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TWO COINCIDENCE THEOREMS OF VIETORIS MAPS

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Abstract. In 1959, Nikaidô established a remarkable coincidence theorem in a compact Hausdorff topological space, which generalizes and gives a unified treatment to the results of Gale regarding the existence of economic equilibrium and theorems in game problems. The main purpose of the present paper is to deduce several generalized key results based on this very powerful result together with some KKM property. Indeed, we shall simplify and reformulate a few coincidence theorems on acyclic multifunctions as well as some Gòrniewicz-type fixed point theorems. Beyond the realm of monotonicity nor metrizability, the results derived here generalize and unify various earlier ones from classic optimization theory. In the sequel, we shall deduce two versions of Nikaidô's coincidence theorem about Vietoris maps from different approaches.

1. Introduction and preliminaries

An *acyclic* space is a nonempty compact Hausdorff path connected topological space whose n-th homology group is zero for each $n = 1, 2, 3, \ldots$. Homology, taken over any fixed field of coefficients, is in terms of either Vietoris or Čech cycles, as in Begle [1, 2, 3]. For example, any nonempty compact convex set and any compact contractible space are acyclic. A function τ from M to N is called a *Vietoris map* if τ is onto and the inverse image $\tau^{-1}(q)$ is acyclic for each $q \in N$. In this paper we will establish two coincidence theorems on Vietoris maps from different approaches together with fixed point theorems and several corollaries for acyclic multifunctions. Beyond the realm of monotonicity and convexity on operators, the results derived here generalize and unify various earlier ones from classical optimization theory, as will be indicated below. For this purpose, we shall adopt a technical result from Nikaidô [15]. Indeed, a remarkable coincidence theorem, due to Nikaidô, is proved by using a result of Begle [1, 2, 3] plus the outline of Knaster-Kuratowski-Mazurkiewicz' proof of Brouwer's fixed point theorem [12].

Nikaidô's Coincidence Theorem [15, Theorem 3]. *Let M be a compact Hausdorff topological space, N a finite-dimensional compact convex set, and σ and τ*

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continuous functions from M to N. If τ is a Vietoris map, then there exists some $p \in M$ such that $\sigma(p) = \tau(p)$.

In particular, when $T = \tau^{-1}$ and $f = \sigma$, Nikaidô's coincidence theorem implies

Gòrniewicz's Fixed Point Theorem [10]. *Let P be an n-simplex in a topological vector space X and Y any compact Hausdorff topological space. If $T : P \longrightarrow Y$ is an acyclic multifunction and $f : Y \longrightarrow P$ is a continuous function, then $f \circ T : P \longrightarrow P$ has a fixed point.*

Accordingly, there exist $x \in P$ and $y \in T(x)$ such that $x = f(y)$. Such a pair (x, y) is called a coincidence for T and f. For multifunctions $T : X \longrightarrow Y$ and $S : Y \longrightarrow X$, we define the *coincidence* for T and S to be a pair $(x, y) \in X \times Y$ with $y \in T(x)$ and $x \in S(y)$.

We digress briefly now to list a little notation and review some definitions. Suppose that C and D are subsets of topological spaces X and Y, respectively. In this paper, we shall use *LC spaces* to indicate the class of locally convex Hausdorff topological vector spaces. A *multifunction* T from C to D, written as $T : C \longrightarrow D$, is simply a function which assigns each point x of C to a (possibly empty) subset $T(x)$ of D. The *domain, range, graph* and *inverse* of T are defined, respectively, by

$$\begin{aligned} D(T) &:= \{x \in C;\ T(x) \neq \emptyset\}, \\ R(T) &:= \{y \in D;\ y \in T(x) \text{ for some } x \in D(T)\}, \\ G(T) &:= \{(x, y) \in C \times D;\ y \in T(x)\}, \end{aligned}$$

and

$$G(T^{-1}) := \{(y, x) \in D \times C;\ (x, y) \in G(T)\}.$$

A multifunction $T : C \longrightarrow D$ is *upper semicontinuous at x* provided for each open set V containing $T(x)$, there exists an open set U containing x such that $T(y)$ is contained in V whenever y is in U. We shall say T is *upper semicontinuous* (u.s.c.) provided T is u.s.c. at each x. The multifunctions T will be called *acyclic* provided T is u.s.c. and $T(x)$ is acyclic for each x. It is known from a Künneth theorem (see Massey [14]) that the cartesian product of two acyclic multifunctions is acyclic. We say that T is *closed* if the graph of T is closed in $C \times D$. It is also known that any u.s.c. compact-valued multifunction $T : C \longrightarrow D$ is closed. Conversely, if T is closed and D is compact, then T is also u.s.c.. When $R(T)$ is contained in some compact subset of D, we say T is *compact*; in other words, $clR(T)$ is compact in D. Therefore, any compact closed multifunction is u.s.c.. It is clear that T is compact whenever D is compact. As well, when C is compact, any u.s.c. compact-valued multifunction T is compact. Further, T is said to have the *local intersection property* if, for each $x \in C$ with $T(x) \neq \phi$, there exists an open neighborhood $N(x)$ of x such that

$$\bigcap_{z \in N(x)} T(z) \neq \phi.$$

The following proposition provides a relation between Vietoris maps and acyclic multifunctions.

Proposition 1.1. *Let M and N be nonempty subsets of Hausdorff topological spaces X and Y, respectively, and let τ be a continuous Vietoris map from M to N. If M is compact, then the multifunction $\tau^{-1} : N \longrightarrow M$ is acyclic.*

Proof. . Let $T = \tau^{-1}$. Since τ is a Vietoris map, $T(q) = \tau^{-1}(q)$ is nonempty and acyclic for each $q \in N$. Notice that

$$G(\tau) = \{(y, x) \in M \times N;\ \tau(y) = x\} = \{(y, x) \in M \times \tau(M);\ \tau(y) = x\}$$

is a closed subset of $M \times \tau(M)$, as τ is continuous. It follows that

$$G(T) = G(\tau^{-1}) = \{(x, y) \in N \times M;\ y \in \tau^{-1}x\} = \{(x, y) \in N \times M;\ \tau(y) = x\}$$

is also closed. This shows that T is a closed multifunction. Since $R(T)$ is contained in the compact set M, it follows that T is u.s.c., and hence T is an acyclic multifunction. $\square$

For a subset C of X, the closure and the convex hull of C will be denoted by clC and coC, respectively. For such a pair (T, C) and a function $\phi : C \times C \times D \longrightarrow R \cup \{+\infty\}$, we shall also consider an auxiliary problem, called the *generalized variational inequality problem*:

$\mathbf{G}VI(T, C, \phi)$: Find $\overline{x} \in C$ and $\overline{y} \in T(\overline{x})$ such that $\phi(x, \overline{x}, \overline{y}) \geq 0,\ \forall\ x \in C$.

In particular, if $\phi(z, x, y) := \langle z - x, y\rangle,\ \forall\ (z, x, y) \in C \times C \times D$, the problem $GVI(T, C, \phi)$ reduces to the usual variational inequality $VI(T, C)$. Furthermore, when X is a reflexive Banach space, with dual X^*, and T is the subdifferential of a convex function $f : X \longrightarrow R\ \cup\ \{+\infty\}$; i.e.,

$$T(x) = \partial f(x) := \{y \in X^*;\ f(z) - f(x) \geq \langle z - x, y\rangle,\ \forall z \in X\},$$

it is easy to see that $(\overline{x}, \overline{y})$ solves $VI(T, C)$ only if $\overline{x}$ solves the abstract convex programming problem $\min\{f(x);\ x \in C\}$.

Finally, we expose a general continuous selection theorem. A *locally selectionable multifunction* $T : C \longrightarrow D$ is a multifunction such that for each $x \in C$ there exist an open neighborhood U_x of x and a continuous mapping $f_x : C \longrightarrow D$ with

$$f_x(y) \in T(y),\ \forall\ y \in U_x \cap C.$$

In virtue of partition of unity, we follow mainly an idea from Wu and Shen [18] to establish a unified continuous selection theorem as follows.

Proposition 1.2. *Let $S : C \longrightarrow D$ be a multifunction, where C is a nonempty subset of a Hausdorff topological space X, and D is a nonempty convex subset of a topological vector space Y. If K is a compact subset of C and any one of the following properties holds, then there exists a continuous selection f from S on K; that is, $f(x) \in S(x),\ \forall\ x \in K$.*

(I) *There exists a multifunction $A : K \longrightarrow D$ satisfying*

(i) *$A(x)$ is nonempty and $coA(x) \subset S(x)$ for each $x \in K$,*

(ii) *A is locally selectionable.*

(II) *There exists a multifunction $A : K \longrightarrow D$ satisfying*

(i) *$A(x)$ is nonempty and $coA(x) \subset S(x)$ for each $x \in K$,*

(ii) *A has the local intersection property.*

(III) *There exists a multifunction $A : K \longrightarrow D$ satisfying*

(i) *$A(x)$ is nonempty and $coA(x) \subset S(x)$ for each $x \in K$,*

(ii) *$A^{-1}(y)$ is open for each $y \in D$.*

Proof. (I) Since A is locally selectionable, for any $x \in K \subset C$, there exist an open neighborhood U_x of x and a continuous mapping $f_x : K \longrightarrow D$ such that

$$f_x(y) \in A(y),\ \forall\ y \in U_x \cap K.$$

Since $\{U_x;\ x \in K\}$ forms an open covering of the compact set K, there is a finite subcover $\{U_{x_1}, U_{x_2}, \cdots, U_{x_n}\}$ of K. Thus, there is a partition of unity subordinated to this subcover; that is, there are continuous functions $\varphi_k : K \longrightarrow [0, 1],\ k = 1, 2, ..., n$, such that

(i) for each k, $\varphi_k(y) = 0,\ \forall\ y \notin U_{x_k}$,

(ii) $\sum\limits_{k=1}^{n} \varphi_k(y) = 1,\ \forall\ y \in K$.

Define a mapping $f : K \longrightarrow Y$ by

$$f(y) := \sum_{k=1}^{n} \varphi_k(y) f_{x_k}(y), \quad \forall\ y \in K.$$

Then, f is clearly continuous. Note that for $y \in K$ with $\varphi_k(y) \neq 0$, the condition (i) yields $y \in U_{x_k}$, and hence $f_{x_k}(y) \in A(y)$. It follows that

$$f(y) = \sum_{k=1}^{n} \varphi_k(y) f_{x_k}(y) \in \sum_{k=1}^{n} \varphi_k(y) A(y) \subset coA(y) \subset S(y), \quad \forall\ y \in K.$$

This shows that f is a continuous selection from S on K.

(II) Since A has the local intersection property, for each $x \in K$, there exists an open neighborhood $N(x)$ of x such that

$$F(x) := \bigcap_{z \in N(x)} A(z) \neq \emptyset.$$

Since K is compact, there is a finite open cover $\{N(x_i);\ i \in I\}$ of K and a partition of unity subordinated to this cover, say $\{f_i;\ i \in I\}$, such that

(i) $f_i(x) = 0,\ \forall\ x \notin N(x_i)$ for each $i \in I$,

(ii) $\sum\limits_{i \in I} f_i(x) = 1,\ \forall\ x \in K$.

Now, we choose any $y_i \in F(x_i)$ for each $i \in I$, and define $f : K \longrightarrow Y$ by

$$f(x) := \sum_{i \in I} f_i(x) y_i,\ \forall\ x \in K.$$

Then f is clearly continuous. Moreover, for each $x \in K$ and each $i \in I$, if $f_i(x) \neq 0$, then $x \in N(x_i)$. It follows that $y_i \in F(x_i) \subset A(x)$, and hence

$$f(x) \in co\{y_i;\ f_i(x) \neq 0\} \subset coA(x) \subset S(x), \quad \forall\, x \in K.$$

This yields that f is a continuous selection from S on K.

(III) Since $A^{-1}(y)$ is open for each $y \in D$, then for each $x \in C$ with $A(x) \neq \emptyset$, we can choose a $y \in A(x)$ and let $N(x) = A^{-1}(y)$. Then $N(x)$ is an open neighborhood of x and $y \in \bigcap_{z \in N(x)} A(z)$. Hence, A has the local intersection property, and applying Part (II) the proof is complete. □

As an application of Nikaidô's Coincidence Theorem, a new coincidence theorem is obtained as follows. This result will be further extended; see Theorem 2.5 and Theorem 3.5.

Theorem 1.1. *Let C be a nonempty compact subset of a Hausdorff topological space X and D a nonempty convex subset of a topological vector space Y. If $S : C \longrightarrow D$ is a multifunction satisfying any one of conditions* (II)∼(III) *in Proposition 1.2 with $K = C$, and $T : D \longrightarrow C$ is an acyclic multifunction, then there is a coincidence for S and T; that is, there is some $(\bar{x}, \bar{y}) \in C \times D$ such that $\bar{y} \in S(\bar{x})$ and $\bar{x} \in T(\bar{y})$.*

Proof. Following the proof of Proposition 1.2, we have a finite subset $\{y_i;\ i \in I\}$ of D and a continuous selection f from S. Let $N := co\{y_i;\ i \in I\}$. Then N is a finite-dimensional compact convex subset of D. Since T is u.s.c., the image $T(N)$ is compact. Thus, the closed subset

$$M := (T(N) \times N)\ \cap\ G(T^{-1})$$

of the compact set $T(N) \times N$ is also compact. Let σ and τ be two maps from M into N defined by $\sigma(x, y) := f(x)$ and $\tau(x, y) := y$. Then they are continuous and τ is a Vietoris map, since

$$\tau^{-1}(y) = \{(x, y) \in M; \tau(x, y) = y\} = T(y) \times \{y\}$$

is an acyclic set for each $y \in N$. Applying Nikaidô's coincidence theorem, we have some $(\bar{x}, \bar{y}) \in M \subset C \times D$ such that $\sigma(\bar{x}, \bar{y}) = \tau(\bar{x}, \bar{y})$. It follows that $\bar{x} \in T(\bar{y})$ and $\bar{y} = \sigma(\bar{x}, \bar{y}) = f(\bar{x}) \in S(\bar{x})$. □

2. The first version of Nikaidô's cioncidence theorem

We begin with a technical result regarding the existence of Fan-type elements [7, 8]. The versatile tool to prove it is adapted from Nikaidô [15], where there is a consequence of the Lefschetz fixed point theorem. For the literature, see also [3, 4, 5, 7, 8, 11, 13, 16, 17].

Lemma 2.1. *Let C be a nonempty compact convex subset of a LC space X, D a subset of a Hausdorff topological space Y, and $T : C \longrightarrow D$ an acyclic* *multifunction. Suppose that*

$$F(z,x) = \{y \in T(x);\ \varphi_1(z,y) \geq \varphi_2(x,y)\}, \quad \forall\ z,x \in C,$$

where $\varphi_1, \varphi_2 : C \times D \longrightarrow R \cup \{+\infty\}$ *are functions satisfying*

(i) $\varphi_1(x,y) \geq \varphi_2(x,y),\ \forall\ (x,y) \in G(T)$,

(ii) *for each* $y \in D$ *the map* $x \mapsto \varphi_1(x,y)$ *is quasiconvex on* C,

(iii) *for each* $x \in C$ *the map* $y \mapsto \varphi_1(x,y)$ *is continuous on* D,

(iv) *the function* φ_2 *is lower semicontinuous (l.s.c.) on* $C \times D$.

Then there exists a Fan-type element $\bar{x} \in C$ *such that*

$$\bigcap_{z \in C} F(z,\bar{x}) \neq \emptyset.$$

Proof. First, we observe that $G(T)$ is compact, as it is a closed subset of the compact set $C \times T(C)$. Let α and β be the natural projections of the graph $G(T)$ of T onto C and $T(C)$, respectively. That is, for any $p := (x,y) \in G(T)$ we have $\alpha(p) := x$ and $\beta(p) := y$. Thus for all $z \in C$ the sets

$$A(z) := \{p \in G(T);\ \varphi_1(z,\beta(p)) \geq \varphi_2(p)\}$$

are each nonempty and compact. To complete the proof we need to show that

$$\bigcap \{A(z);\ z \in C\} \neq \emptyset. \tag{1}$$

For this, it will suffice to show just that

$$\bigcap \{A(z_i);\ i \in I\} \neq \emptyset, \tag{2}$$

for any nonempty finite subset $\{z_i;\ i \in I\}$ of C. Assume on the contrary that there is a finite subset $\{z_i;\ i \in I\}$ of C such that

$$\bigcap \{A(z_i);\ i \in I\} = \emptyset.$$

Then for each $p \in G(T)$,

$$f_I(p) := \min\{\varphi_1(z_i,\beta(p));\ i \in I\} < \varphi_2(p).$$

More specifically, since $\varphi_2 - f_I$ is l.s.c. on the compact set $G(T)$, for each $p \in G(T)$ there exists some $j \in I$ such that

$$\varphi_1(z_j,\beta(p)) = f_I(p) < \epsilon + f_I(p) \leq \varphi_2(p), \tag{3}$$

where ε is a positive number given by

$$\varepsilon := \inf\{\varphi_2(p) - f_I(p);\ p \in G(T)\} > 0.$$

Let

$$\theta_i(p) := \max\{0, \epsilon + f_I(p) - \varphi_1(z_i,\beta(p))\}.$$

It follows from (3) that the formula

$$\theta(p) := \frac{\sum\limits_{i\in I} \theta_i(p) z_i}{\sum\limits_{i\in I} \theta_i(p)}$$

specifies a well-defined continuous function from $G(T)$ to the set $S := co\{z_i;\ i \in I\}$. Also, the projection α maps $L := G(T) \cap (S \times T(C))$ into S continuously, with $\alpha^{-1}(x) = \{x\} \times T(x)$, an acyclic subset of L for each $x \in S \subset C$. Therefore Nikaidô's coincidence theorem yields some $\bar{p} := (\bar{x}, \bar{y}) \in L$ such that $\theta(\bar{p}) = \alpha(\bar{p})$. Let $J := \{i \in I;\ \theta_i(\bar{p}) > 0\}$. Then J is nonempty by (3). Since for any $i \in J$ we have $\theta_i(\bar{p}) > 0$, it follows that

$$\varepsilon + f_I(\bar{p}) - \varphi_1(z_i, \beta(\bar{p})) > 0.$$

Thus, by the definition of ϵ, we have

$$\varphi_2(\bar{p}) - f_I(\bar{p}) \geq \epsilon > \varphi_1(z_i, \beta(\bar{p})) - f_I(\bar{p}),\ \forall\ i \in J.$$

From this, we deduce

$$\varphi_2(\bar{p}) > \varphi_1(z_i, \beta(\bar{p})),\ \forall\ i \in J. \tag{4}$$

Notice that the summation in $\theta(\bar{p})$ can be taken just over J. It follows that

$$\bar{x} = \alpha(\bar{p}) = \theta(\bar{p}) \in co\{z_i;\ i \in J\} \subset C.$$

Since the function $\varphi_1(\cdot, \bar{y})$ is quasiconvex on C, by (4) we then have

$$\varphi_1(\bar{x}, \bar{y}) \leq \max\{\varphi_1(z_i, \bar{y});\ i \in J\} < \varphi_2(\bar{p}) \leq \varphi_1(\bar{x}, \bar{y}).$$

This contradiction yields (2), and hence the proof is complete. □

Remark that for a function $\phi : C \times C \times D \longrightarrow R \cup \{+\infty\}$, we may consider the sets

$$F(z, x) = \{y \in T(x);\ \phi(z, x, y) \geq 0\}, \quad \forall\ z, x \in C.$$

By an argument analogous to the above technical lemma, there exists a Fan-type element $\bar{x} \in C$ such that

$$\bigcap_{z \in C} F(z, \bar{x}) \neq \emptyset.$$

Thus, we can obtain an existence theorem of solutions to $GVI(T, C, \phi)$ as follows.

Theorem 2.1. *Let C be a nonempty compact convex subset of a LC space X, D a subset of a Hausdorff topological space Y, and $T : C \longrightarrow D$ an acyclic multifunction. Suppose that $\phi : C \times C \times D \longrightarrow R \cup \{+\infty\}$ is a function satisfying*

(i) $\phi(x, x, y) \geq 0,\ \ \forall\ (x, y) \in G(T)$,

(ii) *for each $(x, y) \in G(T)$, the map $z \mapsto \phi(z, x, y)$ is quasiconvex,*

(iii) *for each $z \in C$, the map $(x, y) \mapsto \phi(z, x, y)$ is continuous.*

Then there is a solution to $GVI(T, C, \phi)$.

Now, we will prove a reverse form of Gòrniewicz-type fixed point theorem. Indeed, the theorem unifies and relaxes almost all of the well-known fixed point theorems.

Theorem 2.2. *Let C be a nonempty compact convex subset of a LC space X, and D a nonempty subset of a Hausdorff topological space Y. If $f : C \longrightarrow D$ is a continuous function, and $T : D \longrightarrow C$ is an acyclic multifunction, then there exists a fixed point to the composite $T \circ f$.*

Proof. Since X is locally convex, there is a local base β for X consisting of closed symmetric convex neighborhoods of 0 in X. For $U \in \beta$, we define

$$F_U := \{z \in C;\ z \in T \circ f(z) + U\}.$$

It is clear that $z \in C$ is a fixed point of $T \circ f$ if and only if

$$z \in \bigcap \{F_U;\ U \in \beta\}. \tag{5}$$

Since C is compact, it will suffice to show that the sets F_U are closed and satisfy the finite intersection property. Note that for any finite collection $\{U_1, U_2, \ldots, U_n\}$ from β, there is a $U \in \beta$ such that $F_U \subset F_{U_k}$, $\forall\ k = 1, 2, \ldots, n$. Therefore, it will suffice just to show that each F_U is both closed and nonempty.

Assume on the contrary that F_U is empty for some $U \in \beta$. We then have

$$x - y \notin U, \quad \forall\ (x, y) \in G(T \circ f).$$

Hence, the Minkowski functional

$$\mu_U(x) := \inf\{\lambda > 0;\ x \in \lambda U\}$$

is continuous and satisfies

$$\mu_U(x - y) \geq 1,\ \forall\ (x, y) \in G(T \circ f). \tag{6}$$

It follows that the functional $\alpha_U : C \longrightarrow R$, defined by

$$\alpha_U(x) := \min\{\mu_U(x - y);\ y \in T \circ f(x)\}, \quad \forall\ x \in C,$$

satisfies that $\alpha_U(x) \geq 1$, $\forall\ x \in C$. Applying Lemma 2.1 to $(T \circ f, X, X, C, C)$ in place of (T, X, Y, C, D) with $\varphi_1(x, y) = \mu_U(x - y)$, and $\varphi_2(x, y) = \alpha_U(x)$, we obtain some $\bar{x} \in C$ and $\bar{y} \in T \circ f(\bar{x})$ such that

$$\mu_U(x - \bar{y}) \geq \alpha_U(\bar{x}), \quad \forall\ x \in C.$$

Since $\bar{y} \in T \circ f(\bar{x}) \subset C$, we deduce a contradiction:

$$0 = \mu_U(\bar{y} - \bar{y}) \geq \alpha_U(\bar{x}) \geq 1.$$

This implies that all the sets F_U are nonempty. It remains to show that each F_U is closed. Define $\Delta := \{(x, x);\ x \in C\}$, and let $T_U : C \longrightarrow C$ be the multifunction given by

$$T_U(x) := T \circ f(x) + U,\ \forall\ x \in C.$$

Observe that $F_U = p(\Delta \cap G(T_U))$, where p denotes the projection of $C \times C$ onto the first coordinate. Notice that $T \circ f$ is u.s.c.. Since C is compact, the graph

$G(T_U)$ is closed and hence compact in the compact set $C \times C$. It follows that each F_U is compact and hence closed. Thus, the proof is complete. □

Using the above technical result, we can establish several consequences. By taking $C = D$ and $f(x) = x$, $\forall\, x \in C$, we first obtain a rather generalized fixed point theorem. Indeed, many well-known results can be considered as our consequences such as Brouwer's fixed point theorem, Kakutani's fixed point theorem, Browder's fixed point theorem, and Fan-Glicksberg's fixed point theorem.

Corollary 2.1. *If C is a nonempty compact convex subset of a LC space X, then any acyclic multifunction T from C into itself has a fixed point.*

As an application, we can extend a new coincidence theorem [6, Theorem 2.1] to the case that the image $S(x)$ need not be convex and the lower section $S^{-1}(y)$ is not necessarily open.

Theorem 2.3. *Let C be a nonempty compact convex subset of a LC space X and D a nonempty convex subset of a Hausdorff topological vector space Y. If $S : C \longrightarrow D$ is a multifunction satisfying any one of conditions* (I)∼(III) *in Proposition 1.2 with $K = C$, and $T : D \longrightarrow C$ is an acyclic multifunction, then there is a coincidence for S and T; that is, there is some $(\bar{x}, \bar{y}) \in C \times D$ such that $\bar{y} \in S(\bar{x})$ and $\bar{x} \in T(\bar{y})$.*

Proof. Following the proof of Proposition 1.2, the multifunction S admits a continuous selection f. Thus, by Theorem 2.2, there exists a fixed point x to $T \circ f$. Let $y = f(x)$. Then $x \in T \circ f(x) = T(y)$ and $y = f(x) \in S(x)$. This completes the proof. □

We remark that Theorem 2.2 is a particular case of Corollary 2.1, as well as Corollary 2.2, since the composite $T \circ f$ is an acyclic multifunction from C to itself. By Proposition 1.1, we can use Theorem 2.2 to establish a general form of Nikaidô's coincidence theorem. Therefore, all the above theorems are equivalent logically to Nikaidô's coincidence theorem in any locally convex Hausdorff topological vector space.

Theorem 2.4. (The first version of Nikaidô's coincidence theorem) *Let M be a nonempty compact convex subset of a LC space X, N a nonempty subset of a Hausdorff topological space Y, and σ and τ continuous functions from M to N. If τ is a Vietoris map, then there exists some $p \in M$ such that $\sigma(p) = \tau(p)$.*

Proof. Let $C = M$, $D = N$, and define $T = \tau^{-1}$ and $f = \sigma$. By Proposition 1.1, T is an acyclic multifunction. Applying Theorem 2.2, we have a fixed point p to the composite $T \circ f$; that is, $p \in T \circ f(p) = \tau^{-1}(\sigma(p))$. It follows that $\sigma(p) = \tau(p)$. □

3. The second version of Nikaidô's coincidence theorem

A multifunction $S : C \longrightarrow X$ is called a *KKM mapping* if $coA \subset S(A)$ for each finite subset A of C. In [7, 8], Fan proved the following celebrated lemma, which asserts that, given any convex set C in X and a closed-valued KKM mapping $S : C \longrightarrow X$, if $S(x)$ is compact for at least one $x \in C$, then $\bigcap_{x\in C} S(x) \neq \emptyset$. This lemma generalizes a classical finite-dimensional result of Knaster, Kuratowski, and Mazurkiewicz. Since then, many results in the direction have been obtained; see for example [5, 7, 13, 16, 17]. Following [5], we generalized the above property to the following form : if $S, T : C \longrightarrow D$ are two multifunctions such that $T(coA) \subset S(A)$ for each finite subset A of C, then we call S a *generalized KKM mapping* with respect to (w.r.t.) T. We say that $T : C \longrightarrow D$ has the *KKM property*, if $S : C \longrightarrow Y$ is a generalized KKM mapping w.r.t. T, then the family $\{clS(x);\ x \in C\}$ has the finite intersection property. We shall denote

$$KKM(C, D) = \{T;\ T : C \longrightarrow D \text{ has the } KKM \text{ property}\}.$$

Using this terminology, we have the following basic property.

Lemma 3.1. *Let C be a nonempty convex subset of a LC space X, and let Y, Z be two topological spaces.*

(1) *If $T \in KKM(C, C)$ is compact and closed, then T has a fixed point.*

(2) *If $T : C \longrightarrow Y$ is an acyclic multifunction, then $T \in KKM(C, Y)$.*

(3) *If $T \in KKM(C, Y)$ and f is continuous from Y to Z, then $f \circ T \in KKM(C, Z)$.*

Proof. Part (1) is a result of [5, Theorem 2]. Part (2) is a consequence of [17, Corollary 2]. To prove (3), we let $S : C \longrightarrow Z$ be a generalized KKM mapping w.r.t. $f \circ T$. Then for any finite subset $\{x_1, x_2, ..., x_n\}$ of C, we have

$$f \circ T(co\{x_1, x_2, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} S(x_i).$$

Hence

$$T(co\{x_1, x_2, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} f^{-1}S(x_i).$$

It follows that $f^{-1} \circ S$ is a generalized KKM mapping w.r.t. T. Since $T \in KKM(C, Y)$, the family $\{cl(f^{-1} \circ S(x));\ x \in C\}$ has the finite intersection property, and hence $\{clS(x);\ x \in C\}$ has the finite intersection property. □

To establish our main results, we begin with a generalized fixed point theorem of Gòrniewicz-type equipped with the KKM property.

Theorem 3.1. *Let C be a nonempty convex subset of a LC space X and D a nonempty subset of a topological space Y. If $T \in KKM(C, D)$ is a compact closed multifunction and $f : D \longrightarrow C$ is a continuous function, then $f \circ T : C \longrightarrow C$ has a fixed point.*

Proof. By Lemma 3.1(3), $f \circ T \in KKM(C, C)$. Since T is compact and closed, $f \circ T$ is also compact and closed. It follows from Lemma 3.1(1) that $f \circ T$ has a fixed point. □

Corollary 3.1. (Generalized Gòrniewicz Fixed Point Theorem) *Let C be a nonempty convex subset of a LC space X and D a nonempty subset of a topological space Y. If $T : C \longrightarrow D$ is a compact acyclic multifunction and $f : D \longrightarrow C$ is a continuous function, then $f \circ T : C \longrightarrow C$ has a fixed point.*

Corollary 3.2. *If C is a nonempty compact convex subset of a LC space X, then any acyclic multifunction T from C to itself has a fixed point.*

In virtue of the property of continuous selections, we have a general coincidence theorem for KKM mappings.

Theorem 3.2. *Let C be a nonempty subset of a Hausdorff topological space X and D a nonempty convex subset of a LC space Y. If $S : C \longrightarrow D$ is a multifunction satisfying any one of conditions* (I)∼(III) *in Proposition 1.2 with $K = clT(D)$, and $T \in KKM(D, C)$ is a compact closed multifunction, then there exists a coincidence for S and T.*

Proof. Since T is compact, the set $clT(D)$ is a compact subset of C. By Proposition 1.2, the restriction of the multifunction S to the compact set $clT(D)$ admits a continuous selection f. Thus, by Theorem 3.1, there exists a fixed point x to $f \circ T$. Let $x = f(y)$ for some $y \in T(x)$. Then $x = f(y) \in coA(y) \subset S(y)$. This completes the proof. □

Comparing with Theorem 2.4, the following version requires convexity of N and local convexity on Y, but M need not to be convex and X need not to be locally convex.

Theorem 3.3. (The second version of Nikaidô's coincidence theorem) *Let M be a nonempty compact subset of a Hausdorff topological space X, N a nonempty convex subset of a LC space Y, and σ and τ be continuous functions from M to N. If τ is a Vietoris map, then there exists some $p \in M$ such that $\sigma(p) = \tau(p)$.*

Proof. Let $C = N, D = M$, and define $T = \tau^{-1}$. By Proposition 1.1, T is an acyclic multifunction. Since M is compact, T is compact. Applying Corollary 3.1, we have a fixed point p to the composite $f \circ T$; that is, $p \in f \circ T(p)$. Let $q \in T(p)$ such that $p = f(q)$. It follows that $\sigma(q) = f(q) = p = \tau(q)$. □

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