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SUPPLY-DEMAND THEOREMS FOR FINITE PROBABILISTIC AUTOMATA

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Abstract. In this paper, we show that there are supply-demand theorems for finite probabilistic automata, but here the notion of state is understood in a rather general sense. It is notion of hyperstate.

1. Introduction

In [4, 5] we have shown that there are supply-demand theorems for the finite automata, for the automata with a time-variant structure and for the Petri nets. They describe a nice relation between state growth speed of an automaton (a supply) and (non-equivalent) word growth speed of the language which is accepted by this automaton (a demand). Applying the supply-demand theorems for different processing systems we get again the well-known necessary conditions, but now on an united point of view, for the classes of languages accepted by finite automata, finite automata with a time-variant structure, $\varphi(t)$-automata with a time-variant structure, Petri nets and Petri nets with a time-variant structure.

In this paper we show that there are also the supply-demand theorems for finite probabilistic automata, but here the notion of state is understood in a rather general sense. It is the notion of hyperstate.

The definitions of finite probabilistic automaton and language acceptable by it are recalled in Section 2. Section 3 deals with the notion of hyperstate, the supply-demand theorem and the growth speed theorem of a finite probabilistic automaton. Finally, in Section 4 some other supply-demand theorems for a finite probabilistic automaton are considered.

2. Preliminaries

We recall some notions. *A finite probabilistic automaton* (FPA) is given by a list

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$$A = (I, S, \pi_0, M, F),$$

where

I is *the input alphabet*;

S is *the finite set of states*, $S = \{s_1, \cdots, s_n\}$;

π_0 is *the initial state distribution vector*;

F is *the set of final states*, $F \subseteq S$.

For $a \in I$, $M(a)$ is a stochastic matrix of order n, whose component $m_{ij}(a)$ is *the transition probability* of A from the state s_i to the state s_j when the imput symbol is a.

Let I^* be the set of all words over the alphabet I. For each $u = a_1 \cdots a_k \in I^*$ we define

$$\begin{cases} M(\Lambda) = E, \\ M(u) = M(a_1) \cdots M(a_k). \end{cases}$$

E is the unity matrix of order n. Let η_F denote the n-dimensional colum vector whose i-th component is equal to 1 if $s_i \in F$, and to 0 if $s_i \notin F$. We define the function $p_A : I^* \to [0, 1]$ as follows. For any $u \in I^*$, we set

$$p_A(u) = \pi_0 M(u) \eta_F.$$

Let λ be a real number, $0 \le \lambda < 1$. The set of words

$$L(A, \lambda) = \{u \in I^* \mid p_A(u) > \lambda\}$$

is called the *FPA-language* (or *stochastic language*) over alphabet I, defined by the probabilistic automaton A and *the cut point* λ.

A set $L \subseteq I^*$ is called *FPA-language* if there are a finite probabilistic automaton A and a cut point λ $(0 \le \lambda < 1)$ such that

$$L = L(A, \lambda).$$

The set of all FPA-languages is denoted by $\mathcal{L}(FPA)$.

3. Hyperstate and supply-demand Theorem for finite probabilistic automata

Let I be a nonempty finite alphabet and $L \subseteq I^*$. For any finite set $\Omega \subset I^*$, $|\Omega| = N < +\infty$, we define the relation $R_\Omega \pmod L$ in I^* as follows:

$$uR_\Omega v \pmod L \Leftrightarrow \forall\, \omega \in \Omega : u\omega \in L \leftrightarrow v\omega \in L, \quad \forall\, u, v \in I^*.$$

It is easy to show that the relation $R_\Omega \pmod L$ is reflexive, symmetric and transitive. Therefore, it is an equivalent relation in I^* and we define

$$G_L(\Omega) = \operatorname{Rank} R_\Omega \pmod L.$$

$G_L(\Omega)$ is also the number of non-equivalent words that are needed distinguish during representating L by an automaton (a demand). Therefore $G_L(\Omega)$ is called a *R-representative complexity* of language L on Ω.

First we notice a simple property of $G_L(\Omega)$:

$$1 \leq G_L(\Omega) \leq 2^{|\Omega|} = 2^N, \quad \forall\, \Omega \subseteq I^*.$$

Now we estimate $G_L(\Omega)$ for some languages L and for some sets Ω.

Example 1. Let $|I| = k \geq 2$ and $c \notin I$. We define

$$L_1 = \{\tau_1 c \tau_2 c \cdots c \tau_n c \tau_0 \mid \forall\, i : \tau_i \in I^*;\ \exists \tau_i = \tau_0\}.$$

We choose $\Omega = \{\tau_1, \tau_2, \cdots, \tau_N | \tau_i \in I^*\}$. Now each subset $A = \{\tau_{i_1}, \tau_{i_2}, \cdots, \tau_{i_k}\} \subseteq \Omega$ is associated with a word

$$u_A = \tau_{i_1} c \tau_{i_2} c \cdots c \tau i_k c,$$

It is easy to see that

$$\forall \omega \in \Omega : (u_A \omega \in L_1 \leftrightarrow \omega \in A).$$

Therefore

$$G_{L_1}(\Omega) = 2^{|\Omega|} = 2^N.$$

Example 2. Let $I = \{a\}$ and

$$L_2 = \{a^k \mid v_k = 1;\ k = 1, 2, ...\},$$

where $v_1 v_2 \cdots v_k \cdots$ is the dictionary ordering of all words over the alphabet $\{0, 1\}$ and $l(v_i) = 1$, $i = 1, 2, ...$, i.e.

$$v_1 v_2 \cdots v_k \cdots = 0.1.00.01.10.11.000.001. \cdots$$

and $v_1 = 0$, $v_2 = 1$, $v_3 = 0$, ..., $v_9 = 1$, $v_{10} = 1$, $\cdots$. Therefore

$$L_2 = \{a^2, a^6, a^7, a^9, a^{10}, \cdots\}.$$

We choose $\Omega = \{a^1, a^2, a^3, \cdots, a^N\}$, $N < +\infty$. For each subset

$$A = \{a^{i_1}, a^{i_2}, \cdots, a^{i_k}\} \subseteq \Omega,$$

we define an associated vector ξ_A and an associated word u_A as follows: $\xi_A = (\xi_1, \cdots, \xi_N)$ with $\xi_i = 1$ if $a^i \in A$, and $\xi_i = 0$ if $a^i \notin A$, $i = 1, \cdots, N$. Since $v_1 v_2 \cdots v_k \cdots$ is the dictionary ordering of all words over the alphabet $\{0, 1\}$, there is an integer h such that $\xi_A = (v_{h+1}, v_{h+2}, \cdots, v_{h+N})$. Then we set $u_A = a^h$.

For example if we choose $\Omega = \{a^1, a^2, a^3\}$ and $A = \{a^1, a^2\}$ then $\xi_A = (1, 1, 0)$ and $(1, 1, 0) = (v_{29}, v_{30}, v_{31})$ and $u_A = a^{28}$.

Now we can verify that

$$\forall\, \omega \in \Omega : (u_A \omega \in L_2 \leftrightarrow \omega \in A).$$

For instance, in our above example, if we choose

$$\omega = a^1 \in A \quad \text{then} \quad u_A \omega = a^{28} a^1 = a^{29} \in L_2,$$

$$\omega = a^3 \notin A \quad \text{then} \quad u_A \omega = a^{28} a^3 = a^{31} \notin L_2.$$

Therefore $G_{L_2}(\Omega) = 2^{|\Omega|} = 2^N$.

Let $A = (I, S, \pi_0, M, F)$ be a finite probabilistic automaton with $|S| = n$, $L = L(A, \lambda)$, $0 \leq \lambda < 1$ and $\Omega = \{\omega_1, \cdots, \omega_N\} \subset I^*$, $N < +\infty$.

According to the definition of $R_\Omega \,(\mathrm{mod}\, L)$, for any $u, v \in I^*$ we have

$$uR_\Omega v \,(\mathrm{mod}\, L) \Leftrightarrow \forall\, \omega \in \Omega : u\omega \in L \leftrightarrow v\omega \in L,$$
$$uR_\Omega v \,(\mathrm{mod}\, L) \Leftrightarrow \forall\, \omega \in \Omega : p_A(u\omega) > \lambda \leftrightarrow p_A(v\omega) > \lambda,$$
$$uR_\Omega v \,(\mathrm{mod}\, L) \Leftrightarrow \forall\, \omega \in \Omega : \pi_0 M(u)M(\omega)\eta_F > \lambda \leftrightarrow \pi_0 M(v)M(\omega)\eta_F > \lambda.$$

For each word $u \in I^*$, we define a corresponding point $\alpha(u)$ by

$$\alpha(u) = \pi_0 M(u) = (x_1^u, \cdots, x_n^u) \in R^n,$$

and for each word $\omega \in \Omega$, we define a corresponding vector $\beta(\omega)$ by

$$\beta(\omega) = M(\omega)\eta_F = (a_1^\omega, \cdots, a_n^\omega)^T.$$

Therefore

$$u\omega \in L \Leftrightarrow a_1^\omega x_1^u + \cdots + a_n^\omega x_n^u > \lambda.$$

Now in Euclidean space R^n, we consider N $(n-1)$-dimentional hyperplanes given by equations:

$$a_1^\omega x_1 + \cdots + a_n^\omega x_n = \lambda, \quad \omega \in \Omega. \tag{1}$$

It is easy to see that two words $u, v \in I^*$ are equivalent by relation $R_\Omega \,(\mathrm{mod}\, L)$ if and only if two their corresponding points $\alpha(u) = (x_1^u, \cdots, x_n^u)$ and $\alpha(v) = (x_1^v, \cdots, x_n^v)$ lie in the same connected domain in R^n determined by N hyperplanes (1). Thus, the finite probabilistic automaton A use the connected domains determined by N hyperplanes (1) to remember the non-equivalent words of language $L = L(A, \lambda)$. Therefore we have the following definition:

Definition 1. Each connected domain determined by the hyperplanes (1) is called a *R-hyperstate on* Ω of the finite probabilistic automaton A. The number of R-hyperstates on Ω of A is called the *R-growth funtion on* Ω of A and denoted by $g_A(\Omega)$.

There is a nice relation between the R-growth function on Ω of a finite probabilistic automaton (a supply) and the R-representative complexity on Ω of the language accepted by this automaton (a demand). These relations are called *the supply-demand theorems.*

Theorem 1. (Supply-demand theorem for FPA). *Let A be a finite probabilistic automaton, $L = L(A, \lambda)$, $0 \le \lambda < 1$. Then for any finite set $\Omega \subset I^*$ we have*

$$G_L(\Omega) \le g_A(\Omega).$$

Proof. Let $A = (I, S, \pi_0, M, F)$, $|S| = n$ and $L = L(A, \lambda)$, $0 \le \lambda < 1$. We shall prove that

$$G_L(\Omega) \le g_A(\Omega), \quad \forall\, \Omega \subset I^*;\ |\Omega| = N.$$

To prove this we assume the contrary, i.e. $\exists\, \Omega$, $|\Omega| = N : G_L(\Omega) > g_A(\Omega)$. Then, there are $u, v \in I^*$ such that $u\overline{R}_\Omega v \,(\mathrm{mod}\, L)$, but two their corresponding points $\alpha(u) = (x_1^u, \cdots, x_n^u)$ and $\alpha(v) = (x_1^v, \cdots, x_n^v)$ lie in the same connected domain in R^n determined by N hyperplanes (1). It means that $\forall\, \omega \in \Omega$:

$$\pi_0 M(u)M(\omega)\eta_F > \lambda \leftrightarrow \pi_0 M(v)M(\omega)\eta_F > \lambda,$$
$$p_A(u\omega) > \lambda \leftrightarrow p_A(v\omega) > \lambda,$$
$$u\omega \in L \leftrightarrow v\omega \in L,$$

We obtain $uR_\Omega v \,(\mathrm{mod}\, L)$. This conflicts with the hypothesis $u\overline{R}_\Omega v \,(\mathrm{mod}\, L)$. Therefore

$$G_L(\Omega) \leq g_A(\Omega), \quad \forall \Omega \subset I^*;\ |\Omega| = N < +\infty.$$

□

Theorem 2. (Growth speed theorem for FPA). *If A is a finite probabilistic automaton with n states and Ω is any finite set of words over input alphabet, then*

$$g_A(\Omega) = O(P_n(|\Omega|)),$$

where P_n is some polynominal of degree n.

Thus, the R-growth function on any finite set of the words of any FPA is bounded by a certain polynominal. This is an essential limitation of the FPA. We shall use this limitation to present languages not acceptable by any FPA.

We shall need the following lemma which is a sharpening of P. D. Dieu's lemma in [7].

Lemma 1. *Let $\xi(n, N)$ denote the maximal number of connected domains determined by N $(n-1)$-dimensional hyperplanes in R^n. Then*

$$\xi(n, N) = C_N^0 + C_N^1 + \cdots + C_N^n,$$

where $C_N^k = 0$, $\forall\, k > N$.

Proof. If $n = 1$ then

$$\xi(1, N) = N + 1 = C_N^0 + C_N^1.$$

If $n \geq 2$ then we distinguish two cases:

If $N = 2$, then

$$\begin{aligned} \xi(n, 2) &= 4 = 1 + 2 + 1 \\ &= C_2^0 + C_2^1 + C_2^2 = C_2^0 + C_2^1 + C_2^2 + C_2^3 + \cdots + C_2^n \\ &= C_N^0 + C_n^1 + \cdots + C_N^n. \end{aligned}$$

If $N > 2$, we shall prove by induction on the number N. We consider N $(n-1)$-dimensional hyperplanes $H_1, H_2, \cdots, H_{N-1}, H_N$. The first $(N-1)$ hyperplanes $H_1, H_2, \cdots, H_{N-1}$ can determine in R^n at most $\xi(n, N-1)$ connected domains by the inductive assumption. We denote these domains by $D_1, D_2, \cdots, D_r$, $r \leq \xi(n, N-1)$. Now the hyperplane H_N can be considered as a space R^{n-1}. The maximal number of connected domains in H_N determined by the hyperplanes $H_1, H_2, \cdots, H_{N-1}$ is equal to the maximal number of connected domains in H_N determined by the intersections of H_N with $H_1, H_2, \cdots, H_{N-1}$. It is $\xi(n-1, N-1)$

by induction assumption. Every of these domains in H_N lies entirely in some D_i and divices D_i at most by two connected domains in R^n. Therefore we have

$$\xi(n,N) = \xi(n,N-1) + \xi(n-1,N-1).$$

By induction, we have

$$\xi(n,N-1) = C^0_{N-1} + C^1_{N-1} + \cdots + C^n_{N-1},$$
$$\xi(n-1,N-1) = C^0_{N-1} + C^1_{N-1} = \cdots + C^{n-1}_{N-1}.$$

It follows that

$$\xi(n,N) = (C^0_{N-1}) + (C^1_{N-1} + C^0_{N-1}) + \cdots + (C^n_{N-1} + C^{n-1}_{N-1}).$$

Applying the Pascal's equality $C^k_N + C^{k-1}_N = C^k_{N+1}$ and remarking that $C^0_{N-1} = C^0_N = 1$ we obtain

$$\xi(n,N) = C^0_N + C^1_N + \cdots + C^n_N.$$

□

Now we prove the growth speed theorem for FPA.

Proof of Theorem 2. Let $A = (I, S, \pi_0, M, F)$ be a FPA with $|S| = n$ and $\Omega \subset I^*$, $|\Omega| = N < +\infty$. According to definition 1, we have $g_A(\Omega) \leq \xi(n,N)$. By Lemma 1 we have

$$\xi(n,N) = C^0_N + C^1_N + \cdots + C^n_N.$$

Therefore

$$g_A(\Omega) \leq C^0_N + C^1_N + \cdots + C^n_N = P_n(N).$$

It follows that

$$g_A(\Omega) = O(P_n(N)).$$

□

Corollary 1. (Necessary condition for $\mathcal{L}(FPA)$). *Let $L \subseteq I^*$ and $L \in \mathcal{L}(FPA)$. There exists a contant integer n such that for any set $\Omega \subset I^*$, $|\Omega| = N < +\infty$, we have*

$$G_L(\Omega) = O(N^n).$$

Proof. Since $L \in \mathcal{L}(FPA)$ there exists a FPA A with n states and a cut point λ, $0 \leq \lambda < 1$, such that $L = L(A, \lambda)$. Applying Theorem 1 and Theorem 2 we obtain

$$G_L(\Omega) \leq g_A(\Omega) = O(P_n(N)).$$

It follows that

$$G_L(\Omega) = O(N^n).$$

□

Example 3. We consider the following languages

$$L_1 = \{\tau_1 c \tau_2 c \cdots c \tau_n c \tau_0 \mid \forall\, i : \tau_i \in I^*;\ \exists\, \tau_i = \tau_0\}.$$
$$L_2 = \{a^k \mid v_k = 1; \quad k = 1, 2, ...\},$$

where $v_1 v_2 \cdots v_k \cdots$ is dictionary ordering of all words over alphabet $\{0, 1\}$ and $l(v_i) = 1$, $i \geq 1$. By choosing the adequate sets Ω, $|\Omega| = N < +\infty$, we have shown in Example 1 and Example 2 that

$$G_{L_1}(\Omega) = 2^{|\Omega|} = 2^N > O(N^n),$$
$$G_{L_2}(\Omega) = 2^{|\Omega|} = 2^N > O(N^n).$$

According to the Corollary 1, we obtain $L_1, L_2 \notin \mathcal{L}(FPA)$.

4. Some other supply-demand Theorems for FPA

In this part we consider other equivalent relations in I^*.

Let $L \subseteq I^*$, $\Omega \subset I^*$, $|\Omega| = N < +\infty$ and $u, v, u_1, v_1, u_2, v_2 \in I^*$. We define the relations $L_\Omega \,(\mathrm{mod}\, L)$, $B_\Omega(\mathrm{mod}\, L)$ and $P_\Omega(\mathrm{mod}\, L)$ as follows:

$$uL_\Omega v \,(\mathrm{mod}\, L) \Leftrightarrow \forall\, \omega \in \Omega : (\omega u \in L \leftrightarrow \omega v \in L),$$
$$uB_\Omega v \,(\mathrm{mod}\, L) \Leftrightarrow \forall\, \omega, \tau \in \Omega : (\omega u \tau \in L \leftrightarrow \omega v \tau \in L),$$
$$(u_1, v_1) P_\Omega (u_2, v_2) (\mathrm{mod}\, L) \Leftrightarrow \forall\, \omega \in \Omega : (u_1 \omega v_1 \in L \leftrightarrow u_2 \omega v_2 \in L).$$

It is easy to verify that $L_\Omega(\mathrm{mod}\, L)$, $B_\Omega(\mathrm{mod}\, L)$ and $P_\Omega(\mathrm{mod}\, L)$ are also equivalent relations in I^*. So we define

$$H_L(\Omega) = \mathrm{Rank}\ L_\Omega\,(\mathrm{mod}\, L),$$
$$I_L(\Omega) = \mathrm{Rank}\ B_\Omega\,(\mathrm{mod}\, L),$$
$$K_L(\Omega) = \mathrm{Rank}\ P_\Omega\,(\mathrm{mod}\, L).$$

$H_L(\Omega)$, $I_L(\Omega)$, $K_L(\Omega)$ are called *L-representative complexity, B-representative complexity, P-representative complexity* of language L on the set Ω, respectively.

Let $A = (I, S, \pi_0, M, F)$ be a FPA with

$$|S| = n, \quad L = L(A, \lambda), \quad 0 \leq \lambda < 1,$$
$$\Omega \subset I^*, \quad \Omega = \{\omega_1, \cdots, \omega_N\}, \quad N < +\infty.$$

We consider the following cases:

(1) *Case of* $L_\Omega\,(\mathrm{mod}\, L)$

For $u, v \in I^*$, we have

$$uL_\Omega v\,(\mathrm{mod}\, L) \Leftrightarrow \forall\, \omega \in \Omega : \omega u \in L \leftrightarrow \omega v \in L,$$
$$uL_\Omega v\,(\mathrm{mod}\, L) \Leftrightarrow \forall\, \omega \in \Omega : p_A(\omega u) > \lambda \leftrightarrow p_A(\omega v) > \lambda,$$
$$uL_\Omega v\,(\mathrm{mod}\, L) \Leftrightarrow \forall\, \omega \in \Omega : \pi_0 M(\omega) M(u) \eta_F > \lambda \leftrightarrow \pi_0 M(\omega) M(v) \eta_F > \lambda.$$

For each word $u \in I^*$, we define a corresponding point $\gamma(u)$ by

$$\gamma(u) = M(u)\eta_F = (x_1^u, \cdots, x_n^u)^T \in R^n,$$

and for each word $\omega \in \Omega$, we define a corresponding vector $\zeta(\omega)$ by

$$\zeta(\omega) = \pi_0 M(\omega) = (b_1^\omega, \cdots, b_n^\omega).$$

Therefore

$$\omega u \in L \Leftrightarrow b_1^\omega x_1^u + \cdots + b_n^\omega x_n^u > \lambda.$$

Now in Euclidean space R^n we consider N $(n-1)$-dimentional hyperplanes given by the equations

$$b_1^\omega x_1 + \cdots + b_n^\omega x_n = \lambda, \quad \omega \in \Omega. \tag{2}$$

Definition 2. Each connected domain determined by the hyperplanes (2) is called a *L-hyperstate on* Ω of the finite probabilistic automaton A. The number of L-hyperstates on Ω of A is called the *L-growth function on* Ω of A and denoted by $h_A(\Omega)$.

(2) *Case of* $B_\Omega \,(\mathrm{mod}\, L)$

For $u, v \in I^*$, we have

$$\begin{aligned} uB_\Omega v \,(\mathrm{mod}\, L) &\Leftrightarrow \forall\, \omega, \tau \in \Omega : \omega u \tau \in L \leftrightarrow \omega v \tau \in L, \\ uB_\Omega v \,(\mathrm{mod}\, L) &\Leftrightarrow \forall\, \omega, \tau \in \Omega : p_A(\omega u \tau) > \lambda \leftrightarrow p_A(\omega v \tau) > \lambda, \\ uB_\Omega v \,(\mathrm{mod}\, L) &\Leftrightarrow \forall\, \omega, \tau \in \Omega : \pi_0 M(\omega) M(u) M(\tau) \eta_F > \lambda \\ &\leftrightarrow \pi_0 M(\omega) M(v) M(\tau) \eta_F > \lambda. \end{aligned}$$

For each word $u \in I^*$, we define a corresponding point $\xi(u)$ by

$$\xi(u) = (x_{11}^u, \cdots, x_{n1}^u, x_{12}^u, \cdots, x_{n2}^u, \cdots, x_{1n}^u, \cdots, x_{nn}^u) \in R^{n^2},$$

with $M(u) = (x_{ij})_{n\times n}$ and for each two words $\omega, \tau \in \Omega$, we define two corresponding vectors:

$$\alpha(\omega) = \pi_0 M(\omega) = (a_1^\omega, \cdots, a_n^\omega) \quad \text{and} \quad \beta(\omega) = M(\tau)\eta_F = (b_1^\tau, \cdots, b_n^\tau)^T.$$

We can see that:

$$\omega u \tau \in L \Leftrightarrow \sum_{i,j=1}^{n} a_i^\omega b_j^\tau x_{ij} \;>\; \lambda.$$

Now in Euclidean space R^{n^2} we consider N^2 (n^2-1)-dimentional hyperplanes given by the equations:

$$\sum_{i,j=1}^{n} a_i^\omega b_j^\tau x_{ij} \;=\; \lambda, \quad \omega, \tau \in \Omega. \tag{3}$$

Definition 3. Each connected domain determined by the hyperplanes (3) is called a *B-hyperstate on* Ω of the finite probabilistic automaton A. The number of B-hyperstates on Ω of A is called the *B-growth function on* Ω of A and denoted by $i_A(\Omega)$.

(3) *Case of* P_Ω(modL)

For $u_1, v_1, u_2, v_2 \in I^*$, we have

$$(u_1,v_1)P_\Omega(u_2,v_2)\,(\mathrm{mod}\,L) \Leftrightarrow \forall\,\omega\in\Omega : u_1\omega v_1\in L \leftrightarrow u_2\omega v_2\in L,$$
$$(u_1,v_1)P_\Omega(u_2,v_2)\,(\mathrm{mod}\,L) \Leftrightarrow \forall\,\omega\in\Omega : p_A(u_1\omega v_1) > \lambda \leftrightarrow p_A(u_2\omega v_2) > \lambda,$$
$$(u_1,v_1)P_\Omega(u_2,v_2)\,(\mathrm{mod}\,L) \Leftrightarrow \forall\,\omega\in\Omega : \pi_0 M(u_1)M(\omega)M(v_1)\eta_F > \lambda$$
$$\leftrightarrow \pi_0 M(u_2)M(\omega)M(v_2)\eta_F > \lambda.$$

For each pair of words $(u,v)\in I^*\times I^*$, we define a pair of corresponding points $(\alpha(u),\beta(v))$ as follows:

$$\alpha(u)=\pi_0M(u)=(x_1^u,\cdots,x_n^u)^T,\quad \beta(v)=M(v)\eta_F=(y_1^v,\cdots,y_n^v)^T,$$

and for each word $\omega\in\Omega$, we define a corresponding vector $\theta(\omega)$ by

$$\theta(\omega)=(c_{11}^\omega,\cdots,c_{n1}^\omega,\cdots,c_{1n}^\omega,\cdots,c_{nn}^\omega),$$

with $M(\omega)=(c_{ij}^\omega)_{n\times n}$.

We can see that

$$u\omega v\in L \Leftrightarrow \sum_{i,j=1}^{n} c_{ij}^\omega x_i^u y_j^v \;>\; \lambda.$$

Now in Euclidean space R^{n^2} we consider N (n^2-1)-dimentional hyperplanes given by the equations

$$u\omega v\in L \Leftrightarrow \sum_{i,j=1}^{n} c_{ij}^\omega x_i^u y_j^v \;=\; \lambda,\quad \omega\in\Omega. \tag{4}$$

Definition 4. Each connected domain determined by the hyperplanes (4) is called a *P-hyperstate on Ω* of the finite probabilistic automaton A. The number of P-hyperstates on Ω of A is called the *P-growth function on Ω* of A and denoted by $k_A(\Omega)$.

Theorem 3. (Other supply-demand theorem for FPA). *Let A be a finite probabilistic automaton, $L=L(A,\lambda)$, $0\le\lambda<1$. Then for any finite set of words Ω over input alphabet, we have*

a) $H_L(\Omega) \le h_A(\Omega)$,

b) $I_L(\Omega) \le i_A(\Omega)$,

c) $K_L(\Omega) \le k_A(\Omega)$.

Proof. The proof is analogous to that of Theorem 1. □

Theorem 4. (Other growth speed theorem for FPA). *If A is a finite probabilistic automaton with n states and Ω is any finite set of words over input alphabet, $|\Omega|=N<+\infty$, then*

a) $h_A(\Omega) = O(P_n(N))$,

b) $i_A(\Omega) = O(P_{n^2}(N^2))$,

c) $k_A(\Omega) = O(P_{n^2}(N))$,

where P_n is some polynominal of degree n.

Proof. The proof is analogous to that of Theorem 2. □

Corollary 2. (Other necessary conditions for FPA). *Let $L \subseteq I^*$ and $L \in \mathcal{L}(FPA)$. There exists a constant integer n such that for any set $\Omega \subset I^*$, $|\Omega| = N < +\infty$, we have:*

a) $H_L(\Omega) = O(N^n)$,

b) $I_L(\Omega) = O(N^{2n^2})$,

c) $K_L(\Omega) = O(N^{n^2})$.

Proof. The proof is analogous to that of Corollary 1. □

Now by Corollary 2, we continue to give other examples of languages which do not belong to the class $\mathcal{L}(FPA)$.

Example 4. Let $|I| = k \geq 2$ and $c \notin I$. We define

$$L_4 = \{\tau_0 c \tau_1 c \cdots c \tau_n \mid \forall\, i : \tau_i \in I^*;\ \exists\, \tau_i = \tau_0\}.$$

We choose

$$\Omega = \{\tau_1, \tau_2, \cdots, \tau_N \mid \tau_i \in I^*\}, \quad N < +\infty.$$

Each subset $A = \{\tau_{i_1}, \tau_{i_2}, \cdots, \tau_{i_k}\} \subseteq \Omega$ is associated with a word

$$u_A = c\tau_{i_1} c \tau_{i_2} c \cdots c\tau i_k.$$

It is easy to see that

$$\forall\, \omega \in \Omega\ :\ (\omega u_A \in L_4 \leftrightarrow \omega \in A).$$

Therefore

$$H_{L_4}(\Omega) \ = \ 2^{|\Omega|} = 2^N \ > \ O(N^n).$$

According to Corollary 2, it follows that $L_4 \notin \mathcal{L}(FPA)$.

Example 5. Let $|I| = k \geq 2$ and $c \notin I$. We define

$$L_5 = \{\tau_0 c \tau_1 c \cdots c \tau_n c \tau_0' \mid \forall\, i : \tau_i \in I^*;\ (\exists\, \tau_i = \tau_0)\Lambda(\exists \tau_j = \tau_0')\}.$$

We choose

$$\Omega = \{\tau_1, \tau_2, \cdots, \tau_N \mid \tau_i \in I^*\}, \quad N < +\infty.$$

Each subset $A = \{\tau_{i_1}, \tau_{i_2}, \cdots, \tau_{i_k}\} \subseteq \Omega$ is associated with a word

$$u_A = c\tau_{i_1} c \tau_{i_2} c \cdots c\tau i_k c.$$

It is easy to verify that

$$\forall\, \omega, \tau \in \Omega\ :\ \omega u_A \tau \in L_5 \leftrightarrow ((\omega \in A)\Lambda(\tau \in A)).$$

Therefore

$$I_{L_5}(\Omega) \ = \ 2^{|\Omega|} = 2^N \ > \ O(N^{2n^2}).$$

According to Corollary 2, it follows that $L_5 \notin \mathcal{L}(FPA)$.

Example 6. Let $|I| = k \geq 2$ and $b, c \notin I$. We define

$$L_6 = \{\tau_1 b \cdots b\tau_n b\tau_0 c\tau_1^{,} c \cdots c\tau_m^{,} \mid \forall\, j : \tau_i, \tau_j^{,} \in I^*;\ (\exists\, au_i = \tau_0)\Lambda(\exists \tau_j^{,} = \tau_0)\}.$$

We choose

$$\Omega = \{\tau_1, \tau_2, \cdots, \tau_N \mid \tau_i \in I^*\}, \quad N < +\infty.$$

Each subset $A = \{\tau_{i_1}, \tau_{i_2}, \cdots, \tau_{i_k}\} \subseteq \Omega$ is associated with a pair of words (u_A, v_A) as follows:

$$u_A = \tau_{i_1} b\tau_{i_2} b \cdots b\tau i_k b,$$
$$v_A = c\tau_{i_1} c\tau_{i_2} c \cdots c\tau i_k.$$

We can verify that

$$\forall \omega \in \Omega \ :\ (u_A \omega v_A \in L_6 \leftrightarrow \omega \in A).$$

Therefore

$$K_{L_6}(\Omega) \ =\ 2^{|\Omega|} = 2^N \ >\ O(N^{n^2}).$$

According to Corollary 2, it follows that $L_6 \notin \mathcal{L}(FPA)$.

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