k}} \sum_{n\geq 0}\frac{r^n n^n}{n!} < \infty \quad \text{for } \|x\| < \epsilon_{p+1}.
\end{aligned}$$

Step 2. We will prove the theorem for the case where E is an arbitrary nuclear Frechet space having $(\tilde{\Omega})$. We keep the notations of Theorem 2.1. Choose $\alpha = \alpha_p$ and $\delta > 0$ such that

$$M(\alpha,p) = \sup\Big\{\|f(x,z)\|_p : \|x\| < \delta, \|z\|_\alpha < 1\Big\} < +\infty.$$

For $\alpha = \alpha_p$, choose $\beta, d, C > 0$, and a compact set K in E for which

$$\|.\|_\beta^{*(1+d)} \leq C\|.\|^*_K\|.\|^{*d}_\alpha. \tag{8}$$

By scaling K we may assume that $C = 1$ and hence

$$\|.\|_\beta^{*(1+s)} \leq \|.\|^*_K\|.\|^{*s}_\alpha \quad \forall s\geq d. \tag{9}$$

Choose an arbitrary sequence $\{s_k\}\uparrow\infty$ and $0<\delta<1$ such that

$$d_k = \frac{s_k}{\delta} \geq d, \quad \forall k\geq 1,$$

and

$$\epsilon_k = \gamma_k - \frac{1-\gamma_k}{1+d_k} > 0, \quad \forall k \geq 1,$$

where $\gamma_k = \dfrac{1}{2(1+s_k)}\,$.

By applying the result proved in Case 1 to $f|_{D\cap(B\times E(K))}$ we can find $\epsilon > 0$ such that f is bounded on

$$\Big\{(x,y)\in B\times E(K) : \|x\| < \epsilon, \|y\| < \epsilon\Big\}.$$

As in the proof of Theorem 2.1, for each

$$m \in \mathbf{M} = \Big\{m = (m_j)\in \mathbf{N}^{\mathbf{N}} : m_j \neq 0 \;\text{ only for finitely many } j\Big\}$$

we set

$$
\begin{aligned}
a_m(x) &= \Big(\frac{1}{2\pi i}\Big)^n \int\limits_{|\rho_1|=\tilde{\mu}_1} \int\limits_{|\rho_2|=\tilde{\mu}_2} \cdots \int\limits_{|\rho_n|=\tilde{\mu}_n} \frac{g(x, \rho_1 z_1 + \rho_2 z_2 + \cdots + \rho_n z_n)}{\rho^{m+1}} d\rho \\
&= \omega \underbrace{\Big[\frac{1}{\lambda^m} \Big(\frac{1}{2\pi i}\Big)^n \int\limits_{|\theta_1|=\mu_1} \int\limits_{|\theta_2|=\mu_2} \cdots \int\limits_{|\theta_n|=\mu_n} \frac{f(x, \theta_1 y_1 + \theta_2 y_2 + \cdots + \theta_n y_n)}{\theta^{m+1}} d\theta \Big]}_{b_m(x)}.
\end{aligned}
$$

It follows that

$$
\sup\Big\{ \|b_m(x)\|_q : \|x\| < \epsilon \Big\} \leq \frac{N(q)}{\lambda^m \mu^m}, \quad \forall\, m \in M \ \forall\, q \geq p,
$$

where

$$
N(q) = \sup\Big\{ \|f(x,y)\|_q : \ \|x\| < \epsilon, y = \sum_{j=1}^{\infty} \xi_j y_j \text{ and } |\xi_j| \leq \tilde{\mu_j} \Big\}.
$$

Since $F \in (LB_\infty)$, there exists $p \geq 1$ such that

$$
\begin{gathered}
\forall q \exists k_q \geq q, C_q > 0, \forall \omega \in F, \exists q \leq k \leq k_q : \\
\|\omega\|_q^{1+d_k} \leq C_q \|\omega\|_k \, \|\omega\|_p^{d_k}.
\end{gathered}
$$

Put

$$
F_k = \{\omega \in F : ||\omega||_q^{1+d_k} \leq C_q ||\omega||_k ||\omega||_p^{d_k}\}, \quad q \leq k \leq k_q.
$$

As in Case 1, without loss of generality we may assume that F_k are disjoint. This implies that the sets $J^k_{m,n}$ given by

$$
J^k_{m,n} = \{(i_1, i_2, \ldots, i_n; m) \in \mathbf{I}^n \times \mathbf{M} : P_n(b_m)(e_{i_1}, e_{i_2}, \ldots, e_{i_n}) \in F_k\}
$$

where $P_n(b_m)$ is defined by the Taylor expansion of b_m at $0 \in B$

$$
b_m(x) = \sum_{n=0}^{\infty} P_n(b_m)(x),
$$

are also disjoint.

Then we have the estimates

$$\begin{aligned}
A &= \sum_{m\in M} r^m \|b_m(x)\|_q \prod_{j=1}^{\infty} \|\varphi_j\|_\beta^{*m_j} \\
&\le \sum_{\mathbf{M}} r^m \sum_{n\ge 0} \sum_{i_1,i_2,\ldots,i_n\in I} \|P_n(b_m)(e_1,\ldots,e_n)\|_q \times |e^*_{i_1}(x)|\ldots|e^*_{i_n}(x)| \prod_{j=1}^{\infty} \|\varphi_j\|_\beta^{*m_j} \\
&\le \sum_{M} r^m \sum_{n\ge 0} \sum_{q\le k\le k_q} \sum_{J^k_{m,n}} (\lambda)^{2m\gamma_k} \times \|P_n(b_m)(e_1,\ldots,e_n)\|_q |e^*_{i_1}(x)|\ldots|e^*_{i_n}(x)| \\
&= \sum_{q\le k\le k_q} \sum_{M} r^m \sum_{n\ge 0} \sum_{J^k_{m,n}} \Big(\lambda^m \|P_n(b_m)(e_1,\ldots e_n)\|_q\Big)^{\gamma_k} \times \\
&\quad \times \lambda^{m\gamma_k} \|P_n(b_m)(e_1,\ldots e_n)\|_q^{1-\gamma_k} |e^*_{i_1}(x)|\ldots|e^*_{i_n}(x)| \\
&\le \sum_{q\le k\le k_q} \sum_{M} r^m \sum_{n\ge 0} \sum_{J^k_{m,n}} \Big(\frac{N(q)}{\mu^m}\Big)^{\gamma_k} \Big(\frac{n^n}{n!}\Big)^{\gamma_k} \epsilon^{-n\gamma_k} \times \lambda^{m\gamma_k} C_q^{\frac{1-\gamma_k}{1+d_k}} \\
&\quad \times \|P_n(b_m)(e_1,\ldots,e_n)\|_k^{\frac{1-\gamma_k}{1+d_k}} \|P_n(b_m)(e_1,\ldots,e_n)\|_p^{\frac{(1-\gamma_k)d_k}{1+d_k}} |e^*_{i_1}(x)|\ldots|e^*_{i_n}(x)| \\
&\le \sum_{q\le k\le k_q} \sum_{M} r^m \sum_{n\ge 0} \sum_{J^k_{m,n}} \Big(\frac{N(q)}{\mu^m}\Big)^{\gamma_k} \Big(\frac{n^n}{n!}\Big)^{\gamma_k} \\
&\quad \times \epsilon^{-n\gamma_k} \times \lambda^{m\gamma_k} C_q^{\frac{1-\gamma_k}{1+d_k}} \epsilon^{-n(1-\gamma_k)} \times \Big(\frac{N(k)}{\lambda^m\mu^m}\Big)^{\frac{1-\gamma_k}{1+d_k}} \\
&\quad \times \Big(\frac{n^n}{n!}\Big)^{\frac{1-\gamma_k}{1+d_k}} \times \Big(\frac{M(\alpha,p)}{\mu^m}\Big)^{\frac{(1-\gamma_k)d_k}{1+d_k}} \Big(\frac{n^n}{n!}\Big)^{\frac{(1-\gamma_k)d_k}{1+d_k}} |e^*_{i_1}(x)|\ldots|e^*_{i_n}(x)| \\
&= \sum_{q\le k\le k_q} \underbrace{N(q)^{\gamma_k} C_d^{\frac{1-\gamma_k}{1+d_k}} N(k)^{\frac{1-\gamma_k}{1+d_k}} M(\alpha,p)^{\frac{(1-\gamma_k)d_k}{1+d_k}}}_{D_k} \\
&\quad \times \sum_{M} r^m \sum_{n\ge 0} \Big(\frac{1}{\mu^m}\Big)^{\gamma_k} \lambda^{m\gamma_k} \Big(\frac{1}{\lambda^m\mu^m}\Big)^{\frac{1-\gamma_k}{1+d_k}} \times \\
&\quad \times \Big(\frac{1}{\mu^m}\Big)^{\frac{(1-\gamma_k)d_k}{1+d_k}} \sum_{i_1,i_2,\ldots,i_n\in I} \epsilon^{-n} \Big(\frac{n^n}{n!}\Big)^{\gamma_k} \Big(\frac{n^n}{n!}\Big)^{\frac{1-\gamma_k}{1+d_k}} \Big(\frac{n^n}{n!}\Big)^{\frac{(1-\gamma_k)d_k}{1+d_k}} \Big(\sum_{i\in I} |e^*_i(x)|\Big)^n \\
&= \sum_{q\le k\le k_q} D_k \sum_{n\ge 0} \epsilon^{-n} \Big(\frac{n^n}{n!}\Big) \|x\|^n \sum_{M} r^m \lambda^{m(\gamma_k - \frac{1-\gamma_k}{1+d_k})} \frac{1}{\mu^{m(\gamma_k + \frac{1-\gamma_k}{1+d_k} + \frac{(1-\gamma_k)d_k}{1+d_k})}} \\
&= \sum_{q\le k\le k_q} D_k \sum_{n\ge 0} \epsilon^{-n} \Big(\frac{n^n}{n!}\Big) \|x\|^n \sum_{M} \Big(\frac{\lambda^{\gamma_k - \frac{1-\gamma_k}{1+d_k}}}{\mu}\Big)^m r^m.
\end{aligned}$$

On the other hand, since $\lambda = (\lambda_j) \in s$ we have $\Big(\dfrac{\lambda_j^{\epsilon_k}}{\mu_j}\Big)_j \in l^1$ for $k \ge 1$. Hence

$\sup\left\{\frac{\lambda_j^{\epsilon_k}}{2R\mu_j}\right\} \leq \dfrac{1}{2}$ for

$$R = \max_{q \leq k \leq k_q} \sum_{j=1}^{\infty} \frac{\lambda_j^{\epsilon_k}}{\mu_j} \cdot$$

Since $\left(\dfrac{\lambda_j^{\epsilon_k}}{2R\mu_j}\right) \in l_1$, it follows that

$$\sum_M \lambda^{m\epsilon_k} r^m \frac{1}{\mu^m} = \prod_{j=1}^{\infty} \frac{1}{1 - \dfrac{\lambda_j^{\epsilon_k}}{2R\mu_j}} < \infty$$

for $0 < r < \dfrac{1}{2}R$.

Hence

$$A \leq \sum_{q \leq k \leq k_q} D_k \sum_{n \geq 0} \epsilon^{-n} \left(\frac{n^n}{n!}\right) t^n \prod_{j=1}^{\infty} \frac{1}{1 - \dfrac{\lambda_j^{\epsilon_k}}{2R\mu_j}} < \infty$$

for all $\|x\| < t$ sufficiently small and all $0 < r < \dfrac{1}{2R} \cdot$

Thus f is bounded on $\left\{(x, y) : \|x\| < t, \|y\|_\beta < r < \dfrac{1}{2R}\right\}$, a neighbourhood of $0 \in B \times E$. □

3. The mixed hartogs theorem

The well-known Hartogs theorem on the holomorphicity of separately holomorphic functions was extended to the infinite dimensional case by several authors. In particular this theorem is true for classes of Frechet spaces and dual Frechet-Schwartz spaces. However, the problem is more complicated in the mixed case. In this section, first by using Theorem 2.1 we will prove the following result.

Theorem 3.1. *Let E and F be Freschet-Schwarts spasces having $(\tilde{\Omega})$ and (DN) respectively. Assume that E is nuclear. Then every separately holomorphic function on $D \times F'$, an open set in $E \times F'$, is holomorphic.*

Proof. Let $f : D \times F' \to \mathcal{C}$ be a separately holomorphic function and $\omega_0 \in D \times F'$. Without loss of generality we can assume that $w_0 = 0$. Consider the function $f_D : D \to \mathcal{H}(F')$ defined by $f_D(x) = f(x, .)$.

From the Frechet-Schwartz property of F, we have the continuity of f_D and hence it is holomorphic. Since $\mathcal{H}(F') \in (DN)$ [4] and $E \in (\tilde{\Omega})$, by using Theorem 2.1 we see that f_D is locally bounded on D. Thus without loss of generality we may assume that f_D is bounded on D.

Similarly, we can consider the function

$$f_{F'} : F' \to \mathcal{H}^\infty(D), \quad f_{F'}(u) = f(., u),$$

where $H^\infty(D)$ is the Banach space of bounded holomorphic functions on D. It is easy to see that $f_{F'}$ is holomorphic. Indeed, noting that $f_{F'} : F' \to \mathcal{H}^\infty(D)$ is Gateaux holomorphic we can write the Taylor expansion of $f_{F'}$ at $0 \in F'$ as

$$f_{F'}(u) \;=\; \sum_{n\geq 0} P_n f_{F'}(u).$$

Let $k \geq 1$. Since f_D is bounded on D and $\{u \in F'_k : \|u\|^*_k \leq r\}$, $r > 0$ is relatively compact in F'_{k+1}, we have

$$C_r = \sup\Big\{|f(x,u)| : x \in D,\ \|u^*\|^*_k \leq r\Big\} <\ \infty.$$

It follows that

$$\begin{aligned}
&\sup\Big\{|P_n f_{F'}(u)(x)| : x \in D,\ \|u\|^*_k \leq s\Big\}\\
&= \sup\bigg\{\bigg|\frac{1}{2\pi i}\int\limits_{|\lambda|=r} \frac{f_{F'}(\lambda u)(x)}{\lambda^{n+1}} d\lambda\bigg| : x \in D,\ \|u\|^*_k \leq s\bigg\}\\
&= \sup\bigg\{\bigg|\frac{1}{2\pi i}\int\limits_{|\lambda|=r} \frac{f(x,\lambda u)}{\lambda^{n+1}} d\lambda\bigg| : x \in D,\ \|u\|^*_k \leq s\bigg\}\\
&\leq \frac{C_{rs}}{r^{n+1}} \quad \forall\ n \geq 0 \quad \forall r, s > 0.
\end{aligned}$$

Hence the series $\sum\limits_{n\geq 0} P_n f(u)$ is convergent in $\mathcal{H}^\infty(D)$ uniformly on every compact subset of F'_k. Therefore $f_{F'}|_{F'_k} : F'_k \to \mathcal{H}^\infty(D)$ is holomorphic. Since F is Frechet-Schwartz, $f_{F'}$ is holomorphic [1, p. 61]. This yields the local boundedness of f on $D \times F'$. On the other hand, since f is Gateaux holomorphic, by [1, Corollary 2.9] f is holomorphic. □

Theorem 3.2. *Let E and F be Frechet-Schwartz spaces having $(\overline{\overline{\Omega}})$ and (DN), respectively. Assume that E is nuclear. Then every separately holomorphic function on an open set $E \times D$ in $E \times F'$, is holomorphic.*

Here we recall that $E \in (\overline{\overline{\Omega}})$ if

$$\forall\ p\ \ \exists\ q\ \ \forall\ k\ \forall\ d\ \exists\ C > 0 : \|.\|^{*1+d}_q \leq C\|.\|^*_k \|.\|^{*d}_p.$$

Proof. Let $f : E \times D \to \mathbf{C}$ be a seperately holomorphic function and ω_0 be a point of $E \times D$. Without loss of generality we may assume that $\omega_0 = 0$.

Since separately holomorphic functions defined on open subsets of a product of dual Frechet-Schwartz spaces (DFS-spaces) are holomorphic [1, Example 2.14], we deduce that $f|_{M\times D}$ is holomorphic for all finite domensional subspaces M of E. This in particular implies that the function $\hat{f} : E \to \mathcal{H}(D)$ associated to f:

$$\hat{f}(z)(u) = f(z,u), \quad z \in E, u \in D,$$

is Gateaux holomorphic. On the other hand, we observe the following facts:

(i) $f|_{E\times D\cap F'_k}$ is holomorphic for every $k \geq 1$, where $D \cap F'_k$ is considered as an open subset of F'_k.

(ii) Every compact subset of D is contained and compact in $D \cap F'_k$ for some $k \geq 1$, since F is a DFS-space.

Combining these facts and noting that E is nuclear, we see that $\hat{f}$ is bounded on every bounded subset of E. Therefore $\hat{f}$ is holomorphic.

According to a result of Vogt ([12]), there exists a Banach space B such that F is a subspace of $B\hat{\otimes}_\pi s$, where s is the space of rapidly decreasing sequences.

Consider the restriction map $R : B'\hat{\otimes}_\pi s' \to F'$. Since F is a Frechet-Schwartz space, the map R is open.

We choose an open polydics

$$\mathcal{D}_a = \Big\{\xi = (\xi_j) \in s' : \sup |\xi_j| a_j < 1\Big\} \subset s',$$

where $a = (a_j) \in s$, $a_j \geq 0$ for all $j \geq 1$ such that $\overline{conv(V \otimes \mathcal{D}_a)} \subset R^{-1}(D)$, where V denotes the unit ball of B.

Take $k \geq 1$ sufficiently large such that $\sum\limits_{j=1}^{\infty} \dfrac{1}{j^k} \leq 2$. Put $c = (2j^k a_j) \in s$.

It is then easy to check that

$$\mathcal{D}_c^V = \Big\{\sum_{j=1}^{\infty} x_j \otimes \xi_j e_j^* : \ \bar{x} = (x_j) \subset V, \xi = (\xi_j) \in \mathcal{D}_c\Big\}$$

is a neighbourhood of $0 \in B'\hat{\otimes}_\pi s'$ and contained in $\overline{conv(V \otimes \mathcal{D}_a)}$.

Consider the Frechet space $\mathcal{H}_b(\mathcal{D}_c^V)$ of holomorphic functions h on $\mathcal{D}_c^V$ satisfying

$$\|h\|_{\tilde{K}} = \sup\Big\{\Big|h\Big(\sum_{j\geq 1} x_j \otimes \xi_j e_j^*\Big)\Big| : \bar{x} = (x_j) \subset V, \xi = (\xi_j) \in K\Big\} < \infty,$$

for all compact sets K in $\mathcal{D}_c$, where

$$\tilde{K} = \Big\{\sum_{j\geq 1} x_j \otimes \xi_j e_j^* : (x_j) \subset V, (\xi_j) \in K\Big\}.$$

Observe that $R(\tilde{K})$ is bounded and contained in $\overline{R(conv(V \otimes D_c))} \subset D$. Since every bounded set in F' is relatively compact, $R(\tilde{K})$ is relatively compact in D. It follows that R induces a continuous linear map $\hat{R} : \mathcal{H}(D) \to \mathcal{H}_b(D_c^V)$. Consider the fnction $g = f_0(id_E \times R)$ on D_c^V. It is easy to see that $\hat{g} = \hat{R}\hat{f} : E \to \mathcal{H}_b(D_c^V)$ and thus $\hat{g}$ is holomorphic.

As in the proof of Theorem 3.1, it suffices to prove the following assertions.

Assertion 1.

$$\mathcal{H}_b(\mathcal{D}_c^V) \text{ has } (\underline{DN}).$$

Assertion 2. $\hat{g}$ is of uniformly bounded type.

Proof of Assertion 1.

a) According to the proof of Proposition 3.6 in [6] we can find a matrix $\mathcal{Q} = [q_{j,k}], q_{j,k} \ \geq 0$, satisfying

(i) $\forall\ n\ \exists\ k, \epsilon > 0 \qquad q_{j,n}^{1+\epsilon} \leq q_{j,k} q_{j,1}^{\epsilon}\ \forall\ j \geq 1.$

(ii) $\forall n\ \exists k > n$:

$$\sup \frac{q_{j,n}}{q_{j,k}} < 1 \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{q_{j,n}}{q_{j,k}} < \infty.$$

(iii) The map $\theta : \mathcal{H}(\mathcal{D}_a) \to \Lambda(M, Q^M)$ given by $\theta(f) = \big(a_m(f0)_{m \in M}$ with

$$\Big(a_m(f) = \Big(\frac{1}{2\pi i}\Big)^n \int\limits_{|\lambda_1|=r_1} \cdots \int\limits_{|\lambda_n|=r_n} \frac{f\Big(\sum\limits_{j=1}^{n} \lambda_j e_j^*\Big)}{\lambda^{m+1}} d\lambda\Big), \quad 0 < r_j < \frac{1}{a_j}\ \forall\ j \geq 1,$$

is an isomorphism of $\mathcal{H}(\mathcal{D}_a)$ and $\Lambda(M, Q^M)$, where

$$\Lambda(M, Q^M) = \Big\{\xi = (\xi_m)_{m \in M} :\ \|\xi\|_k = \sup\{|\xi_m| q_k^m :\ m \in M\} < \infty \forall\ \ k \geq 1\Big\}$$

and $q_k^m = q_{1,k}^{m_1} \ldots q_{n,k}^{m_n}$ for $m = (m_1, \ldots, m_n, \ldots, 0, \ldots) \in M$.

From (i) and (iii) we deduce that $\mathcal{H}_b(D_c) \in \underline{(DN)}$.

It follows from the isomorphicity of θ that for each k there exist $C_k > 0$, $l_k \geq k$, such that

$$\begin{aligned} |||\varphi|||_k \ &:= \sup\Big\{\Big|a_m(\varphi_{\bar{x}})\Big| q_k^m :\ \bar{x} \subset B, m \in \mathbf{M}\Big\} \\ &\leq \sup\Big\{\Big|\varphi\Big(\sum_{j=1}^{\infty} x_j \otimes \xi_j e_j^*\Big)\Big| :\ \bar{x} \subset B, \xi \in N_k\Big\} = \|\varphi\|_{\tilde{N}_k} \ := \ \|\varphi\|_{l_k}, \end{aligned}$$

where

$$N_k = \Big\{(\xi_j) :\ |\xi_j| \leq q_{j,k} \forall\ j \geq 1\Big\}.$$

and

$$\varphi_{\overline{x}}(\xi) = \varphi\Big(\sum_{j=1}^{\infty} x_i \otimes \xi_j e_j^*\Big).$$

Hence $|||.|||_k$ is a continuous semi-norm on $\mathcal{H}_b(\mathcal{D}_c^V)$ for $k \geq 1$.

On the other hand, since for every $n \geq 1$ there exists $k > n$ such that

$$\sum_{j=1}^{\infty} \frac{q_{j,n}}{q_{j,k}} < \infty \quad \text{and} \quad 0 \leq \ \sup_{j \geq 1} \frac{q_{j,n}}{q_{j,k}} < 1,$$

for every $\varphi \in \mathcal{H}_b(\mathcal{D}_c^V)$ we have

$$\begin{aligned}\|\varphi\|_n &\le \sup\Big\{\sum_{\mathbf{M}} |a_m(\varphi_{\bar{x}})||\xi^m| \ : \bar{x} \subset V, \xi \in N_n\Big\} \\ &\le |||\varphi|||_k \times \sum_{m\in M}\Big(\frac{q_n}{q_k}\Big)^m \\ &= |||\varphi|||_k \times \prod_{j=1}^{\infty}\sum_{j\ge 1}^{\infty}\Big(\frac{q_{j,n}}{q_{j,k}}\Big)^{m_j} \\ &= |||\varphi|||_k \times \prod_{j=1}^{\infty}\Big(1-\frac{q_{j,n}}{q_{j,k}}\Big)^{-1},\end{aligned}$$

where the last equality follows from (ii).

Since $\{N_k\}$ is an exhaustion sequence of compact sets in $\mathcal{D}_c$, $\{|||.|||_k\}$ is a fundamental system of semi-norms on $\mathcal{H}_b(\mathcal{D}_c^V)$ and hence $\mathcal{H}(D_c^V)$ is isomorphic to a subspace of $\lambda_V(\mathbf{M}, Q^{\mathbf{M}})$, where

$$\Lambda_V(\mathbf{M}, Q^{\mathbf{M}}) = \{(\xi_{m,\bar{x}})_{m\in\mathbf{M},\bar{x}\subset V} \subset \mathbf{C} : \\ \sup\{|\xi_{m,\bar{x}}|q_k^m : m \in \mathbf{M}, \bar{x} \subset V\} < \infty, \forall k \ge 1\}.$$

Morever, from (i) of (a) it follows that $\lambda_V(\mathbf{M}, Q^{\mathbf{M}}) \in (\underline{DN})$; hence $\mathcal{H}_b(\mathcal{D}_c^V) \in (\underline{DN})$.

Proof of Assertion 2.

Since $E \in (\overline{\overline{\Omega}})$ we can find a compact balanced convex subset B of E for which

$$(\overline{\overline{\Omega}})_B \ \forall\ p\exists\ q\forall\ d > 0\exists\ C > 0 : \|.\|_1^{*1+d} \le C\|.\|_B^{*}\|.\|_p^{*d}.$$

On the other hand, since $\mathcal{H}_b(\mathcal{D}_c^V) \in (\underline{DN})$, by [5] we see that $\tilde{g}$ is of uniformly bounded type. □

4. Some examples

We will establish in this section some examples on the existence of a Frechet valued holomorphic function which is *not* locally bounded.

Example 4.1. *Let X be a complex space having a non-bounded holomorphic function σ on X and B a Banach space of infinite dimension. Then there exists a holomorphic function $f : B \to \mathcal{H}(X)$ which is not locally bounded.*

Proof. Without loss of generality we may assume that $\sup Re\sigma(z) = +\infty$, $z \in \mathbf{C}$. By [4] there exists a sequence $\{u_n\} \subset B^*$ such that $u_n(x) \to 0$ for $x \in B$ but $0 < \delta \le \|u_n\| \le 1 \ \ \forall\ n \ge 1$.

Then the formula

$$f(x)(z) = \sum_{n\ge 0} e^{n\sigma(z)}[u_n(x)]^n \text{ for } x \in B,\ z \in X,$$

defines a $\mathcal{H}(X)$-valued holomorphic function on B. Note that is not locally bounded at $0 \in B$. Otherwise, there exists $\epsilon > 0$ such that

$$M_r = \sup\Big\{|f(x)(z) : \|x\| \leq \epsilon, z \in K_r\Big\} < \infty$$

for all $r > 0$, where $\{K_r\}$ is an exhausion sequence of compact sets in X.

This yields

$$\begin{aligned} C_{n,r} &= \sup\Big\{|e^{n\sigma(z)}[u_n(x)]^n| : \|x\| \leq \epsilon, z \in K_r\Big\} \\ &= \sup\Big\{|P_n f(x)(z)| : \|x\| \leq \epsilon\ , z \in K_r\Big\} \\ &= \sup\Big\{\Big|\frac{1}{2\pi i}\int\limits_{|\lambda|=1}\frac{f(\lambda x)(z)}{\lambda^{n+1}}dz\Big| : \|x\| \leq \epsilon\ , z \in K_r\Big\} \leq M_r \end{aligned}$$

for all $r > 0$, $n \geq 0$.

Choose a sequence $(x_n) \subset B$ with $\|x_n\| \leq \epsilon$ such that $|u_n(x_n)| \geq \delta\epsilon\ \forall\ n \geq 1$. Then for $r > 0$ sufficiently large we arrive at a contradiction:

$$\begin{aligned} \infty &= \sup\Big\{|e^{\sigma(z)}u_n(x_n)|^n : n \geq 1, z \in K_r\Big\} \\ &\leq \sup\Big\{|e^{\sigma(z)}u_n(x)|^n : \|x\| \leq \epsilon, z \in K_r\Big\} = C_{n,r} \leq M_r < \infty. \end{aligned}$$

□

Example 4.2. *Let $\alpha = (\alpha_j)$ be an exponent sequence and B a Banach space of infinite dimention. Then there exists a non-locally bounded holomorphic function from B into $\Lambda_\infty(\alpha)$.*

Proof. Choose a sequence $j_k \nearrow \infty$ such that $[\alpha_{j_k}] < [\alpha_{j_{k+1}}]$ for $k \geq 1$ and let $\{u_n\}$ be chosen as in Example 4.1. It is easy to see that the map

$$\Lambda_\infty(\alpha_{j_k}) \ni (\eta_{j_k}) \mapsto (\xi_j) \in \Lambda_\infty(\alpha),$$

where

$$\xi_j \ = \ \begin{cases} \eta_{j_k} & \text{if } j = j_k \\ 0 & \text{if } j \neq j_k \end{cases} \qquad \forall\ k \geq 1,$$

defines $\Lambda_\infty(\alpha_{j_k})$ as a subspace of $\Lambda_\infty(\alpha)$.

Note that

$$\Lambda_\infty(\alpha_{j_k}) \cong \overline{span}\Big(z^{[\alpha_{j_k}]}\Big)_{k=1}^\infty \subset \mathcal{H}(\mathbf{C}).$$

and the sequence $\{u_n\}$ converges weakly to $0 \in B$. Then it converges uniformly on every compact set of B. Hence the series

$$\sum_{n\geq 1}[u_n(x)]^n \sum_{k\geq 1}\frac{n^{[\alpha_{j_k}]}z^{[\alpha_{j_k}]}}{[\alpha_{j_k}]!}$$

converges uniformly on compact sets in $B \times \mathbf{C}$. It implies that the formula

$$f(x)(z) = \sum_{n \geq 1} [u_n(x)]^n \sum_{k \geq 1} \frac{n^{[\alpha_{j_k}]} z^{[\alpha_{j_k}]}}{[\alpha_{j_k}]!}$$

defines a holomorphic function $f : B \to \Lambda_\infty(\alpha_{j_k})$. If f is locally bounded on B, then there exists $\epsilon > 0$ such that for every $r > 0$ and $n \geq 0$

$$\begin{aligned} C_{n,r} &= \sup\Big\{ |u_n(x)|^n \Big| \sum_{k \geq 1} \frac{n^{[\alpha_{j_k}]} z^{[\alpha_{j_k}]}}{[\alpha_{j_k}]!} \Big| : \|x\| < \epsilon, |z| < r \Big\} \\ &= \sup\Big\{ |P_n f(x)(z)| : \|x\| < \epsilon, |z| < r \Big\} \leq M_r, \end{aligned}$$

where

$$M_r = \sup\Big\{ |f(x)(z)| : ||x|| < \epsilon, |z| < r \Big\}.$$

By taking $r = \dfrac{2}{\delta\epsilon}$ and putting for each $m \geq 1$

$$A_m = \Big| u_{[\alpha_{j_m}]}(x_{[\alpha_{j_m}]}) \Big|^{[\alpha_{j_m}]} \frac{[\alpha_{j_m}]^{[\alpha_{j_m}]} r^{[\alpha_{j_m}]}}{[\alpha_{j_m}]!}$$

we have

$$\sup\Big\{ A_m : m \geq 1 \Big\} \geq \sup\Big\{ \delta^{[\alpha_{j_m}]} \epsilon^{[\alpha_{j_m}]} \frac{[\alpha_{j_m}]^{[\alpha_{j_m}]} 2^{[\alpha_{j_m}]}}{[\alpha_{j_m}]! \delta^{[\alpha_{j_m}]} \epsilon^{[\alpha_{j_m}]}} : m \geq 1 \Big\} \quad = \infty.$$

On the other hand, for $g_n(z) = \sum_{k \geq 1} \dfrac{n^{[\alpha_{j_k}]} z^{[\alpha_{j_k}]}}{[\alpha_{j_k}]!}$ we have

$$\begin{aligned} \Big| \frac{n^{[\alpha_{j_m}]} z^{[\alpha_{j_m}]}}{[\alpha_{j_m}]!} \Big| \;=\; \Big| P_{[\alpha_{j_m}]} g_n(z) \Big| \;&=\; \Big| \frac{1}{2\pi i} \int |\lambda| = 1 \frac{g_n(\lambda z)}{\lambda^{[\alpha_{j_m}]+1}} d\lambda \Big| \\ &\leq \sup\Big\{ |g_n(u)| : |u| \leq r \Big\}, \quad \forall |z| < r. \end{aligned}$$

In particular, we obtain

$$\frac{[\alpha_{j_m}]^{[\alpha_{j_m}]} r^{[\alpha_{j_m}]}}{[\alpha_{j_m}]!} \leq \sup\Big\{ \Big| \sum_{k \geq 1} \frac{[\alpha_{j_m}]^{[\alpha_{j_k}]} z^{[\alpha_{j_k}]}}{[\alpha_{j_k}]!} \Big| : |z| < r \Big\}.$$

Thus we have

$$\begin{aligned} \infty &= \sup\Big\{ A_m : m \geq 1 \Big\} \\ &\leq \sup\Big\{ |u_{[\alpha_{j_m}]}(x)|^{[\alpha_{j_m}]} \Big| \sum_{k \geq 1} \frac{[\alpha_{j_m}]^{[\alpha_{j_k}]} z^{[\alpha_{j_k}]}}{[\alpha_{j_k}]!} \Big| : |z| < r \Big\} \\ &\leq \sup C_{[\alpha_{j_m}],r} \leq M_r < \infty. \end{aligned}$$

We arrive at a contradiction and the proof is complete. $\square$

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