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STRONG MINIMALITY OF GAUSSIAN-SUMMING NORM

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Dedicated to Nguyen Duy Tien on the occassion of his 60th birthday

ABSTRACT. By means of a sequence $\varphi. := (\varphi_n)_{n\in\mathbb{N}}$ of square-integrable functions a notion of a $\varphi.$-summing operator is defined. It is shown that if $\inf_n \|\varphi_n\|_2 > 0$, then any $\varphi.$-summing operator is Gaussian-summing. This recovers a previously known result, which asserts the same in case when $\varphi. := (\varphi_n)_{n\in\mathbb{N}}$ is an orthonormal sequence.

1. Introduction

Let $\varphi. := (\varphi_n)_{n\in\mathbb{N}}$ be a sequence of square-integrable functions given on a positive measure space $(\Omega, \mathcal{A}, \nu)$ such that $\|\varphi_n\|_2 > 0$, $n = 1, 2, \ldots$. Fix a continuous linear operator T defined on a Banach space X with values in a Banach space Y and a natural number n. We denote by $\|T\|_{n,\varphi.}$ the least constant $c \geq 0$ such that for any $x_1, \ldots, x_n \in X$ the following inequality holds:

$$(1.1)\qquad \Big(\int_\Omega \Big\|\sum_{k=1}^n Tx_k\varphi_k(\omega)\Big\|^2 d\nu(\omega)\Big)^{1/2} \leq c \sup_{x^*\in B_{X^*}} \Big(\sum_{k=1}^n |x^*(x_k)|^2\Big)^{1/2}.$$

The mapping $T \to \|T\|_{n,\varphi.}$ is a norm on the space $L(X, Y)$ of all continuous linear operators. In [PieWe, (3.11.1)] (when $\varphi. := (\varphi_n)_{n\in\mathbb{N}}$ is an orthonormal sequence) the quantity $\|T\|_{n,\varphi.}$ is denoted by $\pi(T|(\varphi_1, \ldots, \varphi_n))$ and the mapping $T \to \pi(T|(\varphi_1, \ldots, \varphi_n))$ is called a *Parseval ideal norm*.

The operator T will be called $\varphi.$*-summing* (or $\varphi.$*-bounding*) if

$$\|T\|_{\varphi.} := \sup_n \|T\|_{n,\varphi.} < \infty.$$

The set of all $\varphi.$-summing operators $T : X \to Y$ will be denoted $\Pi_{\varphi.}(X, Y)$. It seems that in [PieWe] no special notation is fixed for this class. The mapping $T \to \|T\|_{\varphi.}$ is a norm on $\Pi_{\varphi.}(X, Y)$ and is called $\varphi.$*-summing norm.*

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If $g. := (g_n)_{n\in\mathbb{N}}$ is a sequence of independent standard Gaussian random variables given on a probability space, then the class $\Pi_{g.}(X, Y)$ of $g.$-summing operators will coincide with the class of *Gaussian-summing* (or γ-summing) operators introduced in [LiPie] (see also [PieWe, (4.15.7)]).

If $r. := (r_n)_{n\in\mathbb{N}}$ is the sequence of Rademacher functions given on $[0, 1]$ with the Lebesgue measure, then the class $\Pi_{r.}(X, Y)$ of $r.$-summing operators will coincide with the class $\Pi_{as}(X, Y)$ of *almost summing* operators introduced in [DJT].

If $\varphi. := (\varphi_n)_{n\in\mathbb{N}}$ is a sequence of independent identically distributed symmetric random variables given on a probability space, then the class $\Pi_{\varphi.}(X, Y)$ of $\varphi.$-summing operators appeared in [BTV].

The class of $\varphi.$-summing operators when $\varphi. := (\varphi_n)_{n\in\mathbb{N}}$ is an arbitrary orthonormal sequence was explicitly defined and studied in details in [Ba].

In [DJT, Theorem 12.12] was shown that for any pair of Banach spaces one has the coincidence $\Pi_{as}(X, Y) = \Pi_{g.}(X, Y)$ together with the inequalities

$$\left(\frac{2}{\pi}\right)^{1/2}\|T\|_{r.} \le \|T\|_{g.} \le \|T\|_{r.}.$$

It is known also that for any pair of Banach spaces and *for any orthonormal sequence* $\varphi. := (\varphi_n)_{n\in\mathbb{N}}$ the inclusion $\Pi_{\varphi.}(X, Y) \subset \Pi_{g.}(X, Y)$ remains true and the inequality $\|T\|_{g.} \le \|T\|_{\varphi.}$ holds (cf. [PieWe, (4.15.3) THEOREM]; see also [GeJu, Remark 3.10] where this fact is mentioned as known, but the proof is given too. In [Ba, p.16] the considered result also is presented as known and in its connection is quoted [GeJu, Remark 3.10] and [Se, Theorem 5.5]).

The above results motivate the appearance of the present note. We will show that for any pair of Banach spaces and *for any normalized sequence* $\varphi. := (\varphi_n)_{n\in\mathbb{N}}$ the inclusion $\Pi_{\varphi.}(X, Y) \subset \Pi_{g.}(X, Y)$ remains true and the inequality $\|T\|_{g.} \le \|T\|_{\varphi.}$ takes place as well (Theorem 3.1).

It arises naturally the problem of non-triviality of the class $\Pi_{\varphi.}(X, Y)$. It turns out that for a given pair of infinite-dimensional Banach spaces X, Y we have $\Pi_{\varphi.}(X, Y) \ne \{0\}$ if and only if $\Pi_{\varphi.}(\mathbb{R}, \mathbb{R}) \ne \{0\}$. The proof of this and the other related results will appear elsewhere.

2. Auxiliary resuts

Thereafter $\mathbb{K}$ will denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. The considered normed or inner-product spaces will be supposed to be defined over $\mathbb{K}$. The norm of a normed space, resp., the scalar product of an inner-product space will be denoted by $||\cdot||$ and $(\cdot|\cdot)$, respectively. Also $||T||$ will stand for the ordinary norm of a continuous linear operator T acting between normed spaces.

For a normed space X,

- X^* will stand for the topological dual space,

- We put:

$$B_X := \{x \in X : ||x|| \le 1\}, \quad S_X := \{x \in X : ||x|| = 1\},$$

- $\mathcal{FD}(X)$ will stand for the collection of all finite-dimensional non-zero vector subspaces of X.

For normed spaces X and Y,

- $L(X,Y)$ is the normed space of all continuous linear operators $T : X \to Y$ and $L(X) := L(X,X)$,
- $K(X,Y)$ will stand for the set of all compact linear operators $T : X \to Y$ and $K(X) := K(X,X)$.

For a Hilbert space H,

- $\mathcal{O}_1(H)$ is the set of all orthonormal sequences in H and $\mathcal{O}_2(H)$ is the set of all orthonormal basis of H,
- $\mathcal{U}(H)$ will stand for the set of isometric surjective linear operators $u : H \to H$.

The following proposition collects the important known statements about a finite-dimensional Hilbert space.

Proposition 2.1. *Let E be a finite-dimensional Hilbert space with $n := \dim(E) \ge 1$.*

(a) *$\mathcal{U}(E)$ with the topology induced by operator norm is a compact metrizable topological group, which carries the unique tranlation-invariant probability measure m given on its Borel σ-algebra.*

The measure m is called the (normalized) Haar measure of $\mathcal{U}(E)$.

(b) *There exists the unique $\mathcal{U}(E)$-invariant probability measure $\mathbf{s}$ given on the Borel σ-algebra of E such that $\mathbf{s}(S_E) = 1$.*

The measure $\mathbf{s}$ is called the uniform distribution on S_E.

(c) *If $f : E \to \mathbb{C}$ is a Borel measurable $\mathbf{s}$-integrable (or non-negative) function, then for each fixed $e \in S_E$*

$$\int_{S_E} f(x)ds(x) = \int_{\mathcal{U}(E)} f(ue)dm(u). \tag{2.1}$$

(d) *There exists the unique $\mathcal{U}(E)$-invariant probability measure γ given on the Borel σ-algebra of E, such that*

$$\hat{\gamma}(h) := \int_E \exp(i\mathrm{Re}(x|h))d\gamma(x) = \exp(-\kappa\|h\|^2), \quad \forall h \in E. \tag{2.2}$$

The measure γ is called the standard Gaussian measure. In (2.2) *the parameter κ is $1/2$ in real case and is 1 in complex case.*

(e) *If $f : E \to \mathbb{C}$ is a positively 2-homogeneous Borel measurable γ-integrable (or non-negative) function, then*

$$\int_E f(x)d\gamma(x) = n \int_{S_E} f(x)ds(x). \tag{2.3}$$

Proof. The statements (a) and (b) are well-known.

(c) Fix $e \in S_E$. Denote s_e the image of m under the continuous mapping $u \to ue$ from $\mathcal{U}(E)$ onto S_E. Then by change-variable formula we can write

$$\int_{S_E} f(x)ds_e(x) = \int_{\mathcal{U}(E)} f(ue)dm(u). \tag{2.4}$$

By using translation-invariance of m it is easy to observe that s_e is a $\mathcal{U}(E)$-invariant probability measure. By *uniqueness part* of (b) we get that $s_e = s$. This and (2.4) imply (2.1).

(d) is well-known.

(e) It is easy to observe that

$$\int_E \|x\|^2 d\gamma(x) = n.$$

For each Borel subset $B \subset E$, put

$$\mu_1(B) = \frac{1}{n} \int_B \|x\|^2 d\gamma(x).$$

Then μ_1 is a Borel probability measure on E. Denote μ_2 the image of μ_1 under the continuous mapping $x \to \dfrac{x}{\|x\|}$ from $E \setminus \{0\}$ onto S_E. Since $\gamma(\{0\}) = 0$,

$$\int_E f(x)d\gamma(x) = \gamma(\{0\})f(0) + \int_{E\setminus\{0\}} f\Big(\frac{x}{\|x\|}\Big)\|x\|^2 d\gamma(x) = n \int_{E\setminus\{0\}} f\Big(\frac{x}{\|x\|}\Big)d\mu_1(x).$$

Then by change-variable formula we can write

$$\int_E f(x)d\gamma(x) = n \int_{E\setminus\{0\}} f\Big(\frac{x}{\|x\|}\Big)d\mu_1(x) = n \int_{S_E} f(x)d\mu_2(x). \tag{2.5}$$

By using $\mathcal{U}(E)$-invariance of γ it is easy to observe that μ_2 also is $\mathcal{U}(E)$-invariant probability measure. By *uniqueness part* of (b) we get that $\mu_2 = s$. This and (2.5) imply (2.3). □

Lemma 2.1. *Let E be a non-zero normed space and μ be a positive measure (not necessarily finite) given on the Borel σ-algebra of E such that*

$$0 < \alpha := \int_E \|x\|^2 d\mu(x) < \infty.$$

Then there exists a probability measure μ_2 given on the Borel σ-algebra of E such that $\mu_2(S_E) = 1$ and for any positively 2-homogeneous Borel measurable μ-integrable (or non-negative) function $f : E \to \mathbb{C}$ with $f(0) = 0$, the equality

$$\int_E f(x)d\mu(x) = \alpha \int_{S_E} f(x)d\mu_2(x) \tag{2.6}$$

holds.

Proof. Replace in the proof of Proposition 2.1(e) γ by μ and n by α. ☐

The next statement expresses "the strong minimality" of the Gaussian integrals.

Proposition 2.2. *Let E be a finite-dimensional Hilbert space with $n := \dim(E) \geq 1$ and μ be a positive measure (not necessarily finite) given on the Borel σ-algebra of E such that*

$$0 < \alpha := \int_E \|x\|^2 d\mu(x) < \infty.$$

Then for any positively 2-homogeneous Borel measurable function $f : E \to \mathbb{C}$ with $f(0) = 0$, we have:

$$\int_E |f(x)|ds(x) \leq \frac{1}{\alpha} \sup_{u \in \mathcal{U}(E)} \int_E |f(ux)|d\mu(x) \tag{2.7}$$

and

$$\int_E |f(x)|d\gamma(x) \leq \frac{n}{\alpha} \sup_{u \in \mathcal{U}(E)} \int_E |f(ux)|d\mu(x) \tag{2.8}$$

holds.

Proof. Let μ_2 be the probability measure associated with μ according to Lemma 2.1. Take a positively 2-homogeneous Borel measurable function $f : E \to \mathbb{C}$ with $f(0) = 0$. For each fixed $u \in \mathcal{U}(E)$ we can apply the equality (2.6) to the function $x \to |f(ux)|$ and write:

$$\int_E |f(ux)|d\mu(x) = \alpha \int_{S_E} |f(ux)|d\mu_2(x). \tag{2.9}$$

Hence,

$$\sup_{u \in \mathcal{U}(E)} \int_E |f(ux)|d\mu(x) = \alpha \sup_{u \in \mathcal{U}(E)} \int_{S_E} |f(ux)|d\mu_2(x). \tag{2.10}$$

Since m is a normalized positive measure on $\mathcal{U}(E)$, we have:

$$\sup_{u\in\mathcal{U}(E)}\int_{S_E}|f(ux)|d\mu_2(x)\geq\int_{\mathcal{U}(E)}\Big(\int_{S_E}|f(ux)|d\mu_2(x)\Big)dm(u) \tag{2.11}$$
$$=\int_{S_E}\Big(\int_{\mathcal{U}(E)}|f(ue)|dm(u)\Big)d\mu_2(e).$$

By Proposition 2.1(c) for each fixed $e\in S_E$ we have

$$\int_{\mathcal{U}(E)}|f(ue)|dm(u)=\int_{S_E}|f(x)|ds(x). \tag{2.12}$$

Then, as $m_2(S_E)=1$,

$$\int_{S_E}\Big(\int_{\mathcal{U}(E)}|f(ue)|dm(u)\Big)d\mu_2(e)=\int_{S_E}\Big(\int_{S_E}|f(x)|ds(x)\Big)d\mu_2(e) \tag{2.13}$$
$$=\int_{S_E}|f(x)|ds(x).$$

Now (2.10), (2.11) and (2.13) imply

(2.14)
$$\sup_{u\in\mathcal{U}(E)}\int_{E}|f(ux)|d\mu(x)\geq\alpha\int_{S_E}\Big(\int_{\mathcal{U}(E)}|f(ue)|dm(u)\Big)d\mu_2(e)=\alpha\int_{S_E}|f(x)|ds(x).$$

Clearly (2.14) gives (2.7).

Inequality (2.8) follows from (2.7) and Proposition 2.1(e). □

Let $(\Omega,\mathcal{A},\nu)$ be a positive measure space, X a normed space. Then for any vector-valued function $\xi:\Omega\to X$ for which the scalar-valued function $\omega\to\|\xi(\omega)\|$ is measurable, we put

$$\|\xi\|_2=\Big(\int_{\Omega}\|\xi(\omega)\|^2d\nu(\omega)\Big)^{1/2}.$$

If $\varphi_1,\dots,\varphi_n$ are measurable scalar functions given on Ω and $x_1,\dots,x_n$ are elements of a given normed space X, then the vector-valued function $\omega\to\sum_{k=1}^{n}x_k\varphi_k(\omega)$ will be denoted $\sum_{k=1}^{n}x_k\varphi_k$. Accordingly we have

$$\Big\|\sum_{k=1}^{n}x_k\varphi_k\Big\|_2=\Big(\int_{\Omega}\Big\|\sum_{k=1}^{n}x_k\varphi_k(\omega)\Big\|^2d\nu(\omega)\Big)^{1/2}.$$

A sequence $g_\cdot:=(g_n)_{n\in\mathbb{N}}$ of independent $\mathbb{K}$-valued random variables given on a probability space $(\Omega,\mathcal{A},\mathbb{P})$ is called *a standard Gaussian sequence* if for each natural n the distribution of g_n coincides with the standard Gaussian measure given on $\mathbb{K}$. Note that if $g. := (g_n)_{n\in\mathbb{N}}$ is a standard Gaussian sequence, then it is an orthonormal sequence in the Hilbert space $\mathbf{L}_2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{K})$.

In the sequel, $g. := (g_n)_{n\in\mathbb{N}}$ always will stand for a standard Gaussian sequence.

Lemma 2.2. *Let E be a finite-dimensional Hilbert space with $n := \dim(E) \geq 1$ and $(e_1, \dots, e_n)$ be any fixed orthonormal basis. Then for every Borel measurable γ-integrable (or non-negative) function $f : E \to \mathbb{C}$ we have:*

$$\int\limits_{\Omega} f\Big(\sum_{k=1}^{n} e_k g_k(\omega)\Big) d\mathbb{P}(\omega) = \int\limits_{E} f(x) d\gamma(x). \tag{2.15}$$

In particular, for any normed space Y and any $V \in L(E, Y)$ we have

$$\Big\| \sum_{k=1}^{n} V e_k g_k \Big\|_2 = \Big(\int\limits_{E} \|Vx\|^2 d\gamma(x) \Big)^{1/2}. \tag{2.16}$$

Proof. Put $\xi := \sum\limits_{k=1}^{n} e_k g_k$. Then *the distribution of ξ coincides with the standard Gaussian measure γ on E*. From this, by change-variable formula, we get

$$\int\limits_{\Omega} f\Big(\sum_{k=1}^{n} e_k g_k(\omega)\Big) d\mathbb{P}(\omega) = \int\limits_{\Omega} f(\xi(\omega)) d\mathbb{P}(\omega) = \int\limits_{E} f(x) d\gamma(x),$$

i.e., (2.15) is valid.

Fix now a normed space Y and an operator $V \in L(E, Y)$. Clearly, an application of (2.15) to the function $x \to \|Vx\|^2$ gives

$$\Big\| \sum_{k=1}^{n} V e_k g_k \Big\|_2 = \Big(\int\limits_{E} \|Vx\|^2 d\gamma(x) \Big)^{1/2}.$$

□

The next result in case of an orthonormal sequence $(\varphi_1, \dots, \varphi_n)$ of functions is due to [GeJu].

Lemma 2.3. (cf. [GeJu, Lemma 3.10 (1)]) *Let E be a finite-dimensional Hilbert space with $n := \dim(E) \geq 1$, $(e_1, \dots, e_n)$ is an orthonormal basis of E, $(\varphi_1, \dots, \varphi_n)$ be a sequence of square integrable scalar functionns given on a positive measure space $(\Omega, \mathcal{A}, \nu)$ such that*

$$0 < \alpha_n := \Big(\sum_{k=1}^{n} \|\varphi_k\|_2^2 \Big)^{1/2}.$$

Then for any normed space Y and any $V \in L(E, Y)$ the inequality

$$\Big\| \sum_{k=1}^{n} V e_k g_k \Big\|_2 \leq \frac{\sqrt{n}}{\alpha_n} \sup_{u \in \mathcal{U}(E)} \Big\| \sum_{k=1}^{n} V u e_k \varphi_k \Big\|_2 \tag{2.17}$$

holds.

Proof. Put $\xi = \sum_{k=1}^{n} e_k\varphi_k$. Since $(e_1, \dots, e_n)$ be an orthonormal sequence in E,

$$\|\xi(\omega)\|^2 = \Big\| \sum_{k=1}^{n} e_k\varphi_k(\omega)\Big\|^2 = \sum_{k=1}^{n} |\varphi_k(\omega)|^2, \quad \forall \omega \in \Omega.$$

Hence

$$\int_\Omega \|\xi(\omega)\|^2 d\nu(\omega) = \sum_{k=1}^{n} \int_\Omega |\varphi_k(\omega)|^2 d\nu(\omega) = \alpha_n^2.$$

Denote the image of ν under Borel measurable mapping $\xi : \Omega \to E$ by μ. By change-variable formula we have

$$\alpha_n^2 = \int_\Omega \Big\| \sum_{k=1}^{n} e_k\varphi_k(\omega)\Big\|^2 d\nu(\omega) = \int_\Omega \|\xi(\omega)\|^2 d\nu(\omega) = \int_E \|x\|^2 d\mu(x). \tag{2.18}$$

Let Y be a normed space and $V \in L(E, Y)$. By (2.18), for any fixed $u \in \mathcal{U}(E)$ we obtain

$$\int_\Omega \Big\| \sum_{k=1}^{n} Vue_k\varphi_k(\omega)\Big\|^2 d\nu(\omega) = \int_\Omega \|Vu\xi(\omega)\|^2 d\nu(\omega) = \int_E \|Vux\|^2 d\mu(x).$$

Then

$$\sup_{u\in\mathcal{U}(E)} \Big\| \sum_{k=1}^{n} Vue_k\varphi_k\Big\|_2 = \sup_{u\in\mathcal{U}(E)} \Big(\int_E \|Vux\|^2 d\mu(x)\Big)^{1/2}. \tag{2.19}$$

Therefore, according to (2.16) and Proposition 2.2 applied for the function $x \to \|Vx\|^2$ we have

$$\Big\| \sum_{k=1}^{n} Ve_kg_k\Big\|_2 = \Big(\int_E \|Vx\|^2 d\gamma(x)\Big)^{1/2} \le \frac{\sqrt{n}}{\alpha_n} \sup_{u\in\mathcal{U}(E)} \Big(\int_E \|Vux\|^2 d\mu(x)\Big)^{1/2}.$$

This and (2.19) imply (2.17). □

3. Strong minimality of the Gaussian-summing norm

In what follows, for a normed space X, a natural number n and a finite sequence $(x_1, \dots, x_n)$ of elements of X we shall put

$$\|(x_1, \dots, x_n)\|_2 := \Big(\sum_{k=1}^{n} \|x_k\|^2\Big)^{1/2}$$

and

$$\|(x_1, \dots, x_n)\|_{2,w} := \sup_{x^*\in B_{X^*}} \Big(\sum_{k=1}^{n} |x^*(x_k)|^2\Big)^{1/2}.$$

Let us observe that if X is an inner-product space and $(x_1, \dots, x_n)$ is an *orthonormal sequence* of elements of X, then we have $\|(x_1, \dots, x_n)\|_2 = \sqrt{n}$ and $\|(x_1, \dots, x_n)\|_{2,w} = 1$.

Let $\varphi. := (\varphi_n)_{n\in\mathbb{N}}$ be a sequence of square-integrable functions given on a positive measure space $(\Omega, \mathcal{A}, \nu)$ such that $\|\varphi_n\|_2 > 0,\ n = 1, 2, \ldots$.

Fix normed spaces X, Y, an operator $T \in L(X, Y)$ and a natural number n. We define

(3.1)

$$\|T\|_{n,\varphi.} = \sup\Big\{\Big\|\sum_{k=1}^{n} Tx_k\varphi_k\Big\|_2 : (x_1, \ldots, x_n) \in X^n,\ \|(x_1, \ldots, x_n)\|_{2,w} \le 1\Big\}.$$

It is easy to observe that the functional $T \to \|T\|_{n,\varphi.}$ is a norm on $L(X, Y)$ with the property

$$\big(\min_{k\le n}\|\varphi_k\|_2\big)\|T\| \le \|T\|_{n,\varphi.}, \quad \forall T \in L(X, Y).$$

The operator T will be called *$\varphi.$-summing* (or *$\varphi.$-bounding*) if

$$\|T\|_{\varphi.} := \sup_n \|T\|_{n,\varphi.} < \infty.$$

The set of all $\varphi.$-summing operators $T : X \to Y$ is denoted by $\Pi_{\varphi.}(X, Y)$. The mapping $T \to \|T\|_{\varphi.}$ is a norm on $\Pi_{\varphi.}(X, Y)$ and is called *$\varphi.$-summing norm.* It is easy to see that

$$\big(\inf_{k\in\mathbb{N}}\|\varphi_k\|_2\big)\|T\| \le \|T\|_{\varphi.}, \quad \forall T \in \Pi_{\varphi.}(X, Y).$$

If $g. := (g_n)_{n\in\mathbb{N}}$ is a standard Gaussian sequence, then the class $\Pi_{g.}(X, Y)$ of $g.$-summing operators coincides with the class of *Gaussian-summing* (or γ-summing) operators introduced in [LiPie] (see also [PieWe, (4.15.7)]). Accordingly, the norm $\|\cdot\|_{g.}$ is called *the Gaussian-summing norm* or *γ-summing norm* and for it usually the notation π_γ is used.

If $r. := (r_n)_{n\in\mathbb{N}}$ is the sequence of Rademacher functions given on $[0, 1]$ with the Lebesgue measure, then the class $\Pi_{r.}(X, Y)$ of $r.$-summing operators coincides with the class $\Pi_{as}(X, Y)$ of *almost summing* operators introduced in [DJT]. Accordingly, the norm $\|\cdot\|_{r.}$ is called *the almost summing norm* and for it usually the notation π_{as} is used.

If $\varphi. := (\varphi_n)_{n\in\mathbb{N}}$ is a sequence of independent identically distributed symmetric random variables given on a probability space, then the class $\Pi_{\varphi.}(X, Y)$ of $\varphi.$-summing operators appeared in [BTV].

Lemma 3.1. *Let E be a finite-dimensional Hilbert space with $n := \dim(E)$ and $T \in L(X, Y)$. Then*

$$\|T\|_{n,\varphi.} = \sup\{\|TW\|_{n,\varphi.} : W \in L(E, X),\ \|W\| \le 1\}. \tag{3.2}$$

Proof. Fix arbitrarily $(h_1, \ldots, h_n) \in E^n$, $\|(h_1, \ldots, h_n)\|_{2,w} \le 1$ and $W \in L(E, X)$, $\|W\| \le 1$. Then from $\|W\| \le 1$ we get $\|(Wh_1, \ldots, Wh_n)\|_{2,w} \le 1$. Therefore

$$\Big\|\sum_{k=1}^{n} TWh_k\varphi_k\Big\|_2 \le \|T\|_{n,\varphi.}.$$

Hence $\|TW\|_{n,\varphi.} \le \|T\|_{n,\varphi.}$ and

$$\sup\{\|TW\|_{n,\varphi.} : W \in L(E,X),\ \|W\| \le 1\} \le \|T\|_{n,\varphi.}. \tag{3.3}$$

Fix a finite sequence $(x_1, \dots, x_n) \in X^n$, satisfying $\|(x_1, \dots, x_n)\|_{2,w} \le 1$ and an orthonormal basis $(e_1, \dots, e_n) \in E^n$. Define $W \in L(E,X)$ by putting $We_k = x_k,\ k = 1, \dots, n$. Then $\|W\| = \|(x_1, \dots, x_n)\|_{2,w} \le 1$. (As $\|(e_1, \dots, e_n)\|_{2,w} = 1$),

$$\Big\|\sum_{k=1}^{n} Tx_k\varphi_k\Big\|_2 = \Big\|\sum_{k=1}^{n} TWe_k\varphi_k\Big\|_2 \le \|TW\|_{n,\varphi.}.$$

Hence

$$\Big\|\sum_{k=1}^{n} Tx_k\varphi_k\Big\|_2 \le \sup\{\|TW\|_{n,\varphi.} : W \in L(E,X),\ \|W\| \le 1\}.$$

Consequently,

$$\|T\|_{n,\varphi.} \le \sup\{\|TW\|_{n,\varphi.} : W \in L(E,X),\ \|W\| \le 1\}. \tag{3.4}$$

From (3.3) and (3.4) we obtain (3.2). ☐

Lemma 3.1 reduces the question of computing the norm $\|\cdot\|_{n,\varphi.}$ to the case where the domain is a Hilbert space. This case is treated in the next lemma.

Lemma 3.2. *Let E be a finite-dimensional Hilbert space with $n := \dim(E)$, $(e_1, \dots, e_n)$ be a fixed orthonormal basis of E and $V \in L(E,Y)$. Then there exists an orthonormal basis $(e_1^o, \dots, e_n^o)$ of E such that*

$$\|V\|_{n,\varphi.} = \sup_{u\in\mathcal{U}(E)} \Big\|\sum_{k=1}^{n} Vue_k\varphi_k\Big\|_2 = \Big\|\sum_{k=1}^{n} Ve_k^o\varphi_k\Big\|_2. \tag{3.5}$$

Proof. Since for any fixed $u \in \mathcal{U}(E)$ we have $\|(ue_1, \dots, ue_n)\|_{2,w} = 1$, the inequality

$$\|V\|_{n,\varphi.} \ge \sup_{u\in\mathcal{U}(E)} \Big\|\sum_{k=1}^{n} Vue_k\varphi_k\Big\|_2 \tag{3.6}$$

is evident.

Fix now a finite sequence $(x_1, \dots, x_n)$ in E with $\|(x_1, \dots, x_n)\|_{2,w} \le 1$. Let us show that

$$\Big\|\sum_{k=1}^{n} Vx_k\varphi_k\Big\|_2 \le \sup_{u\in\mathcal{U}(E)} \Big\|\sum_{k=1}^{n} Vue_k\varphi_k\Big\|_2. \tag{3.7}$$

Define $W \in L(E,E)$ by putting $We_k = x_k,\ k = 1, \dots, n$. Then $\|W\| = \|(x_1, \dots, x_n)\|_{2,w} \le 1$. Obviously,

$$\Big\|\sum_{k=1}^{n} Vx_k\varphi_k\Big\|_2 = \Big\|\sum_{k=1}^{n} VWe_k\varphi_k\Big\|_2.$$

It is known that any $W \in L(E)$ is a convex combination of some finite number of operators from $\mathcal{U}(E)$, i.e., $W = \sum_{j=1}^{p} t_j u_j$, where $0 \le t_j \le 1$, $\sum_{j=1}^{p} t_j = 1$ and $u_j \in \mathcal{U}(E)$, $j = 1, \ldots, n$. Using this, we can write

$$\sum_{k=1}^{n} VWe_k\varphi_k = \sum_{k=1}^{n} V\Big(\sum_{j=1}^{p} t_j u_j e_k\Big)\varphi_k = \sum_{j=1}^{p} t_j\Big(\sum_{k=1}^{n} Vu_j e_k\varphi_k\Big)$$

and so,

$$\begin{aligned}\Big\|\sum_{k=1}^{n} VWe_k\varphi_k\Big\|_2 &\le \sum_{j=1}^{p} t_j\Big\|\sum_{k=1}^{n} Vu_j e_k\varphi_k\Big\|_2 \\ &\le \max_{j\le p}\Big\|\sum_{k=1}^{n} Vu_j e_k\varphi_k\Big\|_2 \\ &\le \sup_{u\in\mathcal{U}(E)}\Big\|\sum_{k=1}^{n} Vue_k\varphi_k\Big\|_2.\end{aligned}$$

i.e., (3.7) is true. It is evident that (3.7) implies

$$\|V\|_{n,\varphi.} \le \sup_{u\in\mathcal{U}(E)}\Big\|\sum_{k=1}^{n} Vue_k\varphi_k\Big\|_2. \tag{3.8}$$

From (3.6) and (3.8) we get

$$\|V\|_{n,\varphi.} = \sup_{u\in\mathcal{U}(E)}\Big\|\sum_{k=1}^{n} Vue_k\varphi_k\Big\|_2. \tag{3.9}$$

Clearly, the supremun in (3.9) is attained in some $u_o \in \mathcal{U}(E)$. Put $e_k^o := u_o e_k$, $k = 1, \ldots, n$. Then $(e_1^o, \ldots, e_n^o)$ is an othonormal basis of E for which (3.5) is satisfied. □

In Lemma 3.2, in general, for a given fixed orthonormal basis $(e_1, \ldots, e_n)$ one may have the strict inequality $\|V\|_{n,\varphi.} > \Big\|\sum_{k=1}^{n} Ve_k\varphi_k\Big\|_2$ (see, e.g., [TTV], where the the case of the sequence of Rademacher functions is investigated). However, in the Gaussian case the situation is nicer.

Corollary 3.1. *Let E be a finite-dimensional Hilbert space with $n := \dim(E) \ge 1$, Y a normed space, $V : E \to Y$ a linear operator and $(e_1, \ldots, e_n)$ any fixed orthonormal basis of E. Then*

$$\|V\|_{n,g.} = \Big\|\sum_{k=1}^{n} Ve_k g_k\Big\|_2. \tag{3.10}$$

Proof. By Lemma 3.2 there is an orthonormal basis $(e_1^o, \dots, e_n^o)$ of E such that

$$\|V\|_{n,g.} = \Big\| \sum_{k=1}^{n} V e_k^o g_k \Big\|_2 \tag{3.11}$$

and, by Lemma 2.2, we can write

$$\Big\| \sum_{k=1}^{n} V e_k g_k \Big\|_2 = \Big(\int_E \|Vx\|^2 d\gamma(x) \Big)^{1/2} = \Big\| \sum_{k=1}^{n} V e_k^o g_k \Big\|_2 = \|V\|_{n,g.}.$$

□

For a given finite-dimensional Hilbert space E with $\dim(E) = n$, a given normed space Y and an operator $T \in L(E, Y)$, in general one may have the strict inequality $\|V\|_{\varphi.} > \|V\|_{n,\varphi.}$. However, it is well-known that this cannot happen in Gaussian case. The following statement (*which is not needed for the proof of Theorem 3.1*) contains a proof of this important fact; it contains, in particular, another proof of Corollary 3.1 too.

Lemma 3.3. *Let E be a finite-dimensional Hilbert space with $n := \dim(E) \geq 1$, Y a normed space, $V : E \to Y$ a linear operator and $(e_1, \dots, e_n)$ a fixed orthonormal basis of E. Then*

$$\|V\|_{g.} = \|V\|_{n,g.} = \Big\| \sum_{k=1}^{n} V e_k g_k \Big\|_2 \tag{3.12}$$

and

$$\sqrt{n} \Big(\int_{S_E} \|Vx\|^2 ds(x) \Big)^{1/2} = \|V\|_{n,g.} = \|V\|_{g.}. \tag{3.13}$$

Proof. Observe that (3.13) follows from (3.12) via Lemma 2.2. So it remains to prove (3.12).

The inequalities $\|V\|_{g.} \geq \|V\|_{n,g.} \geq \Big\| \sum_{k=1}^{n} V e_k g_k \Big\|_2$ are evident.

Fix now arbitrarily a natural number n', a finite sequence $(x_1, \dots, x_{n'}) \in E^{n'}$, $\|(x_1, \dots, x_{n'})\|_{2,w} \leq 1$. Let us show that

$$\Big\| \sum_{k=1}^{n'} V x_k g_k \Big\|_2 \leq \Big\| \sum_{k=1}^{n} V e_k g_k \Big\|_2.$$

Fix then a Hilbert space E' with $\dim(E') = n'$ and an orthonormal basis $(e_1^o, \dots, e_{n'}^o)$ of it.

Define $W \in L(E', E)$ by putting $W e_k^o = x_k$, $k = 1, \dots, n'$. Then $\|W\| = \|(x_1, \dots, x_{n'})\|_{2,w} \leq 1$. Clearly,

$$\Big\| \sum_{k=1}^{n'} V x_k g_k \Big\|_2 = \Big\| \sum_{k=1}^{n'} V W e_k^o g_k \Big\|_2.$$

The operator W admits the spectral representation

$$Wx = \sum_{k=1}^{n'} \lambda_k (x|h'_k) h_k, \ \forall x \in E',$$

where $(h'_1, \ldots, h'_{n'})$ is an orthonormal basis of E', $(h_1, \ldots, h_{n'})$ is a finite sequence in E whose non-zero members form an orthonormal sequence in E and

$$1 \geq \|W\| = \lambda_1 \geq \lambda_2 \geq \ldots \lambda_{n'} \geq 0.$$

Applying Lemma 2.2 to E' and to the operator $V' := VW$ we can write

$$\Big\| \sum_{k=1}^{n'} VW e^o_k g_k \Big\|_2 = \Big\| \sum_{k=1}^{n'} VW h'_k g_k \Big\|_2.$$

Then, as $Wh'_k = \lambda_k h_k$, $k = 1, \ldots, n'$,

$$\Big\| \sum_{k=1}^{n'} VW e^o_k g_k \Big\|_2 = \Big\| \sum_{k=1}^{n'} \lambda_k V h_k g_k \Big\|_2.$$

From this, using the contraction principle (see, e.g. [VTC, Lemmma 5.4.1(c), p.298]), we obtain

$$\Big\| \sum_{k=1}^{n'} VW e^o_k g_k \Big\|_2 \leq \big(\max_{k \leq n'} \lambda_k \big) \Big\| \sum_{k=1}^{n'} V h_k g_k \Big\|_2 \leq \Big\| \sum_{k=1}^{n'} V h_k g_k \Big\|_2.$$

Let $(\tilde{h}_1, \ldots, \tilde{h}_n)$ be an othonormal basis of E containing the orthonormal set consisting of the non-zero terms of $(h_1, \ldots, h_{n'})$. Then (e.g., again by the contraction principle):

$$\Big\| \sum_{k=1}^{n'} V h_k g_k \Big\|_2 \leq \Big\| \sum_{k=1}^{n} V \tilde{h}_k g_k \Big\|_2.$$

Applying Lemma 2.2 to E and to the operator V we can write

$$\Big\| \sum_{k=1}^{n} V \tilde{h}_k g_k \Big\|_2 = \Big\| \sum_{k=1}^{n} V e_k g_k \Big\|_2.$$

Consequently,

$$\begin{aligned} \Big\| \sum_{k=1}^{n'} V x_k g_k \Big\|_2 &= \Big\| \sum_{k=1}^{n'} VW e^o_k g_k \Big\|_2 = \Big\| \sum_{k=1}^{n'} \lambda_k V h_k g_k \Big\|_2 \\ &\leq \Big\| \sum_{k=1}^{n} V \tilde{h}_k g_k \Big\|_2 = \Big\| \sum_{k=1}^{n} V e_k g_k \Big\|_2, \end{aligned}$$

which yields the needed inequality. □

The next lemma is a key step in the proof of the main theorem.

Lemma 3.4. *Let E be a finite-dimensional Hilbert space with $n := \dim(E) \geq 1$, $(\varphi_1, \dots, \varphi_n)$ a sequence of square integrable scalar functionns given on a positive measure space $(\Omega, \mathcal{A}, \nu)$ such that*

$$0 < \alpha_n := \Big(\sum_{k=1}^{n} \|\varphi_k\|_2^2 \Big)^{1/2}.$$

Then for any normed space Y and any $V \in L(E, Y)$ the inequality

$$\|V\|_{n,g.} \leq \frac{\sqrt{n}}{\alpha_n} \|V\|_{n,\varphi.} \tag{3.14}$$

holds.

Proof. Fix an orthonormal basis $(e_1, \dots, e_n)$ of E. Then from Corollary 3.1, Lemma 2.3 and the obvious inequality

$$\sup_{u \in \mathcal{U}(E)} \Big\| \sum_{k=1}^{n} V u e_k \varphi_k \Big\|_2 \leq \|V\|_{n,\varphi.}$$

we get

$$\|V\|_{n,g.} = \Big\| \sum_{k=1}^{n} V e_k g_k \Big\|_2 \leq \frac{\sqrt{n}}{\alpha_n} \sup_{u \in \mathcal{U}(E)} \Big\| \sum_{k=1}^{n} V u e_k \varphi_k \Big\|_2 \leq \frac{\sqrt{n}}{\alpha_n} \|V\|_{n,\varphi.}.$$

□

Theorem 3.1. *Let X, Y be normed spaces, $(\varphi_k)_{k \in \mathbb{N}}$ a sequence of square integrable scalar functions given on a positive measure space $(\Omega, \mathcal{A}, \nu)$ such that $\|\varphi_n\|_2 = 1,\ n = 1, 2, \dots$. Then*

(a) *For every natural n and any continuous linear operator $T : X \to Y$ the inequality*

$$\|T\|_{n,g.} \leq \|T\|_{n,\varphi.} \tag{3.15}$$

holds;

(b) *We have $\Pi_{\varphi.}(X, Y) \subset \Pi_{g.}(X, Y)$ and*

$$\|T\|_{g.} \leq \|T\|_{\varphi.}, \quad \forall T \in \Pi_{\varphi.}(X, Y). \tag{3.16}$$

Proof. (a) Let $T \in L(X, Y)$ and n be a natural number. Fix a Hilbert space E with $\dim(E) = n$ and an operator $W \in L(E, X)$. Lemma 3.4, applied to the operator $TW \in L(E, X)$, gives

$$\|TW\|_{n,g.} \leq \|TW\|_{n,\varphi.} \tag{3.17}$$

By (3.17) and Lemma 3.1 we can write

$$\begin{aligned} \|T\|_{n,g.} &= \sup\{\|TW\|_{n,g.} : W \in L(E, X),\ \|W\| \leq 1\} \\ &\leq \sup\{\|TW\|_{n,\varphi.} : W \in L(E, X),\ \|W\| \leq 1\} = \|T\|_{n,\varphi.}. \end{aligned}$$

The statement (b) follows from (a). □

In the same way we can prove also the next result.

Theorem 3.2. *Let X, Y be normed spaces, $(\varphi_k)_{k\in\mathbb{N}}$ a sequence of square integrable scalar functions given on a positive measure space $(\Omega, \mathcal{A}, \nu)$ such that $\|\varphi_n\|_2 > 0,\ n = 1, 2, \dots$ and*

$$\beta := \sup_n \frac{\sqrt{n}}{\Big(\sum_{k=1}^{n} \|\varphi_k\|_2^2\Big)^{1/2}} < \infty.$$

Then

(a) *For every natural n and any continuous linear operator and $T : X \to Y$ the inequality*

$$\|T\|_{n,g.} \leq \beta \|T\|_{n,\varphi.} \tag{3.18}$$

holds;

(b) *We have $\Pi_{\varphi.}(X, Y) \subset \Pi_{g.}(X, Y)$ and*

$$\|T\|_{g.} \leq \beta \|T\|_{\varphi.}, \quad \forall T \in \Pi_{\varphi.}(X, Y). \tag{3.19}$$

Proof. (a) Let $T \in L(X, Y)$ and n be a natural number. Fix a Hilbert space E with $\dim(E) = n$ and an operator $W \in L(E, X)$. Lemma 3.4, applied to the operator $TW \in L(E, X)$, implies

$$\|TW\|_{n,g.} \leq \frac{\sqrt{n}}{\alpha_n} \|TW\|_{n,\varphi.} \leq \beta \|TW\|_{n,\varphi.}. \tag{3.20}$$

By (3.20) and Lemma 3.1 we can write

$$\begin{aligned}\|T\|_{n,g.} &= \sup\{\|TW\|_{n,g.} : W \in L(E, X),\ \|W\| \leq 1\}\\ &\leq \beta \sup\{\|TW\|_{n,\varphi.} : W \in L(E, X),\ \|W\| \leq 1\} = \beta \|T\|_{n,\varphi.}.\end{aligned}$$

The statement (b) follows from (a). □

Remark. In general, for a given sequence $(\varphi_k)_{k\in\mathbb{N}}$ of square integrable scalar functions given on a positive measure space $(\Omega, \mathcal{A}, \nu)$ such that $\|\varphi_n\|_2 = 1$, $n = 1, 2, \dots$ it may happen that $\Pi_{\varphi.}(\mathbb{R}, \mathbb{R}) = \{0\}$ (assume, e.g., that $\nu(\Omega) = 1$ and put $\varphi_n = 1$, $n = 1, 2, \dots$). Therefore, in general, we do not have the equality $\Pi_{\varphi.}(X, Y) = \Pi_{g.}(X, Y)$. A characterization of the orthonormal sequences $(\varphi_k)_{k\in\mathbb{N}}$ for which the coincidence $\Pi_{\varphi.}(X, Y) = \Pi_{g.}(X, Y)$ takes place is given in [Ba, Theorem 4.3, p. 24].

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