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ANOTHER CLASSIFICATION OF QUASI-MARTINGALES IN THE LIMIT

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ABSTRACT. Given a stochastic basic $(\mathcal{A}_n)$, a sequence (X_n) of integrable random variables, adapted to $(\mathcal{A}_n)$ is said to be a quasi-martingale in the limit if for every $\varepsilon > 0$, there exists $p \in N$ such that for every $m \geq p$ there exists $p_m \geq m$ such that for all $n \geq p_m$ we have

$$P\Big(\sup_{p\leq q\leq m} |X_q(n) - X_q| > \varepsilon\Big) < \varepsilon.$$

The main aim of this note is to prove that the class of all quasi-martingales in the limit would be classified into a nondecreasing directed family of subclasses whose smallest element is just the class of mils introduced by M. Talagrand (1985).

1. Notations and Definitions

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space, $(\mathcal{A}_n)$ an increasing sequence of complete sub-σ-fields of $\mathcal{A}$ with $\mathcal{A}_n \uparrow \mathcal{A}$. In this note, we shall consider only sequences (X_n) of random variables with each $X_n \in L^1(\mathcal{A}_n)$, i.e. X_n is $\mathcal{A}_n$-measurable and

$$E(|X|) = \int_\Omega |X_n| dP < \infty.$$

For other related notions of martingale-like sequences, the reader is referred to [2]. In this note, we recall only the following definition.

Definition 1.1. A sequence (X_n) is said to be

a) *a mil* if for every $\varepsilon > 0$, there exists $p \in N$ such that for every $n \geq p$ we have

$$P\Big(\sup_{p\leq q\leq n} |X_q(n) - X_q| > \varepsilon\Big) < \varepsilon,$$

where given $m, n \in N$ with $m \leq n$, $X_m(n)$ denotes the $\mathcal{A}_m$-conditional expectation of X_n (cf. [5]).

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b) *a game which becomes fairer with time* if for every $\varepsilon > 0$, there exists $p \in N$ such that for every $n \geq p$ we have

$$\sup_{p \leq q \leq n} P(|X_q(n) - X_q| > \varepsilon) < \varepsilon.$$

It is clearly that by definition, every mil is a game fairer with time. However, by Theorem 4 [6] the classical Doob's martingale limit theorem still holds for mils. Especially, D. Q. Luu [3] has recently noted that the above results of M. Talagrand would has been extended to the following important generalization of mils.

Definition 1.2. A sequence (X_n) is said to be *a quasi-martingale in the limit* (briefly, *a quasi-mil)* if for every $\varepsilon > 0$, there exists $p \in N$ such that for every $m \geq p$ there exists $p_m \geq m$ such that for all $n \geq p_m$ we have

$$P\Big(\sup_{p \leq q \leq m} |X_q(n) - X_q| > \varepsilon\Big) < \varepsilon. \tag{1.1}$$

As a continuation of [6], [3] and [4], the main aim of this note is to establish another classification of the class of all quasi-mils which is independent of that given in [4].

2. Main results

The first result we begin with is the following example which shows that unlike mils, the class of quasi-mils is independent of games fairer with time.

Example 2.1. Neither the class of games fairer with time nor that of quasi-mils is contained in each other. Let $([0,1], \mathcal{B}_{[0,1)}, P)$ be the Lebesgue probability space on $[0,1)$, where $\mathcal{B}_{[0,1)}$ is the completion of the Borel σ-field w.r.t the Lebesgue measure P. For $m = 0$, set $b_m = 0$, $I_1^0 = [0,1)$ and $\mathcal{A}_m = \{\phi, I_1^0\}$. For $m \geq 1$, set $b_m = \sum_{j=0}^{m-1} 2^j$, Q_m the partition of $[0,1)$ in 2^m intervals $\{I_j^m,\ 1 \leq j \leq 2^m\}$ of equal length and $\mathcal{A}_m$ the σ-algebra generated by Q_m. On the probability space with stochastic basic $(\mathcal{A}_m)$, we shall construct first a game fairer with time (X_n) which is not a quasi-mil. Indeed, for $n = 1$, set $X_n = 0$. For $n \geq 2$, set $X_n = 1$ on the first and $X_n = -1$ on the second interval of Q_m which are contained in $I_j^{(m-1)}$, where (m, j) is the unique pair of $m \geq 1$ and $1 \leq j \leq 2^{m-1}$ with $n = b_{m-1} + j$ and $X_n = 0$ elsewhere. Then it is easily seen that constructed in such a way, the sequence (X_n) has the following properties:

(a) (X_n) converges to zero in L^1,

(b) For all $m, n \in N$, $X_q(n) = 0$ if $q \leq b_{m-1}$ and $X_q(n) = X_n$ if $b_{m-1} < q \leq n$,

(c) (X_n) does not converge to zero, a.s.

By the properties (a), (b) and Chebyshev's inequality, it is easily checked that (X_n) must be a game fairer with time. However, for all $m, n \in N$ with $n > b_m$

we have

$$\sup_{b_{m-1}<q\leq b_m} |X_q(n) - X_q| = \sup_{b_{m-1}<q\leq b_m} |X_q| = 1.$$

Then (X_n) cannot be a quasi-mil.

To construct the converse example, set $n_k = 2^k n_{k-1}$ with $k \in N$ and $n_0 = 1$. Now let define the sequence (Y_n) as follows. For $n \neq n_k$ with $k \in N$, set $Y_n = 0$. For $k \geq 1$ being any but fixed and $n = n_k$, set $Y_n = 2^k$ or $Y_n = -2^k$, resp., on the first interval of Q_n which is contained in the $(2p-1)$-th or in the $2p$-th interval of $Q_{n_{k-1}}$ resp., with $1 \leq p \leq \dfrac{n_{k-1}}{2}$ and $Y_n = 0$ elsewhere. It is not hard to check that defined in such away, we have $P(Y_n \neq 0) = 2^{-k}$. Then (Y_n) converges to zero, a.s. On the other hand, for all $k \geq 1$ we have

$$Y_{n_{k-1}}(n) = 1 \quad \text{or} \quad Y_{n_{k-1}}(n) = -1, \quad \text{resp.,}$$

on the $(2p-1)$-th or $2p$-th interval of $Q_{n_{k-1}}$ resp., with $1 \leq p \leq \dfrac{n_{k-1}}{2}$. It follows that the sequence (Y_n) cannot be a game fairer with time. However for all $k \geq 2$ and $q < n_{k-1}$, we have $Y_q(n) = 0$. It guarantees that if for any $m \in N$ with $m = n_k$ for some $k \in N$ we take $p_m = n_{k+1} + 1$ and for any other m we set $p_m = m + 1$ then for all $p, m, n \in N$ with $m \geq p$ and $n \geq p_m$ we get

$$\sup_{p\leq q\leq m} |Y_q(n) - Y_q| = \sup_{p\leq q\leq m} |Y_q|.$$

This with the almost sure convergence of (Y_n) to zero shows that (Y_n) must be a quasi-mil. It means that the class of quasi-mils is not contained in that of games fairer with time.

To show how large is the class of quasi-mils, we have considered in [4] the set of G of all nondecreasing functions from N to N. Then equipped with the partial order "$\leq'$" given by

$$f =' g \quad \text{iff} \quad \operatorname{card}(\{f \neq g\}) < \infty$$

and

$$f <' g \quad \text{iff} \quad \operatorname{card}(\{f > g\}) < \infty \quad \text{and} \quad \operatorname{card}(\{f < g\}) = \infty,$$

G is easily checked to be a directed set. Further we have pointed out there that a sequence (X_n) is a quasi-mil if and only if it is a *mil of size* g for some $g \in G$, write $(X_n) \in \mathcal{M}^g$, i.e., for every $\varepsilon > 0$ there exists $p \in N$ such that for all $m, n \in N$ with $p < m < m + g(m) \leq n$ we have

$$P\Big(\sup_{p\leq q\leq m} |X_q(n) - X_q| > \varepsilon\Big) < \varepsilon. \tag{2.1}$$

Particularly, it was shown that when g runs over G, the set of all quasi-mils is classified into a nondecreasing family $(\mathcal{M}^g,\ g \in G)$ for which if $f, g \in G$ with $f <' g$ then the class $\mathcal{M}^f$ is strictly contained in $\mathcal{M}^g$. The main aim of this note is to give another classification of the class of all quasi-mils which is independent of that having been just mentioned before.

For this purpose let F denote the set of all functions from N to N. Then it is not hard to check that endowed with the same partial order $\leq'$, F becomes also a directed set. Thus a natural question arises whether or not the above classification can be extended to $(F, \leq')$. The following result gives a negative answer to the question.

Proposition 2.1. *There exists a pair (f, g) of elements of F with $f <' g$ for which $\mathcal{M}^f = \mathcal{M}^g$.*

Proof. First, it is worth noting that the proof of Theorem 2.2 [4] does not depend on the nondecreasing property of the function $g \in G$. Hence, a sequence (X_n) is a quasi-mil if and only if there exists $f \in F$ such that (X_n) is a mil of size f. Now let define two functions f, g as follows:

$$f(m) = k\,\mathrm{mod}^k m + 1$$

and

$$g(m) = \begin{cases} f(m) & \text{if} \quad \mathrm{mod}^k m = 0, \\ f(m) + 1 & \text{if} \quad \mathrm{mod}^k m > 0, \end{cases}$$

where k is a prime number equal to or larger than 2 and mod_m^k means residuation of m devided by k. Then it is evident that $f, g \in F,\ f \leq g$ and $f <' g$. Now for any $h \in G$, set

$$a_k(h) = k + h(k), \quad k \in N,$$

and

$$b_n(h) = \max\{m : m + h(m) \leq n\}, \quad n \geq a_1(h).$$

Clearly, we have

$$b_n(g) \leq b_n(f) < n, \quad n \geq a_1(h). \tag{2.2}$$

We claim more that

$$b_n(f) = b_n(g) = pk,\ p \in N,\ pk + 1 \leq n \leq (p+1)k. \tag{2.3}$$

Indeed, let $p \in N$ be any but fixed. Then by the same definition of f and g we have

$$pk + f(pk) = pk + g(pk) = pk + 1.$$

Hence by (2.2), it follows that

$$b_{pk+1}(f) = b_{pk+1}(g) = pk,$$

and then

$$pk \leq b_n(g) \leq b_n(f), \quad pk + 1 \leq n \leq (p+1)k. \tag{2.4}$$

Now to see (2.3), suppose on the contrary that there exists some $pk+1 \le n \le (p+1)k$ and $j \ge 1$ such that $b_n(f) = pk + j$. Then again by the definitions of f and $b_n(f)$ one obtains

$$\begin{aligned} n \le b_n(f) + f(b_n(f)) &= pk + j + f(pk+j) \\ &= pk + j + kj + 1 \\ &= (p+j)k + (j+1) > n. \end{aligned}$$

This is impossible. Thus by (2.4) we get (2.3) and the claim. Having it in hand, we are in a good position to show that defined as given at the beginning of the construction, the pair (f, g) gives a desired example. To see this, it is useful to note first that $\mathcal{M}^f \subset \mathcal{M}^g$ since $f \le g$. To prove the converse inclusion, let $(X_n) \in \mathcal{M}^g$ and $\varepsilon > 0$ be any but fixed. Then by definition there exists, say $p \ge k$ such that for any $m, n \in N$ with $m \ge p$ and $n \ge m + g(m)$, (2.1) is satisfied, i.e.

$$P\Big(\sup_{p \le q \le m} |X_q(n) - X_q| > \varepsilon\Big) < \varepsilon.$$

Now let $s, n \in N$ with $s \ge p$ and $n \ge s + f(s)$. Then there exists $p_1 \in N$ such that

$$p_1 k + 1 \le n \le (p_1+1)k.$$

Thus by the claim we have

$$b_n(g) = b_n(f) = p_1 k.$$

It follows that

$$b_n(f) = p_1 k \ge s.$$

Therefore by taking $m = b_n(g) = p_1 k$ we have $p \le s \le m$ and

$$n \ge b_n(g) + g(b_n(g)) = m + g(m).$$

Consequently, by (2.1) one obtains

$$P\Big(\sup_{p \le q \le s} |X_q(n) - X_q| > \varepsilon\Big) \le P\Big(\sup_{p \le q \le m} |X_q(n) - X_q| > \varepsilon\Big) < \varepsilon.$$

This means that $(X_n) \in \mathcal{M}^f$, which completes the construction. □

The previous proposition shows that the next classification is independent from Theorem 2.3 of [4]. To see this, let define on F the other partial order $\le^*$, given by $f <^* g$ iff $\operatorname{card}(\{g \le f\}) < \infty$. It is clear that if $f, g \in G$ with $f <^* g$ then $f <' g$. Further, if we choose $f, g \in G$ as

$$f(m) = m, \quad m \in N,$$

and

$$g(2m) = g(2m-1) = 2m, \quad m \in N,$$

then clearly $f <' g$ but one cannot compare f with g in the order $\le^*$. It means that restricted to G, the second order $\le^*$ is strictly weaker than the first one $\le'$. However, even on F we get the following classification of the class of quasi-mils.

Theorem 2.1. *When f runs over the directed set $(F, \leq^*)$, the class of all quasi-mils is classified into a nondecreasing family $(\mathcal{M}^f,\ f \in F)$ for which the smallest subclass $\mathcal{M}^1$ coincides with the set of all mils. Furthermore, for any $f, g \in F$ with $f <^* g$, the subclass $\mathcal{M}^f$ is strictly contained in $\mathcal{M}^g$.*

Proof. The first part of the theorem follows as the first part of Theorem 2.2 [4], where we did not use the increasing property of $g \in G$. The main part of the proof consists in showing the second statement of the theorem. For this purpose, let $f, g \in F$ with $f <^* g$. Then by definition, there exists $n_0 \in N$ such that $f(m) < g(m)$, $m \geq n_0$. To construct a quasi-mil $(X_n) \in \mathcal{M}^g$ which does not belong to $\mathcal{M}^f$, we choose the usual Lebesgue probability space on $[0, 1)$ to be $(\Omega, \mathcal{A}, P)$. Further, for each $n \in N$, set $a_n = \prod\limits_{j \leq n} 2^j$, Q_n the partition of $[0, 1)$ in a_n intervals of equal length and $\mathcal{A}_n$ the complete σ-algebra generated by Q_n.

For simplicity, let define

$$m_0 = \max\big(n_0, a_1(f), a_1(g)\big).$$

Clearly by the definition of $b_n(f)$ given in the proof of the previous proposition, it follows that the sequence $\big(b_n(f),\ n \geq m_0\big)$ does not decrease and the set $\{b_n(f),\ n \geq m_0\}$ is infinite. Let (m_k) denote the strictly increasing sequence renumbering in turn all different elements of $\{b_n(f),\ n \geq m_0\}$. Then it is clear that for every $k \in N$ we have

$$b_n(f) = m_k \quad \text{if and only if} \quad n_k \leq n < n_{k+1}, \tag{2.5}$$

where $n_k = m_k + f(m_k)$.

Now define a desired quasi-mil (X_n) as follows: For $n \neq n_k$, $k \in N$ set $X_n = 0$. For any other $n \in N$, set

$$X_n = \frac{a_n}{a_{b_n(f)}} \quad \text{or} \quad X_n = -\frac{a_n}{a_{b_n(f)}}, \text{ resp.}$$

on the first interval of Q_n which is contained in the $(2s-1)$-th or $(2s)$-th interval of $Q_{b_n(f)}$, resp., with $1 \leq s \leq \dfrac{a_{b_n(f)}}{2}$ and $X_n = 0$, elsewhere. It is easily checked that defined in such a way we have

$$P\big(\{X_n \neq 0\}\big) \leq \frac{a_{b_n(f)}}{a_n} \leq \prod_{j=a_{b_n(f)+1}}^{n} 2^{-j} \leq 2^{-n},$$

noting that by (2.2) $b_n(f) + 1 \leq n$, $n \geq m_0$.

Therefore

$$(X_n) \quad \text{converges to zero, a.s.} \tag{2.6}$$

On the other hand, by taking $n = n_k$, $k \in N$, we get

$$X_{b_n(f)} = 1 \quad \text{or} \quad X_{b_n(f)} = -1, \text{ resp.} \tag{2.7}$$

on the $(2s-1)$-th or $(2s)$-th interval of $Q_{b_n(f)}$, resp. with $1 \leq s \leq \dfrac{a_{b_n(f)}}{2}$. This with (2.6) implies that (X_n) is neither a game fairer with time nor a mil of size f, i.e., $(X_n) \notin \mathcal{M}^f$.

To show that $(X_n) \in \mathcal{M}^g$ we claim that for $n = n_k$, $k \in N$ we have

$$b_n(g) < b_n(f). \tag{2.8}$$

Indeed, by (2.2) suppose on the contrary that there exists some $k \in N$ with $b_{n_k}(g) = b_{n_k}(f)$. Then by (2.5) we have

$$\begin{aligned} n_k = m_k + f(m_k) &= b_{n_k}(f) + f[b_{n_k}(f)] \\ &< b_{n_k}(g) + g[b_{n_k}(g)] \leq n_k. \end{aligned}$$

It is impossible. Thus (2.8) is verified. Hence by (2.7) we get

$$X_q(n) = 0,\ n \geq m_0,\ q \leq b_n(g).$$

Therefore, for any $p, m, n \in N$ with $m_0 \leq p \leq m$ and $n \geq a_m(g)$ we have $b_n(g) \geq m$ and then

$$\sup_{p \leq q \leq m} |X_q(n) - X_q| \leq \sup_{p \leq q \leq b_n(g)} |X_q(n) - X_q| = \sup_{p \leq q \leq b_n(g)} |X_q|.$$

This with (2.6) guarantees that (X_n) is a mil of size g, i.e., $(X_n) \in \mathcal{M}^g$. □

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