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# EXISTENCE OF ABSTRACT SOLUTIONS OF INTEGRO-DIFFERENTIAL OPERATOR EQUATIONS

A. E. HAMZA AND G. MURAZ

Abstract. We consider the integro-differential operator equations having the form

$$\sum_{i=1}^{n} \alpha_i u^{(i)}(t) - \alpha A u(t) + \beta \mu * u(t) = f(t) \quad t \in \mathbb{R},$$

where the free term $f$ belongs to a closed subspace $M$ of $L^\infty(\mathbb{R}, X)$, $A$ is the generator of a $C_0$-semigroup of operators defined on a Banach space $X$, $\mu$ is a bounded Borel measure on $\mathbb{R}$ and $\alpha, \beta, \alpha_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$. Certain conditions will be imposed to guarantee the existence of solutions in the class $M$.

## 1. Introduction

Many authors, e.g, Zhikov-Levitan [14] and Baskakov [6] gave criteria for almost periodicity (a.p.) of solutions of operator differential equations having the form

$$u'(t) - Au(t) = f(t), \quad t \in \mathbb{R} \tag{1.1}$$

where $f$ is (a.p.) from $\mathbb{R}$ to a Banach space $X$ and $A$ is the generator of a $C_0$-semigroup. Others, like Staffans [16], gave criteria for almost periodicity (S-asymptotically almost periodic (S-a.a.p.)) of solutions of convolution equations having the form

$$\mu * \phi(t) = f(t) \ , t \in \mathbb{R} \tag{1.2}$$

where $f$ is a complex a.p (S-a.a.p.) and $\mu$ is a bounded Borel measure on $\mathbb{R}$. Here

$$\mu * \phi(t) = \int_{\mathbb{R}} \phi(t-s)\, d\mu(s), \quad t \in \mathbb{R}.$$

A continuous function from $\mathbb{R}$ to a Banach space $X$ is called (a.p.) if the set of translates $\{f_w : w \in \mathbb{R}\}$ is relatively compact in $C_b(\mathbb{R}, X)$ ($C_b(\mathbb{R}, X)$ is the space of all continuous bounded functions from $\mathbb{R}$ to $X$). It is well-known that an a.p. function is uniformly continuous bounded, i.e. belongs to $C_{ub}(\mathbb{R}, X)$, see [1], [10], [14]. The space of all almost periodic functions is denoted by $AP(\mathbb{R}, X)$. A function $f$ is said to be S-a.a.p. if $f = p + q$, where $p$ is a.p. and $q \in L^\infty(\mathbb{R}, X)$

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such that $\lim_{|t|\to\infty} \|f(t)\| = 0$. The space of all S-a.a.p. functions is denoted by $S - AAP(\mathbb{R}, X)$.

Throughout the paper, $X$ is a complex Banach space with the norm $\| \ \|$. As usual $L^\infty(\mathbb{R}, X)$ denotes the Banach space of all essentially bounded measurable functions with the norm

$$\|f\|_\infty = \operatorname{ess\,sup}_{t\in\mathbb{R}} \|f(t)\|.$$

A function $f$ is called measurable if there exists a sequence of simple functions $\{f_n\}$ such that $f_n \to f$ a.e. with respect to the Lebesgue measure $m$. By a simple function it is meant a function of the form $\sum_{i=1}^{n} x_i \chi_{A_i}$, $x_i \in X$ and $\chi_{A_i}$ is the characteristic function of the Lebesgue measurable set $A_i$ with finite measure. Finally, $M$ denotes a closed subspace of $L^\infty(\mathbb{R}, X)$ satisfying the following conditions:

**(P1)** $M$ is invariant under translations, i.e. $\forall f \in M \ \forall s \in \mathbb{R} \ (f_s \in M)$, where $f_s(t) = f(t + s)$.

**(P2)** $M$ contains the constant functions.

**(P3)** $M$ is invariant under multiplication by characters, i.e. $\forall f \in M \forall \lambda \in \mathbb{R}$ $(\breve{\lambda} \ f \in M)$, where $\breve{\lambda}(t) = e^{i\lambda t}$.

**(P4)** $Au \in M \ \ \forall A \in B(X) \ \ \forall u \in M$, where $B(X)$ is the space of all linear bounded operators on $X$.

We can see that many spaces like $AP(\mathbb{R}, X)$, $AAP(\mathbb{R}, X)$, $S$-$AAP(\mathbb{R}, X)$, $AA(\mathbb{R}, X)$, $AAA(\mathbb{R}, X)$, $S$-$AAA(\mathbb{R}, X)$ and $W(\mathbb{R}, X)$ satisfy conditions (P1)-(P4), for definitions and properties see [11], [12], [1-16].

For a function $u \in L^\infty(\mathbb{R}, X)$, we set

$$I_M(u) = \{f \in L^1(\mathbb{R}) : f * u \in M\}$$

and denote the $M$-spectrum of $u$ by

$$\sigma_M(u) = Z(I_M(u)) = \{\gamma \in \mathbb{R} : \hat{f}(\gamma) = 0 \ \forall f \in I_M(u)\},$$

where

$$\hat{f}(\gamma) = \int_{\mathbb{R}} f(t) e^{-i\gamma t}\, dt.$$

In the case $M = \{0\}, \sigma_M(u) = \sigma(u)$ is the well-known Beurling spectrum.

When $M = AP(\mathbb{R}, \mathbb{C})$, L. H. Loomis [15] proved that if $u \in C_{ub}(\mathbb{R}, \mathbb{C})$ and $\sigma_{AP(\mathbb{R},\mathbb{C})}(u)$ (the set of all non-almost periodicity of $u$) is at most countable, then $u \in AP(\mathbb{R}, \mathbb{C})$. B. Basit generalized this theorem in [5] to a class of bounded uniformly continuous vector-valued functions defined on $\mathbb{R}$ with certain properties satisfied by many known classes.

Some properties of the $M$-spectrum, was shown by A. E. Hamza and G. Muraz [12]. In that paper the following result [12, Theorem 4.2.2] was proved.

**Theorem 1.1.** *If $u$ is uniformly continuous, bounded, such that $\sigma_M(u)$ is at most countable, and for every $\lambda \in \sigma_M(u)$ the function $e^{-i\lambda t}u(t)$ is ergodic, then $u \in M$.*

This theorem plays an essential role in proving the existence of solutions in some classes $M \subseteq L^\infty(\mathbb{R}, X)$ for abstract functional equations defined on $\mathbb{R}$, with free terms in $M$. See the results obtained by A. E. Hamza [11] concerning the equations (1.1) and (1.2). We recall a function $u \in L^\infty(\mathbb{R}, X)$ is called ergodic if there exists $x \in X$ such that

$$\lim_{T\to\infty} \|1/T \int_0^T (u_s - x)\, ds\|_\infty = 0.$$

The space of all ergodic functions is denoted by $E(\mathbb{R}, X)$. It is known that, see [12],

$$AP(\mathbb{R}, X) \subset AAP(\mathbb{R}, X) \subset W(\mathbb{R}, X) \subset E(\mathbb{R}, X).$$

This paper is devoted to the integro-differential operator equation

$$\sum_{i=1}^{n} \alpha_i u^{(i)}(t) - \alpha A u(t) + \beta\mu * u(t) = f(t) \quad t \in \mathbb{R}, \tag{1.3}$$

where the free term $f \in M \bigcap C_{ub}(\mathbb{R}, X)$, $A$ is the generator of a $C_0$-semigroup of linear bounded operators $(T(t))_{t\geq 0}$ defined on $X$ and $\alpha, \beta, \alpha_i \in \mathbb{C}$. We set

$$sp(A, \mu) = \Big\{\lambda \in \mathbb{R} : \alpha A - \Big(\sum_{r=1}^{n} \alpha_r (i\lambda)^r + \beta\hat{\mu}(\lambda)\Big) I \text{ has no bounded inverse on } X\Big\}.$$

Here, $\hat{\mu}(\lambda) = \int_{\mathbb{R}} e^{-i\lambda t}\, d\mu(t)$. Our aim is to show that if $sp(A, \mu)$ is at most countable, then every solution $u$ of equation (1.3) belongs to $M$, provided that the function $e^{-i\lambda t}u(t)$ is ergodic for every $\lambda \in sp(A, \mu)$. In this case, $u$ is said totally ergodic (see ref. on [10]) and the generated Banach space is denoted by $TE(\mathbb{R}, X)$.

When $\alpha_n = \cdots = \alpha_2 = \beta = 0$ and $\alpha_1 = \alpha = 1$, we have $isp(A, \mu) = sp(A) \cap i\mathbb{R}$, where $sp(A)$ is the usual spectrum of $A$, and we get the following result [11, Theorem II.3.6]:

**Theorem 1.2.** *Suppose that $sp(A) \bigcap i\mathbb{R}$ is at most countable. If $u$ is a solution of equation* (1.1) *such that the function $e^{-\lambda t}u(t)$ is ergodic for every $\lambda \in sp(A) \bigcap i\mathbb{R}$, then $u \in M$.*

When $\alpha_n = \cdots = \alpha_1 = \alpha = 0$ and $\beta = 1$, we have $sp(A, \mu) = Z(\mu)$ and we get the following result [11, Theorem II.4.1]:

**Theorem 1.3.** *If $Z(\mu) = \{\alpha \in \mathbb{R} : \hat{\mu}(\alpha) = 0\}$ is at most countable, then every solution $u \in C_{ub}(\mathbb{R}, X)$ of equation* (1.2) *belongs to $M$, provided that the function $e^{-i\alpha t}u(t)$ is ergodic for every $\alpha \in Z(\mu)$.*

When $\alpha_n = \cdots = \alpha_2 = 0$ and $\alpha = \beta = 0$, the solution $u$ of (1.3) is given by

$$u(t) = \int_0^t f(s)\,ds,$$

we obtain Theorem (3.4) in [12]. When $f$ is a.p., Kadets [13] proved that the integral $u$ is a.p. when its range is weakly relatively compact in $X$ and Levitan [14] proved that $u$ is a.p. if $\lim_{T\to\infty} \int_{-T}^{T} u(t+s)\,ds$ exists uniformly on $\mathbb{R}$. Basit [4] extended Levitan's result for recurrent functions. C.Datry and G. Muraz [10] extended the result of Levitan to Banach $G$-modules.

## 2. The integro-differential operator equations

In the sequel we suppose that $M$ is a closed subspace of $L^\infty(\mathbb{R}, X)$ satisfying (P1)-(P4).

Consider the integro-differential operator equation

$$\sum_{i=1}^{n} \alpha_i u^{(i)}(t) - \alpha A u(t) + \beta \mu * u(t) = f(t), \quad t \in \mathbb{R}, \tag{2.1}$$

where the free term $f \in M \bigcap C_{ub}(\mathbb{R}, X)$, $A$ is the generator of a $C_0$-semigroup of linear bounded operators $(T(t))_{t\geq 0}$ defined on $X$ and $\alpha, \beta, \alpha_i \in \mathbb{C}$. We need the following lemmas in proving the main result

**Lemma 2.1.** *Let $A$ be a closed operator defined on $D(A) \subseteq X$. Suppose that $u \in C_b(\mathbb{R}, X)$ such that its range $R(u)$ is a subset of $D(A)$. Set $v = u * \psi$, where $\psi \in L^1(\mathbb{R})$. If $Au \in C_b(\mathbb{R}, X)$, then $R(v) \subseteq D(A)$ and $Av = Au * \psi$.*

*Proof.* At first, suppose that $\psi$ is a continuous function with compact support $[-T, T]$. We have

$$v(t) = \int_{-T}^{T} u(t-s)\psi(s)\,ds, \quad t \in \mathbb{R}.$$

Hence, $v(t) = \lim_{n\to\infty} v_n(t)$ $\forall t \in \mathbb{R}$, where $v_n(t) = 1/n \sum_{i=1}^{n} u(t-s_i)\psi(s_i)$, ($\{s_i\}$ is a partition of $[-T, T]$ with length $1/n$). We have

$$Av_n(t) = 1/n \sum_{i=1}^{n} Au(t-s_i)\psi(s_i)$$

which tends to

$$\int_{-T}^{T} Au(t-s)\psi(s)\,ds = Au * \psi(t)$$

as $n \to \infty$, $t \in \mathbb{R}$. Since $A$ is closed, then $R(v) \subseteq D(A)$ and $Av = Au * \psi$. Now, suppose that $\psi \in L^1(\mathbb{R})$, there exists a sequence $\{\psi_n\}$ of continuous functions with compact support such that $\|\psi_n - \psi\|_1 \to 0$ as $n \to \infty$. Let $v_n = u * \psi_n$, $n \in \mathbb{N}$ and $v = u * \psi$, by the first part, $R(v_n) \subseteq D(A)$ $\forall n$ and $Av_n = Au * \psi_n$, $n \in \mathbb{N}$. Since $\|u * \psi\|_\infty \leq \|u\|_\infty \|\psi\|_1$, then it is clear that $\|v_n - v\|_\infty \to 0$ as $n \to \infty$ and $\|Av_n - Au * \psi\|_\infty \to 0$ as $n \to \infty$. Again, Since $A$ is closed, then $R(v) \subseteq D(A)$ and $Av = Au * \psi$. □

**Lemma 2.2.** *Let $A$ be a closed operator defined on $D(A)$ which is dense in $X$. If $v \in C_b(\mathbb{R}, X)$, then there exists a bounded sequence of trigonometric polynomials $\{v_n\}$ such that*

(i) *$v_n \to v$ locally, i. e. $v_n$ is uniformly convergent to $v$ on every compat subset of $\mathbb{R}$*

(ii) *the range $R(v_n)$ of $v_n$ is a subset of $D(A)$ for every $n$.*

*Proof.* We can see that if $v \in C_b(\mathbb{R}, X)$, [14], then there exists a bounded sequence of trigonometric polynomial $\{u_n\}$ such that $u_n \to v$ locally. Since $D(A)$ is dense in $X$, we can approximate $u_n(t) = \sum_i \breve{\lambda}_{i,n} x_i$ by $v_n = \sum_i \breve{\lambda}_{i,n} y_i$, where $\{y_i\} \subset D(A)$. □

**Lemma 2.3.** *Suppose that $A$ is a closed operator defined on $D(A)$ which is dense in $X$ such that $sp(A) \neq \mathbb{C}$. If $v$ and $Av \in C_b(\mathbb{R}, X)$, then there exists a bounded sequence of trigonometric polynomials $\{v_n\}$ such that*

(i) *$v_n \to v$ locally and $R(v_n) \subseteq D(A)$.*

(ii) *$Av_n \to Av$ locally.*

*Proof.* Since both of $v$ and $Av$ belong to $C_b(\mathbb{R}, X)$, there exist two bounded sequences of trigonometric polynomials $\{u_n\}$ and $\{\theta_n\}$ such that

(1) $u_n \to v$ locally and $R(u_n) \subset D(A)$.

(2) $\theta_n \to Av$ locally and $R(\theta_n) \subset D(A)$.

Since $sp(A) \neq \mathbb{C}$, there exists $\lambda \in \mathbb{C}$ such that $(A - \lambda I)^{-1}$ exists as a bounded operator on $X$. We have

$$(A - \lambda I)^{-1}\theta_n \to (A - \lambda I)^{-1}Av = v + \lambda(A - \lambda I)^{-1}v \text{ locally} . \tag{1}$$

Also we have

$$\lambda(A - \lambda I)^{-1}u_n \to \lambda(A - \lambda I)^{-1}v \text{ locally}. \tag{2}$$

Putting $v_n = (A - \lambda I)^{-1}(\theta_n - \lambda u_n)$, we get by (1) and (2) that $v_n \to v$ locally. Now, we show that $Av_n \to Av$ locally. Indeed, we have

$$\begin{aligned} Av_n &= A(A - \lambda I)^{-1}(\theta_n - \lambda u_n) \\ &= [I + \lambda(A - \lambda I)^{-1}](\theta_n - \lambda u_n). \end{aligned}$$

Hence $Av_n \to Av$ locally. □

**Lemma 2.4.** *If $v \in C_b(\mathbb{R}, X)$ is such that its spectrum $\sigma(v)$ is a compact subset of an interval $[a, b]$, then $v \in C^\infty(\mathbb{R}, X)$ and $v^{(k)} \in C_b(\mathbb{R}, X)$ $\forall k \in \mathbb{N}$. If in addition, $A$ is a closed operator such that $D(A)$ is dense in $X$, $sp(A) \neq \mathbb{C}$*

*and* $Av \in C_b(\mathbb{R}, X)$*, then for every* $\varepsilon > 0$ *there exists a bounded sequence of trigonometric polynomials* $\{v_n\}$ *such that*

(i) $\sigma(v_n) \subseteq [a-\epsilon, b+\epsilon]$.

(ii) ${v_n}^{(k)} \to v^{(k)}$ *locally,* $k \in \mathbb{Z}\geq 0$,

(iii) $Av_n \to Av$ *locally.*

*Proof.* Let $\varepsilon > 0$. Choose $\phi \in L^1(\mathbb{R}) \bigcap C^\infty(\mathbb{R})$ such that

(1) $\phi^{(k)} \in L^1(\mathbb{R})\ \forall k$.

(2) *supp* $\hat{\phi} \subseteq [a-\varepsilon, b+\varepsilon]$.

(3) $\hat{\phi} = 1$ on $[a-\epsilon/2, b+\epsilon/2]$.

We show now that $v * \phi = v$. Indeed, let $h \in L^1(\mathbb{R})$. Then

$$(h * \phi - h\hat{)}(\alpha) = \hat{h}(\alpha)\hat{\phi}(\alpha) - \hat{h}(\alpha) = 0 \quad \forall \alpha \in [a-\epsilon/2, a+\epsilon/2].$$

Thus, $\mathrm{supp}\,(h*\phi - h\hat{)} \bigcap [a,b] = \emptyset$, hence $\sigma(v*(h*\phi-h)) = \emptyset$. This implies that $v*(h*\phi-h) = 0 \quad \forall h \in L^1(\mathbb{R})$, i.e. $(v*\phi - v)*h = 0 \quad \forall h \in L^1(\mathbb{R})$. Therefore, $v*\phi = v$ and then $v \in C^\infty(\mathbb{R}, X)$ and $v^{(k)} \in C_b(\mathbb{R}, X)$. Since both $v$ and $Av$ belong to $C_b(\mathbb{R}, X)$, by Lemma 2.3 there exists a bounded sequence of trigonometric polynomials $\{\psi_n\}$ such that

(i) $\psi_n \to v$ locally and $R(\psi_n) \subseteq D(A)$.

(ii) $A\psi_n \to Av$ locally.

Set $v_n = \psi_n * \phi$. We can see that $\sigma(v_n) \subseteq [a-\varepsilon, a+\varepsilon]$. Also, $v_n \to v$ locally. We have $\lim_{n\to\infty} {v_n}^{(k)} = \lim_{n\to\infty} \psi_n * \phi^{(k)} = v * \phi^{(k)} = v^{(k)}$ locally. Also we have $Av_n = A\psi_n * \phi\ \forall n$. Then $\lim_{n\to\infty} Av_n = Av * \phi = A(v*\phi) = Av$ locally. □

We denote

$$sp(A,\mu) = \Big\{\lambda \in \mathbb{R} : \Big[\alpha A - \Big(\sum_{k=1}^{n} \alpha_k (i\lambda)^k + \beta\hat{\mu}(\lambda)\Big) I\Big] \text{ has no bounded inverse on } X\Big\}.$$

and

$$P_\lambda = \Big[\alpha A - \Big(\sum_{k=1}^{n} \alpha_k (i\lambda)^k + \beta\hat{\mu}(\lambda)\Big) I\Big]^{-1},$$

where $\lambda \notin sp(A,\mu)$.

**Theorem 2.1.** *If* $u$ *is a solution of equation* (2.1)*, then* $\sigma_M(u) \subseteq sp(A,\mu)$.

*Proof.* Let $\lambda_0 \notin sp(A,\mu)$, where $\lambda_0 \in \mathbb{R}$. Then $P_\lambda$ is analytic in a neighbourhood of $\lambda_0$, say $[\lambda_0 - 4\alpha, \lambda_0 + 4\alpha]$. Fix a function $\phi \in L^1(\mathbb{R})$ such that

(i) $\hat{\phi} \in C_c^\infty$ the space of all infinitely differentiable functions with compact support.

(ii) $\hat{\phi}(\lambda) = 1\ \forall \lambda \in [\lambda_0 - 2\alpha, \lambda_0 + 2\alpha]$.

(iii) $\mathrm{supp}\,\hat{\phi} \subseteq (\lambda_0 - 4\alpha, \lambda_0 + 4\alpha)$.

Define the function $G : \mathbb{R} \to B(X)$ by

$$G(\lambda) = \hat{\phi}(\lambda) P_\lambda.$$

We see that

$$G(\lambda) = \begin{cases} P_\lambda, & \text{if } \lambda \in [\lambda_0 - 2\alpha, \lambda_0 + 2\alpha], \\ 0, & \text{if } |\lambda - \lambda_0| \geq 4\alpha. \end{cases}$$

Hence $G$ is infinitely differentiable with compact support. Therefore, $G$ is the Fourier transform of a continuous function $F \in L^1(\mathbb{R}, B(X))$ (i.e. $\int_{\mathbb{R}} \|F(t)\|_{B(X)}\, dt < \infty$). Let $\psi \in L^1(\mathbb{R})$ be such that $\operatorname{supp} \hat{\psi} \subseteq [\lambda_0 - \alpha, \lambda_0 + \alpha]$ and $\hat{\psi}(\lambda_0) = 1$. Set $v = u * \psi$ and $g = f * \psi$. We can see that $v$ is a solution of the equation

$$\sum_{k=1}^{n} \alpha_k v^{(k)}(t) - \alpha A v(t) + \beta \mu * v(t) = g(t), \quad t \in \mathbb{R}.$$

Since $v^{(k)} \in C_b(\mathbb{R}, X)$, $Av \in C_b(\mathbb{R}, X)$. Since $\sigma(v)$ is a compact subset of $[\lambda_0 - \alpha, \lambda_0 + \alpha]$, by Lemma 2.4 there exists a bounded sequence of trigonometric polynomials $\{v_n\}$ such that

(1) $\sigma(v_n) \subseteq [\lambda_0 - 2\alpha, \lambda_0 + 2\alpha]$ $\forall n$.
(2) $v_n \to v$ locally .
(3) $Av_n \to Av$ locally .
(4) $v_n^{(k)} \to v^{(k)}$ locally .

Putting $g_n = \sum\limits_{k=1}^{n} \alpha_k v_n^{(k)} - \alpha A v_n + \beta \mu * v_n$, $n \in \mathbb{N}$, we get $g_n \to g$ locally . Now, we prove that $v_n(t) = -\int_{\mathbb{R}} F(s) g_n(t-s)\, ds$, $n \in \mathbb{N}$, $t \in \mathbb{R}$. Fix $n \in \mathbb{N}$, we write $v_n$ in the form

$$v_n(t) = \sum_{k=1}^{n} P_{\lambda_k} a_k e^{i\lambda_k t}, \quad t \in \mathbb{R},$$

where $\{\lambda_k\} \subseteq [\lambda_0 - 2\alpha, \lambda_0 + 2\alpha]$ and $\{a_k\} \subset X$. A simple calculation shows that

$$\sum_{k=1}^{n} \alpha_k v_n^{(k)}(t) - \alpha A v_n(t) + \beta \mu * v_n(t) = -\sum_{k=1}^{n} a_k e^{i\lambda_k t}.$$

Hence $g_n(t) = -\sum\limits_{k=1}^{n} a_k e^{i\lambda_k t}$, $n \in \mathbb{N}$, $t \in \mathbb{R}$. We have

$$\begin{aligned} \int_{\mathbb{R}} F(s) g_n(t-s)\, ds &= -\sum e^{i\lambda_k t} \int_{\mathbb{R}} F(s) e^{-i\lambda_k s}\, ds\ a_k \\ &= -\sum e^{i\lambda_k t} \hat{F}(\lambda_k) a_k \\ &= -\sum e^{i\lambda_k t} P_{\lambda_k} a_k \\ &= -v_n(t), \qquad n \in \mathbb{N},\ t \in \mathbb{R}. \end{aligned}$$

Take the limit to obtain that $v(t) = -\int_{\mathbb{R}} F(s)g(t-s)\,ds$ , $t \in \mathbb{R}$. Now we show that $v \in M$. Define the function $h : \mathbb{R} \to M$ by $h(s) = F(s)g_{-s}$, ($g \in M$ from Lemma 4.1.2.in [12], see also [5]). The function $h$ is continuous. Indeed, fix $s_0 \in \mathbb{R}$. Let $\epsilon > 0$, there exists $\delta > 0$ such that for every $s \in (s_0 - \delta, s_0 + \delta)$

$$\|F(s) - F(s_0)\|_{B(X)} < \frac{\varepsilon}{2\|g\|_\infty} \quad \text{and} \quad \|g_{-s} - g_{-s_0}\|_\infty < \frac{\varepsilon}{2\|F(s_0)\|_{B(X)}} \cdot$$

Hence we have

$$\begin{aligned} \|h(s) - h(s_0)\|_\infty &= \|F(s)g_{-s} - F(s_0)g_{-s_0}\|_\infty \\ &\leq \|F(s)g_{-s} - F(s_0)g_{-s}\|_\infty + \|F(s_0)g_{-s} - F(s_0)g_{-s_0}\|_\infty \\ &< \frac{\varepsilon}{2\|g\|_\infty}\|g\|_\infty + \frac{\varepsilon}{2\|F(s_0)\|_{B(X)}}\|F(s_0)\|_{B(X)} \\ &= \varepsilon \qquad \forall s \in (s_0 - \delta, s_0 + \delta). \end{aligned}$$

Since $\int_{\mathbb{R}} \|h(s)\|_\infty\,ds < \|g\|_\infty \int_{\mathbb{R}} \|F(s)\|\,ds < \infty$, applying Bochner' theorem [17, p 133] we get $\int_{\mathbb{R}} h(s)\,ds \in M$. Therefore, $v \in M$, i.e $u * \psi \in M$, whence $\lambda_0 \notin \sigma_M(u)$. ☐

Now, we prove our main result concerning the integro-differential operator equation (2.1) which studies the conditions that guarantee the existence of solutions in $M$. For the case $M = AP(\mathbb{R}, X)$ see [6], [14], [16].

**Theorem 2.2.** *Suppose that $sp(A, \mu)$ is at most countable. If $u$ is a solution of equation* (2.1) *such that the function $e^{-i\lambda t}u(t)$ is ergodic for every $\lambda \in sp(A, \mu)$, then $u \in M$.*

*Proof.* Let $u$ be a solution satisfying the condition of the Theorem. Hence, by Theorem 2.1, we get that $\sigma_M(u)$ is at most countable. Therefore $u \in M$ by Theorem 1.1. ☐

## References


[1] L. Amerio and G. Prouse, *Almost Periodic Functions and Functional Equations*, Van Nostrand, 1971.

[2] B. Basit, *Generalizations of two theorems of Kadets on the indefinite integral of abstract almost periodic functions*, Math. Notes **9** (1970), 180–184.

[3] ———, *A connection between the almost periodic functions of Levitan and almost automorphic functions*, Vestnik Mosk. State Univ. **4** (1971), 11–15.

[4] ———, *Note on a theorem of Levitan for the integral of almost periodic functions*, Rend. Inst. di Matem. Univ. Trieste 5 (1973), 9-14.

[5] ———, *Spectral charecterization of abstract functions*, Analele stiintifice ale Universitatii AI. I. Guza din Iast XXVIII, S.Ia (1982 f.1).

[6] A. G. Baskakov, *Spectral criteria of almost periodicity of solutions of functional equations*, Math. Notes **24**(2) (1978), 195–205.

[7] A. S. Besicovitch, *Almost Periodic Functions*, Cambridge Press, 1932.

[8] S. Bochner, *A new approach to almost periodicity*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 2039–2043.

[9] ———, *General almost automorphy*, Proc. Nat. Acad. Sci. U.S.A **72** (1975), 3815–3818.

[10] C. Datry and G. Muraz, *Analyse harmonique dans les modules de Banach*, Part I: Propriètés Générales, Bull. Sci. Math. Paris **119** (1995), 299–337. Part II: presque-périodicités et ergodicités, Bull. Sci. Math. Paris **120** (1996), 493–536.
[11] A. E. Hamza, *Solutions of Functional Equations*, Ph. D. Thesis, 1993.
[12] A. E. Hamza and G. Muraz, *Spectral criteria of abstract functions; integral and difference problems*, Acta Math. Vietnam. **23** (1998), 171–184.
[13] M. I. Kadets, *The integration of almost periodic functions with values in Banach spaces*, Funct. Anal. i Pril. **3** (1969), 228–300.
[14] B. M. Levitan and V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge University Press, 1982.
[15] L. H. Loomis, *The spectral characterization of a class of almost periodic functions*, Ann. Math. **72** (1960), 362–368.
[16] O. J. Staffans, *An asymptotically almost periodic solutions of a convolution equation*, Trans. Amer. Math. Soc. **266** (1981), 603-616.
[17] K. Yosida, *Functional Analysis*, Spriger-Verlag Berlin 1974.
[18] S. Zaidman, *Almost Periodic Functions in Abstract Spaces*, Pitman Publ. Inc. London, Research Notes in Math., 1985.

Department of Mathematics,
Faculty of Science, Cairo University, Giza, Egypt

*E-mail address*: ahamza@omantel.net.om

Institut Fourier, UMR 5582,
Université de Grenoble I,
38402 Saint-Martin d'Hères, France

*E-mail address*: gilbert.murazujf-grenoble.fr