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# FIXED POINTS OF SEMIGROUPS OF LIPSCHITZIAN MAPPINGS

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ABSTRACT. In this paper we establish three results on fixed points of semigroups of Lipschitzian mappings. The first one generalizes Lifschitz's fixed point theorem for uniformly Lipschitzian mappings in metric spaces. The second one generalizes Kirk's fixed point theorem for mappings of asymptotically nonexpansive type. The last one generalizes Lim-Xu's fixed point theorem for uniformly Lipschitzian mappings satisfying the Casini-Maluta condition.

## 1. Introduction

The notion of uniformly Lipschitzian mappings, i.e. mappings satisfying

$$d(T^n x, T^n y) \le k d(x, y) \tag{1}$$

for every $x, y$ in a metric space $(M, d)$ and for all $n = 1, 2, \dots$, was introduced and investigated in 1973 by Goebel and Kirk [2]. They showed that if $C$ is a closed convex bounded subset of a Banach space $X$ with the characteristic of convexity $\varepsilon_0(X) < 1$ and $T$ is a uniformly Lipschitzian mapping in $C$ with the coefficient $k$ in (1) less than $\gamma_0$ then $T$ has a fixed point in $C$. Here $\gamma_0$ denotes the unique solution of the equation $\gamma\Big(1 - \delta_X\Big(\dfrac{1}{\gamma}\Big)\Big) = 1$ with $\delta_X$ being the modulus of convexity of $X$. Later, in 1975 Lifschitz generalizes this result in a metric space setting for uniformly Lipschitzian mappings with $k < \kappa(M)$ where the Lifschitz constant of a metric space $M$ is defined by

$$\begin{aligned}\kappa(M) = \sup\big\{\beta > 0 \,|\, \exists \alpha > 1 \,:\, &\forall x, y \in M, \quad \forall r > 0, \;\; d(x,y) < r \Rightarrow \\ &\exists z \in M \,:\, B(x, \alpha r) \cap B(y, \beta r) \subset B(z, r)\big\}.\end{aligned} \tag{2}$$

Here $B(x, r)$ denotes the closed ball centered at $x$ with radius $r$ [6]. Our first result extends Lifschitz's result for semigroups of Lipschitzian mappings.

On the other hand, in 1974 Kirk introduced the notion of mappings of asymptotically nonexpansive type, i.e. mappings satisfying for each $x \in C$,

$$\limsup_n \big(\sup_{y \in C}(\|T^n x - T^n y\| - \|x - y\|)\big) \le 0$$

and established a fixed point theorem for mappings of this type [5]. Our second result extends this result for semigroups of mappings of Lipschitzian type.

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In 1985 Casini and Maluta established an analogous result to Goebel-Kirk's fixed theorem for Banach spaces with $N(X) < 1$ and $k < N(X)^{-1/2}$, where $N(X)$ denotes the constant of uniformly normal structure [1]. In 1989 Ishihara generalizes this result to semigroups of Lipschitzian mappings [4]. Later, in 1995 Lim and Xu proved an analogous result to Casini-Maluta's theorem in a metric space setting [7]. Our third result extends Lim-Xu's theorem to semigroups of Lipschitzian mappings.

Throughout this paper $S$ denotes a left reversible semigroup, i.e. every pair of right ideals in $S$ have nonempty intersection. We introduce an order in $S$ by setting

$$s \geq t \quad \Leftrightarrow \quad \{s\} \cup sS \subset \{t\} \cup tS.$$

Being left reversible, $S$ becomes a directed set, i.e. for every $s, t \in S$ there is $r \in S$ such that $r \geq \{s, t\}$.

Let $C$ be a set and $\mathcal{T} = \{T_s : s \in S\}$ a family of mappings in $C$. If for every $s, t \in S$ we have $T_s T_t = T_{st}$ then $\mathcal{T}$ is called a semigroup of mappings in $C$. In what follows we establish some results on common fixed points of mappings in such semigroups.

## 2. A generalization of Lifschitz's theorem

Let $S$ and $\mathcal{T}$ be as above, for each $s \in S$ and $x \in C$ we denote $\mathcal{T}_s = \{T_s T : T \in \mathcal{T}\}$ and $\mathcal{T}_s(x) = \{T_s T(x) : T \in \mathcal{T}\}$. Now we are able to state our first result.

**Theorem 1.** *Let $M$ be a complete metric space, $S$ a left reversible semigroup, $\mathcal{T} = \{T_s : s \in S\}$ a semigroup of $k_s$-Lipschitzian mappings in $M$ with $\limsup_s k_s = k < \kappa(M)$. Suppose there exist $s_0 \in S$ and $x_0 \in M$ such that $\mathcal{T}_{s_0}(x_0)$ is bounded. Then there exists a common fixed point for all $T_s$ in $\mathcal{T}$.*

*Proof.* Taking any $k' \in (k, \kappa(M))$ there is $s_1 \in S$ such that $k_i \leq k'$ for all $i \geq s_1$. Choose $s_2 \in S$ such that $s_2 \geq \{s_0, s_1\}$. For any $y \in M$ we define

$$r(y) = \inf \big\{ \rho > 0 : \exists x \in M, \ \exists i \geq s_2 \ \text{ such that } \ \mathcal{T}_i(x) \subset B(y, \rho) \big\}.$$

Since $\mathcal{T}_{s_0}(x_0) \subset B(y_0, R)$ for some $y_0 \in M$ and $R < \infty$, we have

$$\mathcal{T}_{s_2}(x_0) \subset B(y, R + d(y, y_0)),$$

so $r(y)$ is well defined for each $y \in M$.

We show that if $r(y) = 0$ then $T_s y = y$, $\forall s \in S$. Indeed, we have $r(y) < \varepsilon$ for every $\varepsilon > 0$. Then by definition of $r(y)$, there are $x \in M$ and $i \geq s_2$ such that

$$d(Tx, y) < \varepsilon, \quad \forall T \in \mathcal{T}_i$$

so for every $T \in \mathcal{T}_i$ we have

$$d(Ty, y) \leq d(Ty, T^2x) + d(T^2x, y) \leq k' d(y, Tx) + d(T^2x, y) \leq \varepsilon(k' + 1).$$

Fix any $s \in S$ and choose $j \in S$ such that $j > \{si, i\}$. Then there are $u, v \in S$ such that $j = siu = iv$. Now we have

$$\begin{aligned} d(T_s y, y) &\leq d(T_s y, T_j y) + d(T_j y, y) = d(T_s y, T_s T_i T_u y) + d(T_i T_v y, y) \\ &\leq k_s d(y, T_i T_u y) + d(T_i T_v y, y) \leq (1 + k_s)(1 + k')\varepsilon. \end{aligned}$$

From this we get $T_s y = y$, $\forall s \in S$.

Now we are going to construct a sequence $\{y_n\}$ in $M$ by induction with an arbitrary $y_1$. Assume we have got $y_1, y_2, ..., y_n$ with $r(y_n) > 0$. Since $k' < \kappa(M)$, there exists $\beta \in (k', \kappa(M))$. From the definition of $\kappa(M)$ (see Introduction) there is $\alpha > 1$ such that (2) holds. Choose $\lambda < 1$ so that

$$\gamma = \min\{\alpha\lambda, \beta\lambda/k'\} > 1.$$

Since $\lambda < 1$, by definition of $r(y_n)$ there is $s \geq s_2$ such that

$$d(y_n, T_s y_n) > \lambda r(y_n). \tag{3}$$

On the other hand, since $\gamma > 1$ there are $x_1 \in M$ and $t \geq s_2$ such that

$$d(y_n, T x_1) \leq \gamma r(y_n), \quad \forall T \in \mathcal{T}_t \tag{4}$$

Choose $u \geq \{st, t\}$ then every $T \in \mathcal{T}_u$ has the form $T = T_s T_t T_i$ for some $i \in S$. Thus for every $T$ in $\mathcal{T}_u$ we have from (4)

$$d(T_s y_n, T x_1) = d(T_s y_n, T_s T_t T_i x_1) \leq k' \gamma r(y_n) \tag{5}$$

because $s \geq s_1$ and $T_t T_i \in \mathcal{T}_t$.

Since $u \geq t \geq s_2$, from (3), (4), (5) we get

$$\mathcal{T}_u(x_1) \subset B(y_n, \alpha\lambda r(y_n)) \cap B(T_s y_n, \beta\lambda r(y_n)) \subset B(z, \lambda r(y_n)) \tag{6}$$

for some $z \in M$. Putting $y_{n+1} = z$ from (6) we obtain

$$r(y_{n+1}) \leq \lambda r(y_n). \tag{7}$$

From (4) and (6) we get

$$d(y_n, y_{n+1}) \leq (\alpha + 1)\lambda r(y_n)$$

which together with (7) shows that $\{y_n\}$ is a Cauchy sequence, hence converges to some $w$ in $M$, and $r(y_n) \to 0$ as $n \to \infty$. We show that $r(w) = 0$. Take any $\varepsilon > 0$ and choose $m$ such that $d(y_m, w) < \varepsilon/2$ and $r(y_m) < \varepsilon/2$. Then there are $x \in M$ and $s \geq s_2$ such that $d(Tx, y_m) < \varepsilon/2$ for all $T \in \mathcal{T}_s$. From this we get $d(Tx, w) < \varepsilon$ for all $T \in \mathcal{T}_s$, hence $r(w) \leq \varepsilon$, i.e. $r(w) = 0$. Thus $T_s w = w$ for all $s \in S$ and the theorem is proved. $\square$

For a Banach space $X$ the Lifschitz characteristic of $X$ is defined as follows

$$\kappa_0(X) = \inf\{\kappa(C) : C \ \text{ is a bounded closed convex subset of } \ X\}.$$

From the above theorem we immediately get the following

**Corollary 1.** *Let $C$ be a bounded closed convex subset of a Banach space $X, S$ a left reversible semigroup, $\{T_s : s \in S\}$ a semigroup of $k_s$-Lipschitzian mappings in $C$ with $\limsup_s k_s < \kappa_0(X)$. Then $\{T_s\}$ has a common fixed point.*

## 3. A generalization of Kirk's theorem

Let us begin this section with some lemmas.

**Lemma 1.** *Let $S$ be a left revesible semigroup, $\{a_s : s \in S\}$ a bounded decreasing net in $\mathbb{R}$ (the real line), $\{a_t : t \in S'\}$ a subnet of $\{a_s\}$. Then we have*

$$\inf\{a_t : t \in S'\} = \inf\{a_s : s \in S\}.$$

*In particular, for each $t \in S$ we have*

$$\inf\{a_{ts} : s \in S\} = \inf\{a_s : s \in S\}.$$

*Proof.* Putting $m_1 = \inf\{a_t : t \in S'\}$, $m_2 = \inf\{a_s : s \in S\}$ we have $m_2 \leq m_1$. For every $\varepsilon > 0$ there is $i \in S$ such that $a_i \leq m_2 + \varepsilon$. Then for every $t \geq i$ we have $a_t \leq a_i \leq m_2 + \varepsilon$. Choose $t \in S'$ so that $t \geq i$, then $m_1 \leq a_t \leq m_2 + \varepsilon$. From this $m_1 \leq m_2$, thus $m_1 = m_2$. The last assertion follows from the fact that $\{a_{ts} : s \in S\}$ is a subnet of $\{a_s : s \in S\}$. □

**Corollary 2.** *Let $S$ be as in Lemma 1, $\{T_s : s \in S\}$ a semigroup of mappings in a Banach space $X$. Then for $x, y \in X$ and $t \in S$ we have*

$$\limsup_s \|T_s x - y\| = \limsup_s \|T_{ts} x - y\|.$$

*Proof.* Putting $a_s = \sup\{\|T_i x - y\| : i \geq s\}$ and using Lemma 1 we get

$$\begin{aligned}\limsup_s \|T_s x - y\| &= \inf_s(\sup\{\|T_i x - y\| : i \geq s\}) = \inf\{a_s : s \in S\} \\ &= \inf\{a_{ts} : s \in S\} = \limsup_s \|T_{ts} x - y\|.\end{aligned}$$

□

**Lemma 2.** *For two bounded positive nets $\{a_s : s \in S\}$, $\{b_s : s \in S\}$ we have*

$$\liminf_s (a_s b_s) \leq \limsup_s a_s \liminf_s b_s.$$

*Proof.* Put $m = \liminf_s (a_s b_s)$, $m_1 = \limsup_s a_s$, $m_2 = \liminf_s b_s$.

Since $m = \sup_s(\inf\{a_t b_t : t \geq s\})$, for every $\varepsilon > 0$ there is $s_1$ such that

$$\inf\{a_t b_t : t \geq s_1\} > m - \varepsilon. \tag{1}$$

Since $m_1 = \inf_s(\sup\{a_t : t \geq s\})$, for the above $\varepsilon$ there is $s_2$ such that

$$\sup\{a_t : t \geq s_2\} < m_1 + \varepsilon. \tag{2}$$

Choosing $s_3 \geq \{s_1, s_2\}$, from (1), (2) for every $s \geq s_3$ we get

$$a_s b_s > m - \varepsilon, \quad a_s < m_1 + \varepsilon. \tag{3}$$

Since $\inf\{b_t : t \geq s_3\} \leq m_2$, there is $i \geq s_3$ such that $b_i < m_2 + \varepsilon$.

From this and (3) we get

$$m - \varepsilon < a_i b_i < (m_1 + \varepsilon)(m_2 + \varepsilon).$$

Letting $\varepsilon \to 0$ we get $m \leq m_1 m_2$. □

**Lemma 3.** *Let $\{a_s : s \in S\}$ be a bounded net in $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ a decreasing continuous function. Then we have*

$$\liminf_s f(a_s) = f(\limsup_s a_s).$$

*Proof.* We have

$$\liminf_s f(a_s) = \sup_s(\inf\{f(a_t) : t \geq s\}). \tag{4}$$

We shall prove that

$$\inf\{f(a_t) : t \geq s\} = f(\sup\{a_t : t \geq s\}). \tag{5}$$

Put $M = \sup\{a_t : t \geq s\}$, $m = \inf\{f(a_t) : t \geq s\}$. Since $a_t \leq M$ for each $t$, we have $f(a_t) \geq f(M)$ for each $t$, hence $m \geq f(M)$. On the other hand, for every $\varepsilon > 0$ there is $i \geq s$ such that $a_i \geq M - \varepsilon$, hence $m \leq f(a_i) \leq f(M - \varepsilon)$. Letting $\varepsilon \to 0$ we get $m \leq f(M)$. Thus (5) is proved.

Similarly, we get $\sup_s f(b_s) = f(\inf_s b_s)$ for any net $\{b_s : s \in S\}$. From this and (4), (5) we have (putting $b_s = \sup\{a_t : t \geq s\}$)

$$\begin{aligned}\liminf_s f(a_s) &= \sup_s f(sup\{a_t : t \geq s\})\\ &= f(\inf_s \sup\{a_t : t \geq s\})\\ &= f(\limsup_s a_s).\end{aligned}$$

The proof is complete. □

**Definition.** Let $C$ be a subset of a Banach space, $S$ a left reversible semigroup. A semigroup of mappings in $C$ $\{T_s : s \in S\}$ is called of Lipschitzian type if there exists a positive net $\{k_s\}$ such that for each $x \in C$ we have $\limsup_s c_s(x) = 0$, where

$$c_s(x) = \max\big\{\sup_{y \in C}(\|T_s x - T_s y\| - k_s\|x - y\|), 0\big\}.$$

Now we are able to state our second result generalizing Theorem 2 in [8].

**Theorem 2.** *Let $X$ be a Banach space with $\varepsilon_0(X) < 1$, $C$ a bounded closed convex subset of $X$, $S$ a left reversible semigroup and $\{T_s : s \in S\}$ a semigroup of mappings in $C$ of Lipschitzian type. If each $T_s$ is continuous and $\limsup_s k_s < \gamma_0$, then the mappings $\{T_s\}$ have a common fixed point, where $\gamma_0$ is defined as in the introduction.*

*Proof.* Since $\varepsilon_0(X) < 1$ we have $\gamma_0 > 1$ and we may assume that $k_s \geq \gamma_1$ $\forall s \in S$, for some $\gamma_1 \in (1, \gamma_0)$. Denote $k = \limsup_s k_s$.

Take $x_0 \in X$ and put $x_s = T_s x_0$, $\forall s \in S$. For each $x \in C$, denote

$$
\begin{aligned}
r(\{x_s\}, x) &= \limsup_s \|x_s - x\|,\\
r(\{x_s\}, C) &= \inf_{x\in C} r(\{x_s\}, x),\\
\mathcal{A}(\{x_s\}, C) &= \{z \in C \,:\, r(\{x_s\}, z) = r(\{x_s\}, C)\}.
\end{aligned}
$$

It is well known that $r(\{x_s\}, \cdot)$ is weakly lower semicontinuous and $\mathcal{A}(\{x_s\}, C)$ is nonempty.

Take $z_1 \in \mathcal{A}(\{x_s\}, C)$ and put $r_1 = r(\{x_s\}, z_1)$. Using the corollary of Lemma 1 we have (for fixed $s \in S$)

$$
\begin{aligned}
&\limsup_t \|T_t x_0 - T_s z_1\| = \limsup_t \|T_s T_t x_0 - T_s z_1\|\\
&\le \limsup_t (k_s \|T_t x_0 - z_1\| + c_s(z_1)) = k_s r_1 + c_s(z_1). \qquad (6)
\end{aligned}
$$

On the other hand, by definition it follows that

$$
\limsup_t \|T_t x_0 - z_1\| = r_1 \le k_s r_1 + c_s(z_1). \tag{7}
$$

By convexity of $C$, for each $s \in S$ we have

$$
r_1 \le \limsup_t \Big\| T_t x_0 - \frac{z_1 + T_s z_1}{2} \Big\|,
$$

hence

$$
r_1 \le \liminf_s \Big( \limsup_t \Big\| T_t x_0 - \frac{z_1 + T_s z_1}{2} \Big\| \Big). \tag{8}
$$

From (6), (7) for each $\varepsilon > 0$ there is $t_0 \in S$ such that

$$
\begin{aligned}
&\sup \big\{ \|T_t x_0 - z_1\| \,:\, t \ge t_0 \big\} \le k_s r_1 + c_s(z_1) + \varepsilon,\\
&\sup \big\{ \|T_t x_0 - T_s z_1\| \,:\, t \ge t_0 \big\} \le k_s r_1 + c_s(z_1) + \varepsilon.
\end{aligned}
$$

From a property of modulus of convexity we get for $t \ge t_0$

$$
\begin{aligned}
&\Big\| \frac{1}{2}(T_t x_0 - z_1) + \frac{1}{2}(T_t x_0 - T_s z_1) \Big\|\\
&\le (k_s r_1 + c_s(z_1) + \varepsilon)\Big(1 - \delta_X \Big( \frac{\|z_1 - T_s z_1\|}{k_s r_1 + c_s(z_1) + \varepsilon} \Big)\Big). \qquad (9)
\end{aligned}
$$

We now show that

$$
\frac{\|z_1 - T_s z_1\|}{k_s r_1 + c_s(z_1)} < 2. \tag{10}
$$

Indeed, we have

$$
\begin{aligned}
\|z_1 - T_s z_1\| &\le \limsup_t \|T_t x_0 - z_1\| + \limsup_t \|T_t x_0 - T_s z_1\|\\
&\le r_1 + k_s r_1 + c_s(z_1), \qquad (11)
\end{aligned}
$$

hence

$$\frac{\|z_1 - T_s z_1\|}{k_s r_1 + c_s(z_1)} \le 1 + \frac{r_1}{k_s r_1 + c_s(z_1)} \le 1 + \frac{1}{k_s} \le 1 + \frac{1}{\gamma_1} < 2.$$

Thus (10) is proved. From this and the continuity of $\delta_X$ on $[0, 2)$, letting $\varepsilon \to 0$ in (9) we get

$$\limsup_t \left\| T_t x_0 - \frac{z_1 + T_s z_1}{2} \right\| \le (k_s r_1 + c_s(z_1)) \Big( 1 - \delta_X \Big( \frac{\|z_1 - T_s z_1\|}{k_s r_1 + c_s(z_1)} \Big) \Big). \tag{12}$$

From (8), (12) and using Lemma 2 we obtain

$$\begin{aligned} r_1 &\le \liminf_s \Big[ (k_s r_1 + c_s(z_1) \Big( 1 - \delta_X \Big( \frac{\|z_1 - T_s z_1\|}{k_s r_1 + c_s(z_1)} \Big) \Big) \Big] \\ &\le \limsup_s (k_s r_1 + c_s(z_1)). \liminf_s \Big( 1 - \delta_X \Big( \frac{\|z_1 - T_s z_1\|}{k_s r_1 + c_s(z_1)} \Big) \Big). \end{aligned}$$

By Lemma 3, this implies

$$r_1 \le k r_1 \Big( 1 - \delta_X \Big( \limsup_s \frac{\|z_1 - T_s z_1\|}{k_s r_1 + c_s(z_1)} \Big) \Big). \tag{13}$$

If $r_1 = 0$, i.e. $\limsup_t \|T_t x_0 - z_1\| = 0$, from the continuity of $T_s$ for each $s \in S$ we get (using Corollary 2)

$$\limsup_t \|T_t x_0 - T_s z_1\| = \limsup_t \|T_s T_t x_0 - T_s z_1\| = 0.$$

Hence from (11) we obtain $\|z_1 - T_s z_1\| = 0$, i.e. $z_1 = T_s z_1 \ \forall s \in S$ and the theorem is proved.

If $r_1 > 0$, then from (13) we have

$$\delta_X \Big( \limsup_s \frac{\|z_1 - T_s z_1\|}{k_s r_1 + c_s(z_1)} \Big) \le 1 - \frac{1}{k}. \tag{14}$$

We consider two possible cases.

If $\delta_X \Big( \limsup_s \frac{\|z_1 - T_s z_1\|}{k_s r_1 + c_s(z_1)} \Big) = 0$, then

$$\limsup_s \frac{\|z_1 - T_s z_1\|}{k r_1} \le \limsup_s \frac{\|z_1 - T_s z_1\|}{k_s r_1 + c_s(z_1)} \le \varepsilon_0(X),$$

hence

$$\limsup_s \|z_1 - T_s z_1\| \le k \varepsilon_0(X) r_1. \tag{15}$$

Since $k < \gamma_0(X)$ it is easy to show that $k\varepsilon_0(X) < 1$.

If $\delta_X \Big( \limsup_s \frac{\|z_1 - T_s z_1\|}{k_s r_1 + c_s(z_1)} \Big) > 0$ then $\limsup_s \frac{\|z_1 - T_s z_1\|}{k_s r_1 + c_s(z_1)} > \varepsilon_0(X)$. Since $\delta_X$ is strictly increasing on $[\varepsilon_0(X), 2)$, from (14) we get

$$\limsup_s \frac{\|z_1 - T_s z_1\|}{k_s r_1 + c_s(z_1)} \le \delta_X^{-1} \Big( 1 - \frac{1}{k} \Big) < \delta_X^{-1} \Big( 1 - \frac{1}{\gamma_0} \Big) = \frac{1}{\gamma_0} \cdot$$

So $\limsup_s \dfrac{\|z_1 - T_s z_1\|}{kr_1} < \dfrac{1}{\gamma_0}$, hence

$$\limsup_s \|z_1 - T_s z_1\| < \frac{kr_1}{\gamma_0}\cdot \tag{16}$$

Putting $\eta = \max\left\{k\varepsilon_0(X), \dfrac{k}{\gamma_0}\right\} < 1$, from (15), (16) we get

$$\limsup_s \|z_1 - T_s z_1\| \le \eta r_1. \tag{17}$$

Take $z_2 \in \mathcal{A}(\{T_s z_1\}, C)$ and put $r_2 = r(\{T_s z_1\}, z_2)$. Then $r_2 \le \eta r_1$.

Continuing this process we obtain a sequence $\{z_n\} \subset C$ satisfying

(i) $z_{n+1} \in \mathcal{A}(\{T_s z_n\}, C)$,

(ii) $r_n = r(\{T_s z_{n-1}\}, z_n) = r(\{T_s z_{n-1}\}, C)$, with $n \ge 1$, $z_0 = x_0$,

(iii) $\limsup_s \|T_s z_n - z_n\| \le \eta r_n$.

We have

$$\begin{aligned}\|z_{n+1} - z_n\| &\le \limsup_t \|z_{n+1} - T_t z_n\| + \limsup_t \|T_t z_n - z_n\| \\ &\le r_{n+1} + \eta r_n \le 2\eta r_n \le \ldots \le 2\eta^n r_1,\end{aligned}$$

hence $\{z_n\}$ is a Cauchy sequence which converges to some $z \in C$.

For each $s \in S$ we have

$$\|T_s z - z\| \le \|z - z_n\| + \|z_n - T_s T_t z_n\| + \|T_s T_t z_n - T_s z\|,$$

hence

$$\|T_s z - z\| \le \|z - z_n\| + \limsup_t \|z_n - T_s T_t z_n\| + \limsup_t \|T_s T_t z_n - T_s z\| \tag{18}$$

Note that

$$\begin{aligned}\limsup_t \|T_t z_n - z\| &\le \limsup_t \big(\|T_t z_n - z_n\| + \|z_n - z\|\big) \\ &\le \eta r_n + \|z_n - z\| \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

From the continuity of $T_s$ we also have

$$\limsup_t \|T_s T_t z_n - T_s z\| \to 0 \quad \text{as} \quad n \to \infty.$$

Taking into account the inequality

$$\limsup_t \|z_n - T_s T_t z_n\| = \limsup_t \|z_n - T_t z_n\| \le \eta r_n,$$

from (18) we finally get $T_s z = z$ for each $s \in S$, and the proof is complete. □

## 4. A generalization of Lim-Xu's theorem

First of all we recall some notions presented in [7].

Let $(X,d)$ be a metric space and $M$ a bounded subset of $X$. Denote

$$\begin{aligned} r(x,M) &= \sup\{d(x,y) : y \in M\} \quad \text{for} \quad x \in X, \\ \delta(M) &= \sup\{d(x,y) : x,y \in M\}, \\ R(M) &= \inf\{r(x,M) : x \in M\}. \end{aligned}$$

A subset of $X$ is said to be admissible if it is an intersection of closed balls. $\mathcal{A}(X)$ denotes the family of all admissible subsets of $X$. For a bounded subset $A$ of $X$, the admissible hull of $A$, denoted by $\operatorname{ad}(A)$, is the intersection of all those admissible subsets of $X$ which contain $A$. We have always

$$r(x, \operatorname{ad}(A)) = r(x,A), \quad \delta(\operatorname{ad}(A)) = \delta(A).$$

The constant of uniformity of normal structure $N(X)$ of $X$ is defined by

$$N(X) = \sup\left\{\frac{R(A)}{\delta(A)} : A \text{ admissible}, \ \delta(A) > 0\right\},$$

and $X$ is said to have uniform normal structure if $N(X) < 1$.

Let $S$ be a semigroup. We say that $S$ is a totally ordered semigroup if it is totally ordered with respect to the order defined by

$$s \geq t \quad \Leftrightarrow \quad \{s\} \cup sS \subset \{t\} \cup tS.$$

A metric space $(X,d)$ is said to have property $(P)$ if for every totally ordered semigroup $S$ and two bounded nets $\{x_s : s \in S\}$, $\{z_s : s \in S\}$ in $X$ there exists $z \in \bigcap_s \operatorname{ad}(z_t : t \geq s)$ such that

$$\limsup_s d(x_s, z) \leq \limsup_t (\limsup_s d(x_s, z_t)).$$

Before stating our last result we need two lemmas.

**Lemma 4.** *Let $(X,d)$ be a complete metric space with $N(X) < 1$, $S$ a totally ordered semigroup, $\{K_s : s \in S\}$ a decreasing net of nonempty admissible closed bounded subset of $X$. Then $\bigcap_s K_s \neq \emptyset$.*

*Proof.* Choose $k \in (N(X), 1)$ and for each closed bounded admissible subset $C$ of $X$ with $\delta(C) > 0$ we denote

$$A(C) = \{x \in C : r(x,C) \leq k\delta(C)\}.$$

Then $A(C)$ is closed bounded and nonempty.

Let $\{K_s : s \in S\}$ be as in the Lemma. We must show that $\bigcap_s K_s \neq \emptyset$.

Put $K_s^1 = \operatorname{ad}(\bigcup_{t \geq s} A(K_t^0))$, where $K_t^0 = K_t$ for each $t \in S$. Then $K_s^1$ is a nonempty closed admissible subset of $K_s^0$ for each $s$ and $K_s^1 \subset K_t^1$ whenever $s \geq t$. We shall prove that $\delta(K_s^1) \leq k\delta(K_s^0)$.

Indeed, for $x, y \in \bigcup_{t \geq s} A(K_t^0)$ we have $x \in A(K_p^0)$, $y \in A(K_q^0)$ with $q \geq p \geq s$. Since $y \in A(K_q^0) \subset K_q^0 \subset K_p^0$, we get

$$d(x,y) \leq r(x, K_p^0) \leq k\delta(K_p^0) \leq k\delta(K_s^0),$$

hence $\delta(K_s^1) \leq k\delta(K_s^0)$. Continuing this process for each $s \in S$ we get a sequence of subsets $\{K_s^i : i = 1, 2, ...\}$ such that $K_s^i \subset K_t^i$ whenever $s \geq t$ and $\delta(K_s^i) \leq k\delta(K_s^{i-1})$ for each $i$. For each $s \in S$ we have $\delta(K_s^i) \leq k^i \delta(K_s^0) \to 0$ as $i \to \infty$. By Cantor's principle there is a unique $x_s \in X$ such that $\bigcap_i K_s^i = \{x_s\}$.

For every $s, t \in S$ we may assume $t > s$. Then

$$\{x_s\} = \bigcap_i K_s^i \supset \bigcap_i K_t^i = \{x_t\}.$$

From this $\bigcap_{i,s} K_s^i \neq \emptyset$, hence $\bigcap_s K_s \neq \emptyset$ and the proof is complete. □

The above proof is essentially due to Maluta (1989).

**Lemma 5.** *Let $(X, d)$ be a complete metric space with $N(X) < 1$ and having property $(P)$, $S$ a totally ordered semigroup and $c > N(X)$. Then for each bounded net $\{x_s : s \in S\}$ in $X$ there is $z \in \bigcap_s \operatorname{ad}(x_t : t \geq s)$ such that*

(i) *$d(z, y) \leq r(\{x_s\}, y)$ for every $y \in X$, where $r(\{x_s\}, y) = \limsup_s d(x_s, y)$,*

(ii) *$r(\{x_s\}, z) \leq c \limsup_s \delta(x_t : t \geq s)$.*

*Proof.* (i) For each $s \in S$ we put $A_s = \operatorname{ad}(x_t : t \geq s)$. By Lemma 4, $A = \bigcap_s A_s \neq \emptyset$. For $z \in A$, $y \in X$ we set

$$M = \limsup_s d(x_s, y) = r(\{x_s\}, y).$$

Then for each $\varepsilon > 0$ there is $t \in S$ such that $\sup_{s \geq t} d(x_s, y) < M + \varepsilon$. Hence

$$d(z, y) \leq r(y, A) \leq r(y, A_t) \leq M + \varepsilon,$$

and (i) follows.

(ii) Fix $s \in S$. Since $R(A_s) \leq N(X)\delta(A_s) < c\delta(A_s)$, there is $z_s \in A_s$ such that $r(z_s, A_s) < c\delta(A_s)$. The net $\{z_s\}$ is bounded, hence by property $(P)$ there is $z \in \bigcap_s \operatorname{ad}(z_t : t \geq s)$ such that

$$\begin{aligned} r(\{x_s\}, z) &\leq \limsup_t r(\{x_s\}, z_t) \leq \limsup_t r(z_t, A_t) \\ &\leq \limsup_t c\delta(A_t) = c \limsup_t \delta(A_t). \end{aligned}$$

This implies (ii) and the lemma is proved. □

Now we are able to state our third result.

**Theorem 3.** *Let $(X,d)$ be a bounded complete metric space with $N(X)<1$ and having property $(P)$, $S$ a totally ordered semigroup, $\{T_s : s\in S\}$ a semigroup of $k$-Lipschitzian mappings with $k<N(X)^{-1/2}$. Then $\{T_s\}$ have a common fixed point.*

*Proof.* Take $c\in(N(X),1)$ such that $\eta=k^2c<1$. For $x\in X$, put $x_s=T_sx$. By Lemma 5 there is $z=z(x)$ such that

(i) $d(z,y)\le r(\{T_sx\},y)$ for all $y\in X$,

(ii) $r(\{T_sx\},z)\le c\limsup_s \delta(T_tx : t\ge s)$.

Put $r(x)=\sup\{d(T_sx,x) : s\in S\}$. We have

$$\limsup_s \delta(T_tx : t\ge s)\le \sup\{d(T_sx,T_tx) : s,t\in S\} \le k\sup_{i\in S} d(T_ix,x)=kr(x).$$

From this and (ii) we get

$$r(\{T_sx\},z)\le ckr(x). \tag{1}$$

From (i) we have

$$r(z)=\sup_s d(T_sz,z)\le \sup_s r(\{T_tx\},T_sz).$$

On the other hand

$$\begin{aligned} r(\{T_tx\},T_sz) &= \limsup_t d(T_tx,T_sz)=\limsup_t d(T_sT_tx,T_sz)\\ &\le k\limsup_t d(T_tx,z)=kr(\{T_tx\},z).\end{aligned}$$

From this and (1) we get

$$r(z)\le ck^2r(x)=\eta r(x).$$

We construct a sequence $\{x_n\}$ by putting $x_1=x$, $x_{n+1}=z(x_n)$, $n\ge 1$. Then

$$r(x_{n+1})\le \eta r(x_n)\le ...\le \eta^n r(x_1)\to 0 \quad \text{as} \quad n\to\infty.$$

On the other hand

$$\begin{aligned} d(x_{n+1},x_n) &\le d(x_{n+1},T_sx_n)+d(T_sx_n,x_n)\\ &\le d(x_{n+1},T_sx_n)+r(x_n), \quad \forall s\in S.\end{aligned}$$

From this we get

$$\begin{aligned} d(x_{n+1},x_n) &\le \limsup_s d(x_{n+1},T_sx_n)+r(x_n)=r(\{T_sx_n\},x_{n+1})+r(x_n)\\ &\le ckr(x_n)+r(x_n)\le (1+ck)\eta^{n-1}r(x_1).\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence. Let $z=\lim_n x_n$. For each $s\in S$ we have

$$\begin{aligned} d(T_sz,z) &\le d(z,x_n)+d(x_n,T_sx_n)+d(T_sx_n,T_sz)\\ &\le d(z,x_n)+r(x_n)+kd(x_n,z)\to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

So $T_s z = z$, $\forall s \in S$ and the proof is complete. $\square$

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## References

[1] E. Casini and E. Maluta, *Fixed points of uniformly Lipschitzian mappings in spaces with uniformly normal structure*, Nonlinear Analysis **9** (1985), 103-108.
[2] K. Goebel and W. A. Kirk, *A fixed point theorem for transformations whose iterates have uniform Lipschitz constant*, Studia Math. **47** (1973), 135-140.
[3] K. Goebel, W. A. Kirk and R. L. Thele, *Uniformly Lipschitzian families of transformations in Banach spaces*, Canad. J. Math. **26** (1974), 1245-1256.
[4] H. Ishihara, *Fixed point theorems for Lipschitz semigroups*, Canad. J. Math. Bull. **32** (1989), 90-97.
[5] W. A. Kirk, *Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type*, Israel J. Math. **17** (1974), 334-346.
[6] E. A. Lifschitz, *Fixed point theorems for operators in strongly convex spaces*, Voronez. Gos. Univ. Trudy Mat. Fak. **16** (1975), 23-28
[7] T. C. Lim and H. K. Xu, *Uniformly Lipschitzian mappings in metric spaces with uniform normal structure*, Nonlinear Analysis **25** (1995), 1231-1235.
[8] D. H. Tan and H. D. Vuong, *On eventually and asymptotically Lipschitzian mappings*, Vietnam J. Math. **30** (2002), 31-42.

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