ACTA MATHEMATICA VIETNAMICA
Volume 28, Number 2, 2003, pp. 215-223

OPTIMALITY CONDITIONS IN REVERSE CONVEX OPTIMIZATION

M. LAGHDIR

Abstract. Necessary and sufficient optimality conditions associated with the problem of minimizing a convex function subject to a reverse convex constraint are obtained in this paper.

1. Introduction

In the present work, our main objective is to establish optimality conditions for the problem of minimizing an extended real-valued convex function over the complement of a convex subset, called usually reverse convex programming. This wide class of problems has received recently particular attention from the point of view of duality (see [7] and [8]).

More generally, we will study the problem in a large class of objective function that can be written as difference of a convex function and an extended real function. More precisely, let X be a topological vector space, $f_1, f_2 : X \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ be two extended real-valued functions with f_1 being convex and S be a nonempty convex subset of X, we are concerned with the problem

$$(\mathcal{P}) \qquad \inf_{X \setminus S} \{f_1(x) - f_2(x)\}.$$

Naturally, this class of problems covers the class of reverse programming problems by taking f_2 identically equal to zero.

Firstly, we shall establish necessary conditions for an extremum of the problem $(\mathcal{P})$ in the case where f_2 is supposed strictly Hadamard differentiable. Secondly, we will state the sufficient conditions when the objective function is DC (that is f_1 and f_2 are both convex).

The paper is organized as follows. In Section 2 we recall some definitions and notations. In Section 3 we establish the necessary and sufficient optimality conditions associated to the problem $(\mathcal{P})$. Finally, in Section 4 we give an illustration of the problem $(\mathcal{P})$ where the reverse constraint is termed by a mapping taking its values in a partially ordered topological vector space.

Received September 19, 2002; in revised form December 4, 2002.
1991 *Mathematics Subject Classification.* 90C48, 49N15, 90C26.
Key words and phrases. Optimality conditions, reverse convex constraint, DC-optimization.

2. Definitions and notations

Let $(X, \| \|)$ be a normed real vector space. A function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is said to be locally Lipschitz at point $\bar{x} \in \operatorname{dom} f$ if there exist some neighbourhood V of $\bar{x}$ and $k > 0$ satisfying

$$|f(x) - f(y)| \leq k \; \|x - y\|, \qquad \forall x, y \in V.$$

In [3], it was shown that when f is locally Lipschitz, the generalized directional derivative

$$v \longrightarrow f^0(\bar{x}, v) := \limsup_{\substack{x \to \bar{x} \\ t \to 0^+}} \frac{f(x + tv) - f(x)}{t},$$

is, for each $\bar{x} \in \operatorname{dom} f$, a finite sublinear function. The following set

$$\partial^c f(\bar{x}) := \{x^* \in X^* : \langle x^*, x \rangle \leq f^0(\bar{x}, v), \quad \forall v \in X\}$$

called generalized subdifferential or Clarke's subdifferential, is a nonempty convex $\sigma(X^*, X)$-compact subset of X^*. When f is convex and continuous at $\bar{x}$ then f is locally Lipschitz and $f^{'}(\bar{x}, v) = f^0(\bar{x}, v)$ for any $v \in X$ where $v \longrightarrow f^{'}(\bar{x}, v)$ is the usual directional derivative defined by

$$v \longrightarrow f^{'}(\bar{x}, v) := \lim_{t \to 0^+} \frac{f(\bar{x} + tv) - f(\bar{x})}{t},$$

and therefore, $\partial^c f(\bar{x})$ is exactly the subdifferential of f in the sense of the convex analysis, usually denoted by $\partial f(\bar{x})$.

Following [11], we say that f is strictly Hadamard differentiable (with gradient $\nabla f(\bar{x})$) if it is finite on a neighbourhood of $\bar{x}$ and for arbitrary $v \in X$, the function

$$(x, t) \longrightarrow \frac{f(x + tv) - f(x)}{t} - \langle \nabla f(\bar{x}), v \rangle$$

converges to zero uniformly on all compact v-sets as $t \longrightarrow 0^+$ and $x \longrightarrow \bar{x}$.

Let S be a nonempty subset of X and consider its distance function, that is the function $d_S : X \longrightarrow \; [0, +\infty[$ defined, for any $x \in X$, by

$$d_S(x) := \inf_{y \in S} \|x - y\|.$$

The Clarke's normal cone to S at $\bar{x}$ is given by

$$N_S^c(\bar{x}) := \operatorname{cl}\Big(\bigcup_{\lambda \geq 0} \lambda \partial^c d_S(\bar{x}) \Big),$$

where "cl" stands for weak star closure in X^*. If S is convex then $N_S^c(\bar{x})$ coincides with the closed normal cone $N_S(\bar{x})$ to S at $\bar{x}$ in the sense of convex analysis.

Let us recall (see [9] and [10]) that a subset S is said to be epi-Lipschitzian at $\bar{x}$ ($\bar{x}$ is a cluster point of S) if there exist some neighbourhood V of $\bar{x}$, $\lambda > 0$ and a nonempty open subset O such that

$$x + ty \in S, \quad \forall\, x \in S \cap V, \quad \forall\, y \in O, \quad \forall\, t \in \,]0, \lambda[.$$

It was demonstrated in [10] that if S is epi-Lipschitzian at $\bar{x}$ and $\bar{x}$ is a boundary point of S then

$$N^c_{X\setminus S}(\bar{x}) = -N_S(\bar{x}).$$

As an example of epi-Lipschitzian subset one can take any nonempty open convex subset of X at any cluster point.

3. Optimality conditions

In this section we investigate the optimality conditions related to the problem $(\mathcal{P})$. At first, we study the necessary optimality conditions given by the following proposition.

Proposition 3.1. *Assume that f_1 is convex, finite and continuous at $\bar{x}$ which is a local minimum of the problem $(\mathcal{P})$ and f_2 is supposed to be strictly Hadamard differentiable at $\bar{x}$, then we have*

(i) $\nabla f_2(\bar{x}) \in \partial f_1(\bar{x}) - N_S(\bar{x})$ *where $\bar{x}$ is a boundary point to S.*

(ii) $\nabla f_2(\bar{x}) \in \partial f_1(\bar{x})$ *where $\bar{x}$ is a topological interior point of $X \setminus S$.*

Proof. (i) By $k > 0$ we denote a common Lipschitz constant of f_1 and f_2. As $\bar{x}$ is a local minimum of $(\mathcal{P})$, by Proposition 2.4.3 in Clarke [3], the function $x \longrightarrow f_1(x) - f_2(x) + kd_{X\setminus S}(x)$ attains its local minimum at $\bar{x}$. So

$$0 \in \partial^c\big(f_1 - f_2 + kd_{X\setminus S}\big)(\bar{x}).$$

Applying the sum rule ([3]) we obtain

$$\nabla f_2(\bar{x}) \in \partial^c f_1(\bar{x}) + N^c_{X\setminus S}(\bar{x}).$$

Since S is an open convex subset, it follows from [10] that it is epi-Lipschitzian at $\bar{x}$ which is a boundary point to S. According to Rockafellar's result [10], we have

$$N^c_{X\setminus S}(\bar{x}) = -N_S(\bar{x}),$$

and thus we get

$$\nabla f_2(\bar{x}) \in \partial f_1(\bar{x}) - N_S(\bar{x}).$$

(ii) If $\bar{x}$ is an topological interior point of $X \setminus S$ then $\bar{x}$ is indeed a local minimum of $(\mathcal{P})$ without constraint and therefore it results from Proposition 3.1 that $\nabla f_2(\bar{x}) \in \partial f_1(\bar{x})$. □

Remark 3.1. 1) In Proposition 3.1, there is no difference to work on $f_1 - f_2$ or $f_1 + f_2$ provided f_2 is smooth.

2) In the above proof, we only need to assume that f_1, f_2 are Lipschitz around $\bar{x}$, and S is epi-Lipschitzian. Of course the formula in (i) should be changed to

$$0 \in \partial^c f_1(\bar{x}) - \partial^c f_2(\bar{x}) - N^c_S(\bar{x}).$$

Before stating the sufficient conditions linked to the problem $(\mathcal{P})$, we need first to recall some notions and results that will be used in the sequel. In [5], Hiriarty-Urruty established that a necessary and sufficient conditions for $\bar{x}$ to be a global solution of the following minimization problem

$$\inf_{x \in X} \{g(x) - h(x)\}$$

is that

$$\partial_\varepsilon h(\bar{x}) \subset \partial_\varepsilon g(\bar{x}), \quad \forall \varepsilon \geq 0, \tag{3.1}$$

where

$$\partial_\varepsilon f(\bar{x}) := \{x^* \in X^* : f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x} \rangle - \varepsilon, \quad \forall x \in X\},$$

denotes the ε-subdifferential of the function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ at $\bar{x}$ and $g, h : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ are two proper convex lower semicontinuous functions. Also we will need the following result due to Hiriart-Urruty et al. [6].

Theorem 3.1. *Suppose that $g, h : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ are convex, proper and lower semicontinuous and $\bar{x} \in \operatorname{dom} g \cap \operatorname{dom} h$. Then for all $\varepsilon > 0$, one has*

$$\partial_\varepsilon (g + h)(\bar{x}) = \operatorname{cl}\Big(\bigcap_{\substack{\varepsilon_1 \geq 0, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \partial_{\varepsilon_1} g(\bar{x}) + \partial_{\varepsilon_2} h(\bar{x}) \Big)$$

where "cl" stands for topological closure operation with respect to weak star topology $\sigma(X^*, X)$.

Let S be a subset of X and let $\Delta_S : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be the function defined, for any $x \in X$, by

$$\Delta_S(x) := d_S(x) - d_{X \setminus S}(x).$$

If S is empty, $\Delta_S \equiv +\infty$ and if $S = X$, $\Delta_S \equiv -\infty$. In other cases Δ_S is a Lipschitz function and its Lipschitz constant $k = 1$. In [5], Hiriart-Urruty proved that Δ_S is obtained by infimal convolution of a function μ_S given by

$$\mu_S(x) := \begin{cases} +\infty, & \text{if } x \in X \setminus S \\ -d_{X \setminus S}(x), & \text{if } x \in S \end{cases}$$

and the norm function $\| \;\; \|$. In [5], it was shown that S is convex if and only if μ_S is convex and hence Δ_S is convex. Let us consider the following auxiliary nonconvex minimization problem

$$(\mathcal{H}) : \inf_{x \in X} \{f_1(x) - f_2(x) + d_{X \setminus S}(x)\}.$$

It is easy to check that if $\bar{x}$ is both a boundary point of S and a global (resp. local) minimum of $(\mathcal{H})$ then it is also a global (resp. local) minimum of the problem $(\mathcal{P})$.

Now, we can state the sufficient optimality conditions related to problem $(\mathcal{P})$.

Proposition 3.2. *Suppose that $f_1, f_2\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ are convex, proper and lower semicontinuous, S is a nonempty open convex subset of X and $\bar{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ is a boundary point of S. If, for each $\varepsilon > 0$, we have*

$$\partial_\varepsilon f_2(\bar{x}) + N_S(\bar{x}) \ \subset\ \partial_\varepsilon f_1(\bar{x}) \tag{3.2}$$

then $\bar{x}$ is a global minimum of $(\mathcal{P})$.

Proof. As previously mentioned that S is an epi-Lipschitzian subset at $\bar{x}$ and according again to Rokafellar's result [10] we have

$$N^c_{X\setminus S}(\bar{x}) = -N_S(\bar{x}),$$

and since

$$\partial^c d_{X\setminus S}(\bar{x}) \ \subset\ N^c_{X\setminus S}(\bar{x}),$$

it follows from (3.2) that

$$\partial_\varepsilon f_2(\bar{x}) - \partial^c d_{X\setminus S}(\bar{x}) \ \subset\ \partial_\varepsilon f_1(\bar{x}), \quad \forall \varepsilon > 0,$$

which implies

$$\partial_\varepsilon f_2(\bar{x}) + \partial d_S(\bar{x}) - \partial^c d_{X\setminus S}(\bar{x}) \ \subset\ \partial_\varepsilon f_1(\bar{x}) + \partial d_S(\bar{x}).$$

As

$$\partial \Delta_S(\bar{x}) \ \subset\ \partial d_S(\bar{x}) - \partial^c d_{X\setminus S}(\bar{x}),$$

we get

$$\partial_\varepsilon f_2(\bar{x}) + \partial \Delta_S(\bar{x}) \ \subset\ \partial_\varepsilon f_1(\bar{x}) + \partial d_S(\bar{x}), \quad \forall \varepsilon > 0,$$

which yields

$$\partial_\varepsilon f_2(\bar{x}) + \partial \Delta_S(\bar{x}) \ \subset\ \partial_\varepsilon \big(f_1 + d_S\big)(\bar{x}), \quad \forall \varepsilon > 0. \tag{3.3}$$

By virtue of Theorem 3.1 we have

$$\partial_\varepsilon \big(f_2 + \Delta_S\big)(\bar{x}) \ \subset\ \overline{\partial_\varepsilon f_2(\bar{x}) + \partial \Delta_S(\bar{x})}, \quad \forall \varepsilon > 0. \tag{3.4}$$

Since $\partial_\varepsilon \big(f_2 + \Delta_S\big)(\bar{x})$ and $\partial_\varepsilon \big(f_2 + d_S\big)(\bar{x})$ are both $\sigma(X^*, X)$-closed, combining (3.3) and (3.4) we obtain

$$\partial_\varepsilon \big(f_2 + \Delta_S\big)(\bar{x}) \ \subset\ \partial_\varepsilon \big(f_1 + d_S\big)(\bar{x}), \quad \forall \varepsilon > 0.$$

Then, we deduce from (3.1) that $\bar{x}$ is a global minimum of $(\mathcal{H})$ and since $\bar{x}$ is a boundary point to S , it follows that $\bar{x}$ is also a global minimum of $(\mathcal{P})$. □

4. Application

In the present section we apply the previously obtained results to the following minimization problem subject to a vector reverse convex constraint

$$(\mathcal{Q}) \quad \begin{cases} \inf f_1(x) - f_2(x) \\ h(x) \notin -\text{int } Y_+, \end{cases}$$

where f_1 and f_2 are two extended real-valued convex functions and $h : X \longrightarrow Y \cup \{+\infty\}$ is a convex and proper mapping taking values in a topological vector real space equipped with a partial ordered induced by a convex cone Y_+ and defined as

$$y_1 \leq_Y y_2 \iff y_2 - y_1 \in Y_+,$$

for any $y_1, y_2 \in Y$. By "int Y_+" we denote the topological interior of the cone Y_+. The convexity of the mapping h is taken with respect to the partial order in the following sense

$$h(\alpha x_1 + (1-\alpha)x_2) \leq_Y \alpha h(x_1) + (1-\alpha)h(x_2)$$

for any $\alpha \in [0,1]$ and any $x_1, x_2 \in X$. Let us notice that the mapping h be authorized to take the value $+\infty$ supposed the greatest element adjoined to Y : $y \leq +\infty,\ \forall\, y \in Y$.

For a given function $g : Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ we denote by $g \circ h$ the composite function defined by

$$(g \circ h)(x) := \begin{cases} g(h(x)) & \text{if } x \in \text{dom } h \\ \sup\limits_{y \in Y} g(y), & \text{otherwise.} \end{cases} \tag{4.1}$$

Throughout, we assume that the positive cone Y_+ is with nonempty topological interior and h is continuous. Let us consider the following subset S of X defined by

$$S := \{x \in X : \ h(x) \in -\text{int } Y_+\} = h^{-1}(-\text{int } Y_+),$$

and the following constraint qualification

$$(C.Q.S) : \quad \exists a \in X \quad \text{such that} \quad h(a) \in -\text{int } Y_+,$$

called usually, the Slater condition. In the sequel, we shall need the following result (see [1]): Under the Slater condition (C.Q.S) we have

$$\partial(\delta_{-Y_+} \circ h)(\bar{x}) = \bigcup_{\substack{y^* \in Y_+^* \\ \langle y^*, h(\bar{x})\rangle = 0}} \partial(y^* \circ h)(\bar{x}), \tag{4.2}$$

where Y_+^* is the polar positive cone defined as

$$Y_+^* := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \quad \forall y \in Y_+\}$$

and the symbol $\langle\ ,\ \rangle$ denotes the bilinear pairing between Y and Y^* (resp. X and X^*).

Remark 4.1. Let us notice that the function $y \longrightarrow \delta_{-Y_+}(y)$ defined on Y be nondecreasing with respect to the partial order associated to the cone Y_+ (see [1]) i.e.

$$y_1 \leq_Y y_2 \Longrightarrow \delta_{-Y_+}(y_1) \leq \delta_{-Y_+}(y_2),$$

and also, it is easy to see that for a given Y_+-convex mapping $h : X \longrightarrow Y \cup \{+\infty\}$, the composite function $\delta_{-Y_+} \circ h : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is also convex. Indeed, (4.2) is a particular form of a general formula established by Combari et al. [1] (see also [2]) in the setting of partially ordered topological vector space by replacing the indicator function δ_{-Y_+} by a convex and Y_+-nondecreasing function.

In order to derive the main results of this section, we will need the following lemma which characterizes the closure of the subset S.

Lemma 4.1. *If we assume that the mapping $h : X \longrightarrow Y \cup \{+\infty\}$ is Y_+-convex, continuous and the cone Y_+ is closed then under the Slater condition we have*

$$\overline{S} = \{x \in X : h(x) \in -Y_+\}$$

where $\overline{S}$ denotes the norm topological closure in X of the subset S.

Proof. It is obvious that $S \subset \{x \in X : h(x) \in -Y_+\}$. From the continuity of the mapping h and the fact that the cone Y_+ is closed, it follows that the subset $\{x \in X : h(x) \in -Y_+\}$ is closed and hence we obtain $\overline{S} \subset \{x \in X : h(x) \in -Y_+\}$.

Conversely, let us consider any $x \in X$ with $h(x) \in -Y_+$ and an element $a \in X$ satisfying $h(a) \in -\text{int } Y_+$ whose existence is guaranteed by the Slater condition. If we set $x_n := \frac{1}{n}a + (1 - \frac{1}{n})x$ for any integer $n \geq 1$, obviously the sequence $(x_n)_{n\geq 1}$ converges to x. By applying the convexity of the mapping h and the convexity of the cone Y_+ we obtain

$$h(x_n) \leq_Y \frac{1}{n}h(a) + (1 - \frac{1}{n})h(x) \in -\text{int } Y_+ - Y_+ \subset -\text{ int } Y_+$$

which yields $x_n \in S$. Hence the equality

$$\overline{S} = \{x \in X : h(x) \in -Y_+\}$$

holds. □

Now, we are ready to state the local necessary optimality conditions related to problem $(\mathcal{Q})$.

Proposition 4.1. *Let us assume that $f_1 : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is convex, proper and lower semicontinuous, $f_2 : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is strictly Hadamard differentiable at $\bar{x}$, $h : X \longrightarrow Y \cup \{+\infty\}$ is continuous and Y_+-convex, the Slater condition (C.Q.S) is satisfied and $\bar{x}$ is a local minimum of $(\mathcal{Q})$. Then we have*

(i) *If $\bar{x}$ is a boundary point of S, then there exists some $y^* \in Y_+^*$ satisfying $\nabla f_2(\bar{x}) \in \partial f_1(\bar{x}) - \partial(y^* \circ h)(\bar{x})$ and $\langle y^*, h(\bar{x})\rangle = 0$;*

(ii) *If $\bar{x}$ is a topological interior point of $X \setminus S$, then we have $\partial f_2(\bar{x}) \subset \partial f_1(\bar{x})$.*

Proof. (i) It is straightforward to check that by means of the convexity and continuity of h that the subset S is convex and open. Also, let us note that S is nonempty by virtue of the Slater condition. Hence, it follows from Proposition 3.1 that when $\bar{x}$ is both a boundary point of S and a local minimum to $(\mathcal{Q})$ we have

$$\nabla f_2(\bar{x}) \in \partial f_1(\bar{x}) - N_S(\bar{x}).$$

By Lemma 4.1, we can write $\delta_{\overline{S}} = \delta_{-Y_+} \circ h$. Since $N_S(\bar{x}) = N_{\overline{S}}(\bar{x})$, we get

$$N_S(\bar{x}) = \partial\delta_{\overline{S}}(\bar{x}) = \partial(\delta_{-Y_+} \circ h)(\bar{x}).$$

Applying formula (4.2) we can conclude that there exist some $y^* \in Y_+^*$ satisfying $\nabla f_2(\bar{x}) \in \partial f_1(\bar{x}) - \partial(y^* \circ h)(\bar{x})$ and $\langle y^*, h(\bar{x})\rangle = 0$.

(ii) We apply the same arguments used in (ii) of Proposition 3.1. □

Concerning the sufficient conditions associated to problem $(\mathcal{Q})$ we have

Proposition 4.2. *Suppose that $f_1,\ f_2 : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ are convex, proper and lower semicontinuous, $h : X \longrightarrow Y\cup\{+\infty\}$ is proper, continuous and Y_+-convex, $\bar{x} \in \text{dom } f_1 \cap \text{dom } f_2$ is a boundary point of S and the Slater condition (C.Q.S) is satisfied. If for any $y^* \in Y_+^*$ satisfying $\langle y^*, h(\bar{x})\rangle = 0$ and*

$$\partial_\varepsilon f_2(\bar{x}) + \partial(y^* \circ h)(\bar{x}) \subset \partial_\varepsilon f_1(\bar{x}), \quad \forall\varepsilon > 0, \tag{4.3}$$

then $\bar{x}$ is a global minimum of $(\mathcal{Q})$.

Proof. As in the proof of Proposition 4.1, observe that the subset $S = \{x \in X : h(x) \in -\text{int } Y_+\}$ is again, under the same assumptions, a nonempty open convex subset of X. Also, as mentioned previously, under the Slater condition we have

$$N_S(\bar{x}) = \partial\delta_{\overline{S}}(\bar{x}) = \partial(\delta_{-Y_+} \circ h)(\bar{x}) = \bigcup_{\substack{y^* \in Y_+^* \\ \langle y^*, h(\bar{x})\rangle = 0}} \partial(y^* \circ h)(\bar{x})$$

and hence condition (4.3) is equivalent to

$$\partial_\varepsilon f_2(\bar{x}) + N_S(\bar{x}) \subset \partial_\varepsilon f_1(\bar{x}), \quad \forall\varepsilon > 0.$$

Thus, by applying Proposition 3.2, we see that $\bar{x}$ is a global minimum of problem $(\mathcal{Q})$. □

Remark 4.2. In the case when $Y = \mathbb{R}$ and $Y_+ = \mathbb{R}_+$ we have $Y_+^* = \mathbb{R}_+$ and the problem $(\mathcal{Q})$ becomes

$$(\mathcal{L}) \quad \begin{cases} \inf f_1(x) - f_2(x) \\ h(x) \geq 0. \end{cases}$$

Noticing that $\partial(\lambda h)(\bar{x}) = \lambda\partial h(\bar{x})$ for any $\lambda > 0$ and $\partial(0 \cdot h)(\bar{x}) = \{0\}$ according to convention (4.1) we derive easily from Proposition 4.1 and Proposition 4.2 the related optimality conditions to the above scalar problem $(\mathcal{L})$.

Acknowledgment

The author would like to thank an anonymous referee for his valuable remarks and suggestions.

References

[1] C. Combari, M. Laghdir and L. Thibault, *Sous-différentiel de fonctions convex composées*, Ann. Sci. Math. Quebec **8** (1994), 119-148.

[2] C. Combari, M. Laghdir and L. Thibault, *A note on subdifferentials of convex composite functionals*, Arch. Math. **67** (1996), 239-252.

[3] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[4] I. Ekland and R. Temam, *Analyse convexe et Problémes variationnels*, Dunod, Paris, 1974.

[5] J. B. Hiriart-Urruty, *Generalized differentiability, duality and optimization for problems dealing with differences of convex functions*. In "Convexity and duality in optimization", Lecture Notes in Economics and Mathematical Systems **256** (1986), pp. 37-70.

[6] J. -B. Hiriarty-Urruty, M. Moussaoui, A. Seeger and M. Volle, *Subdifferential calculus without qualification conditions, using approximate subdifferentials: a survey*, Nonlinear Anal. **24** (1995), 1727-1754.

[7] B. Lemaire, *Duality in reverse optimization*, SIAM J. Optim. **8** (1998), 1029-1037.

[8] B. Lemaire and M. Volle, *Duality in D. C. programming*, Proceedings of the 5th international Symposium on Generalized Convexity, Luniny, 1996.

[9] R. T. Rockafellar, *Clarke's tangent cones and boundaries of closed sets in* $\mathbb{R}^n$, Nonlinear Anal. TMA **3** (1979), 145-154.

[10] R. T. Rockafellar, *Generalized directional derivatives and subgradients of nonconvex functions*, Canadian. J. Math. **32** (1980), 175-180.

[11] R. T. Rockafellar, *Directionally Lipschitzian functions and subdifferential calculus*, Proc. London Math. Soc. **39** (1979), 331-355.

Département de Mathématiques,
Faculté des Sciences
B.P. 20, El-Jadida, Maroc.

E-mail address: laghdir@ucd.ac.ma