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ON THE ASYMPTOTIC STABILITY OF TIME-VARYING DIFFERENTIAL EQUATIONS WITH MULTIPLE DELAYS AND APPLICATIONS

NGUYEN SINH BAY, NGUYEN THE HOAN AND VU NGOC PHAT

Abstract. This paper studies the asymptotic stability of some class of retarded functional differential equations. Based on the stability of the underlying linear system, new sufficient conditions for asymptotic stability of linear retarded systems as well as systems with nonlinear perturbations are derived. The class of systems is allowed to be time-varying and time-delay. The results are applied to the input feedback stabilization problem of some class of linear control systems.

1. Introduction

Stability analysis of dynamical systems with time-delayed states is a topic of practical and theoretical interest, because time-delay states encounter in many dynamical systems described by continuous-time equations (see, e.g., [1, 4, 8, 10, 11, 12] and references therein). Recently, research interest has been focused on the asymptotic stability problem for time-delay systems described by differential retarded equations of the form

$$\dot{x}(t) = f(t, x_t),\ f(t, 0) = 0,\ t \geq 0, \tag{1}$$

with initial condition $(t_0, \phi) : x(s) = \phi(s),\ s \in [t_0 - h, t_0]$, where $h > 0,\ x(t) \in R^n,\ t_0 \in R^+,\ \phi \in \mathcal{C} := \mathcal{C}([-h, 0], R^n)$, x_t denotes the segment on $[t-h, t]$ of vector function $x(.) \in \mathcal{C}(R^+, R^n)$ so that $x_t : [-h, 0] \to R^n$ is defined by $x_t(s) = x(t+s)$, $-h \leq s \leq 0$.

We recall that the zero solution of system (1) (or system (1) itself) is called asymptotically stable if for every $\varepsilon > 0$, for every $t_0 \in R^+$, there is $\delta > 0$ such that for any $\phi \in \mathcal{C}([-h, 0], R^n) : \|\phi\| < \delta$ the solution $x(t)$ with the initial condition (t_0, ϕ) of the system satisfies

(i) $\|x(t)\| < \varepsilon,\ \forall t \geq t_0$,

(ii) $\|x(t)\| \to 0$, as $t \to \infty$.

The asymptotic stability analysis of general system (1) based on the second direct Lyapunov method has gained significant advances over the past years; see, e.g., [5, 7, 12, 15]. It is woth noting that in most of the mentioned papers, sufficient conditions for asymptotic stability are given in terms of the existence of

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Lyapunov functionals and finding such Lyapunov functionals is a difficult task in many cases. In [4, 6, 9, 14], the stability conditions for time-invariant delay systems are formulated in terms of the existence of the solution of Riccati/Lyapunov equations. For linear neutral-type time-invariant equations a similar approach and stability conditions were proposed in [7, 13]. Some authors (see [2] and references therein) have obtained stability conditions via the linear matrix inequality approach, shown to be equivalent to some Riccati equations. In [13, 14, 15, 16], sufficient stability conditions for linear systems with nonlinear or delay perturbations were established by using the stability criteria of the linear underlying systems.

In this paper, we give both time-varying and time-delay stability conditions formulated in terms of the asymptotic stability of the linear underlying systems. The approach allows us to apply the obtained results in the stabilization problem of some classes of linear control systems.

2. Preliminaries

Let us recall some notations and definitions, which will be used throughout the paper.

Let R^+ denote the set of all nonnegative real numbers, $\langle x, y\rangle$ the scalar product of x, y in R^n, A^T the transpose of matrix A, I the identity matrix, A^{-1} the inverse of matrix A, $B_{[0,1]}$ the closed unit ball in R^n, $x_t \in \mathcal{C}$ with

$$\begin{aligned}
\|x_t\| &= \sup_{-h\le s\le 0} \|x_t(s)\|;\\
\lambda_{\min}(A) &= \inf\left\{\langle Ax, x\rangle : x \in B_{[0,1]}\right\},\\
\lambda_{\max}(A) &= \sup\left\{\langle Ax, x\rangle : x \in B_{[0,1]}\right\} := \|A\|.
\end{aligned}$$

A matrix A is said to be positive definite if $\langle Ax, x\rangle \ge 0$, $\forall x \in R^n$ and $\langle Ax, x\rangle > 0$, if $x \ne 0$. It follows from [3] that for a symmetric positive definite matrix A the spectrum $\sigma(A)$ is a bounded closed set contained in the segment $[\lambda_{\min}(A), \lambda_{\max}(A)]$ and we have the following result.

Proposition 2.1. [3] *If A is a symmetric positive definite matrix, then $\lambda_{\min}(A) > 0$ and there is A^{-1} such that the following relations hold*

(i) $\lambda_{\min}(A)\|x\|^2 \le \langle Ax, x\rangle \le \lambda_{\max}(A)\|x\|^2, \quad \forall x \in R^n$;

(ii) $\dfrac{1}{\lambda_{\max}(A)}\|x\|^2 \le \langle A^{-1}x, x\rangle \le \dfrac{1}{\lambda_{\min}(A)}\|x\|^2, \quad \forall x \in R^n$;

(iii) $\lambda_{\min}(A^2) \ge \left[\lambda_{\min}(A)\right]^2$;

(iv) $\lambda_{\max}(A^2) \le \left[\lambda_{\max}(A)\right]^2$.

Consider a homogeneous time-varying system

$$\dot{x}(t) = A(t)x(t), \quad t \ge 0. \tag{2}$$

Let us denote by $S(t)$ the fundamental matrix and by $U(t, s)$ the evolution matrix of the system defined by $U(t, s) = S(t)S^{-1}(s);\ \ t \ge s \ge 0$. It is obvious that if matrix function $A(t)$ is bounded on R^+, i.e., there is $M > 0$ such that

$$\sup_{t\geq 0} \|A(t)\| \leq M < +\infty,$$

then the evolution matrix $U(t,s)$ satisfies the condition

$$\|U(t,s)\| \leq e^{M|t-s|}, \quad \forall t,s \geq 0.$$

Definition 2.1. The zero solution of system (2) is exponentially stable if there exist positive numbers K and δ such that

$$\|U(t,s)\| \leq Ke^{-\delta(t-s)}, \quad \forall t \geq s \geq 0. \tag{3}$$

Definition 2.2. Let K and δ be positive real numbers. A $n \times n-$matrix function $A(t)$ belongs to $BCAS(K,\delta)$ if $A(t)$ is continuous, bounded on R^+ and the evolution matrix $U(t,s)$ of system (2) satisfies condition (3).

The classical Lyapunov Theorem asserts that the zero solution of system (2), where $A(t) = A$ for all $t \geq 0$, is asymptotically stable if only if for every symmetric positive definite matrix P the Lyapunov equation

$$A^TQ + QA = -P$$

has a symmetric positive definite matrix solution Q. In the sequel, we give a time-varying analog of this result. For this, let us define a matrix function $P(t)$ to be uniformly positive definite on R^+ if

$$\exists\, c > 0 : \langle P(t)x, x\rangle \geq c\|x\|^2, \quad \forall t \in R^+,\ \forall x \in R^n.$$

Throughout this paper we denote by $BSUPD(R^n)$ the set of all $n \times n$-matrix functions, which are symmetric, bounded, uniformly positive definite on R^+. Consider the following time-varying Lyapunov matrix equation

$$\dot{Q}(t) + A^T(t)Q(t) + Q(t)A(t) = -P(t),\ t \in R^+. \tag{4}$$

Proposition 2.2. *Assume that $A(t) \in BCAS(K,\delta)$. Then for every $P(t) \in BSUPD(R^n)$, the Lyapunov matrix equation (4) has a solution $Q(t) \in BSUPD(R^n)$ given by*

$$Q(t) = \int_t^\infty U^T(\tau,t)P(\tau)U(\tau,t)d\tau \tag{5}$$

and the following relation holds

$$\frac{P}{2M}\|x\|^2 \leq \langle Q(t)x,x\rangle \leq \frac{PK^2}{2\delta}\|x\|^2, \quad \forall t \in R^+,\ \forall x \in R^n, \tag{6}$$

where $M := \sup_{t\geq 0}\|A(t)\|$; $P := \sup_{t\geq 0}\|P(t)\|$. Conversely, if for any matrix function $P(t) \in BSUPD(R^n)$ there is a solution $Q(t) \in BSUPD(R^n)$ of equation (4) defined by (5), then the zero solution of (2) is asymptotically stable.

Proof. The matrix function $A(t)$ belongs to $BCAS(K,\delta)$, then the evolution matrix $U(\tau,t)$ satisfies condition (3). For any matrix $P(t) \in BSUPD(R^n)$ we consider the matrix function given by (5)

$$Q(t) = \int_t^{\infty} U^T(\tau,t)P(\tau)U(\tau,t)d\tau.$$

By the properties of $U(\tau,t)$ and $P(t)$, the matrix function $Q(t)$ is well defined, $Q(t)$ is symmetric for all $t \geq 0$ and we can show that $Q(t)$ satisfies the Lyapunov equation (4) as follows. Replacing $U(\tau,t) = S(\tau)S^{-1}(t)$ and differentiating both sides of (5) in t, we have

$$\begin{aligned}\dot{Q}(t) = \dot{S}^{T^{-1}}(t)\int_t^{\infty} S^T(\tau)P(\tau)S(\tau)d\tau S^{-1}(t)\\ + S^{T^{-1}}(t)\frac{d}{dt}\Big[\int_t^{\infty} S^T(\tau)P(\tau)S(\tau)d\tau\Big]S^{-1}(t)\\ + S^{T^{-1}}(t)\int_t^{\infty} S^T(\tau)P(\tau)S(\tau)d\tau\dot{S}^{-1}(t).\end{aligned}$$

Since $\dot{S}^{-1}(t) = -S^{-1}(t)A(t)$, $\dot{S}^{T^{-1}}(t) = -A^T(t)S^{T^{-1}}(t)$, we have

$$\dot{Q}(t) = -A^T(t)Q(t) - Q(t)A(t) + S^{T^{-1}}(t)\frac{d}{dt}\Big[\int_t^{\infty} S^T(\tau)P(\tau)S(\tau)d\tau\Big]S^{-1}(t).$$

Therefore

$$\dot{Q}(t) = -A^T(t)Q(t) - Q(t)A(t) - P(t),$$

as desired. We now prove that $Q(t) \in BSUPD(R^n)$. Indeed, we have

$$\langle Q(t)x,x\rangle = \int_t^{\infty} \langle P(\tau)U(\tau,t)x, U(\tau,t)x\rangle d\tau.$$

Since $P(t) \in BSUPD(R^n)$, we have

$$\exists c > 0: \quad \langle P(t)x,x\rangle \geq c\|x\|^2, \quad \forall x \in R^n,\ \forall t \in R^+.$$

Therefore

$$\begin{aligned}\langle Q(t)x,x\rangle &= \int_t^{\infty} \langle P(\tau)U(\tau,t)x, U(\tau,t)x\rangle d\tau\\ &\geq c\int_t^{\infty} \|U(\tau,t)x\|^2 d\tau,\ \forall x \in R^n, \forall t \in R^+.\end{aligned}$$

On the other hand, since

$$\|x\| = \|U(t,\tau)U(\tau,t)x\| \le \|U(t,\tau)\|\|U(\tau,t)x\|,$$
$$\|U(\tau,t)x\| \ge \frac{\|x\|}{\|U(t,\tau)\|},$$

it holds

$$\langle Q(t)x, x\rangle \ge c\int_t^\infty \frac{\|x\|^2}{\|U(t,\tau)\|^2}d\tau.$$

Taking Proposition 2.1 into account we have

$$\begin{aligned}\langle Q(t)x, x\rangle &= \int_t^\infty \langle P(\tau)U(\tau,t)x, U(\tau,t)x\rangle d\tau\\ &\ge c\|x\|^2\int_t^\infty e^{-2M|t-\tau|}d\tau\\ &= c\|x\|^2\int_t^\infty e^{2M(t-\tau)}d\tau\\ &= \frac{c}{2M}\|x\|^2, \quad \forall t\in R^+, \forall x\in R^n,\end{aligned}$$

which shows that $Q(t)\in BSUPD(R^n)$. To prove the second inequality in (6), we deduce from (3) that

$$\begin{aligned}\langle Q(t)x, x\rangle &= \int_t^\infty \langle P(\tau)U(\tau,t)x, U(\tau,t)x\rangle d\tau\\ &\le P\|x\|^2\int_t^\infty \|U(\tau,t)\|^2 d\tau\\ &\le PK^2\|x\|^2\int_t^\infty e^{-2\delta(\tau-t)}d\tau\\ &\le \frac{PK^2}{2\delta}\|x\|^2.\end{aligned}$$

To prove the converse part, we take a Lyapunov function for linear system (2) of the form $V(t,x) = \langle Q(t)x, x\rangle,\ \ x\in R^n$. It is easy to verify that

$$\frac{d}{dt}V(t,x(t)) = -\langle P(t)x(t), x(t)\rangle \le -c\|x(t)\|^2, \quad \forall t\in R^+;$$

hence the zero solution is asymptotically stable. The proof is complete. □

If $V : R^+ \times \mathcal{C} \to R^+$ is continuous and $x(\tau,\phi)(t)$ is the solution of equation (1) through (τ,ϕ), we define

$$\dot{V}(t,\phi) := \overline{\lim_{\eta\to 0^+}} \frac{1}{\eta}[V(t+\eta, x_{t+\eta}(\tau,\phi)) - V(t,\phi)].$$

The function $\dot{V}(t,\phi)$ is upper right-hand derivate of $V(t,\phi)$ along the solution of equation (1).

In the sequel, we will need the following stability theorem of functional differential equations.

Proposition 2.3. [4] *Assume that $f : R^+ \times \mathcal{C} \to R^n$ takes $R^+\times$ (bounded set of $\mathcal{C}$) into bounded set of R^n, and c_1, c_2, c_3 are positive real numbers. If there is a continuous functional $V : R^+ \times \mathcal{C} \to R$ such that*

i) $c_1\|\phi(0)\|^2 \le V(t,\phi) \le c_2\|\phi\|^2, \quad \forall t \ge 0,\ \forall \phi \in \mathcal{C}$;

ii) $\dot{V}(t,\phi) \le -c_3\|\phi(0)\|^2, \quad \forall t \ge 0,\ \forall \phi \in \mathcal{C}$,

where $\dot{V}(t,\phi)$ is the upper right-hand derivate of $V(t,\phi)$ along the solution of equation (1)*. Then the solution $x = 0$ of equation* (1) *is asymptotically stable.*

3. Main results

Consider a linear retarded system of the form

$$\dot{x}(t) = A(t)x(t) + \sum_{i=1}^{r} A_i(t)x(t-h_i) + \sum_{i=1}^{q} \int_{-k_i}^{0} dG_i(s)x(t+s) \tag{7}$$

where $t \ge 0$; $x \in R^n$, $A(t)$, $A_i(t)$, $i = 1,2,\dots,r$ are $n\times n$-matrix functions on R^+; $G_i(s), i = 1,2,...,q$ are $n \times n$-matrix functions with finite total variations on respective segments $[-k_i, 0]$, $0 := k_0 < k_1 < k_2 < \dots < k_q \le h$, $0 < h_1 < h_2 < ... < h_r \le h$ $(h := \max\{h_r, k_q\})$. Throughout this section we assume that the functions $A_i(t)$ are bounded on R^+ and the function $G_i(t)$ satisfies a condition so called Lipschitzian property on segments $[-k_i, 0]$, $i = 1,2,\dots,q$:

$$\|G_i(s) - G_i(s')\| \le L_i|s - s'|, \quad \forall s, s' \in [-k_i, 0],\ i = 1,2,\dots,q. \tag{8}$$

Theorem 3.1. *Assume that $A(t) \in BCAS(K,\delta)$ and $G_i(s), i = 1,2,...,q$ satisfy condition* (8)*. Then system* (7) *is asymptotically stable if*

$$\sum_{i=1}^{r} \sup_{t\ge 0} \|A_i(t)\|^2 + \sum_{i=1}^{q} k_i L_i^2 \le \frac{\delta^2}{\left(r + \sum\limits_{i=1}^{q} k_i\right)K^4}. \tag{9}$$

Proof. Since $A(t) \in BCAS(K,\delta)$, by Proposition 2.2 for the matrix function $P(t) = \alpha I,\ \alpha > 0$, the matrix function $Q(t) \in BSUPD(R^n)$ given by

$$Q(t) = \alpha \int_{t}^{\infty} \|U(\tau,t)\|^2 d\tau$$

satisfies the Lyapunov equation (4). Take the Lyapunov functional $V(t,\phi)$ on $R^+\times\mathcal{C}$ for system (7) defined by

$$V(t,\phi) := \langle Q(t)\phi(0),\phi(0)\rangle + \sum_{i=1}^{r}\int_{-h_i}^{0}\|\phi(\tau)\|^2 d\tau + \sum_{i=1}^{q}\int_{-k_i}^{0}\int_{s}^{0}\|\phi(\tau)\|^2 d\tau ds.$$

Taking $\phi := x_t$, where $x(t)$ is a solution of equation (1) and x_t is defined by $x_t(s) = \phi(s)$; $s\in[-h,0]$, we have

$$V(t,x_t) := \langle Q(t)x(t),x(t)\rangle + \sum_{i=1}^{r}\int_{-h_i}^{0}\|x_t(\tau)\|^2 d\tau + \sum_{i=1}^{q}\int_{-k_i}^{0}\int_{s}^{0}\|x_t(\tau)\|^2 d\tau ds.$$

By some simple calculations and by Proposition 2.2 we can show that

$$\frac{\alpha}{2M}\|x(t)\|^2 \le V(t,x_t) \le \Big(\frac{\alpha K^2}{2\delta} + rh + h\sum_{i=1}^{q}k_i\Big)\|x_t\|^2.$$

Condition (i) of Proposition 2.3 is fulfiled.

Taking derivative of $V(t,x_t)$ in t along the solution of system (7) we get

$$\begin{aligned}\dot V(t,x_t) =& \Big\langle (\dot Q(t) + A^T(t)Q(t) + Q(t)A(t))x(t), x(t)\Big\rangle\\ &+ \sum_{i=1}^{r}\Big(\|x(t)\|^2 - \|x(t-h_i)\|^2\Big) + \sum_{i=1}^{q}\int_{-k_i}^{0} ds\Big(\|x(t)\|^2 - \|x(t+s)\|^2\Big)\\ &+ 2\Big\langle Q(t)x(t), \sum_{i=1}^{r}A_i(t)x(t-h_i)\Big\rangle + 2\Big\langle Q(t)x(t), \sum_{i=1}^{q}\int_{-k_i}^{0} dG_i(s)x(t+s)\Big\rangle.\end{aligned}$$

By completing the square we have

$$-\|x(t-h_i)\|^2 + 2\langle A_i^T(t)Q(t)x(t), x(t-h_i)\rangle \le \|A_i^T(t)\|^2\|Q(t)\|^2\|x(t)\|^2;$$

hence

$$\begin{aligned}&-\sum_{i=1}^{r}\|x(t-h_i)\|^2 + 2\sum_{i=1}^{r}\langle A_i^T(t)Q(t)x(t), x(t-h_i)\rangle \le\\ &\sum_{i=1}^{r}\|A_i^T(t)\|^2\|Q(t)\|^2\|x(t)\|^2 \le \|Q(t)\|^2\sum_{i=1}^{r}\sup_{t\ge0}\|A_i(t)\|^2\|x(t)\|^2.\end{aligned} \tag{10}$$

On the other hand, since $\|G_i(s)-G_i(s')\| \le L_i|s-s'|$ and $k_i\ge0$, for $i=1,2,...,q$, we deduce that

$$\Big\|\int_{-k_i}^{0} dG_i(s)x(t+s)\Big\| \le L_i\int_{-k_i}^{0}\|x(t+s)\|ds$$

and

$$-\sum_{i=1}^{q}\int_{-k_i}^{0}\|x(t+s)\|^2ds+2\sum_{i=1}^{q}\int_{-k_i}^{0}\langle Q(t)x(t),dG_i(s)x(t+s)\rangle$$

$$\leq\sum_{i=1}^{q}k_iL_i^2\|Q(t)\|^2\|x(t)\|^2. \tag{11}$$

Taking (4), (6), (10), (11) and Proposition 2.2 into account, we obtain that

$$\dot V(t,x_t)\leq$$

$$\Big(-P+r+\sum_{i=1}^{q}k_i+\|Q(t)\|^2\sum_{i=1}^{r}\sup_{t\geq 0}\|A_i(t)\|^2+\|Q(t)\|^2\sum_{i=1}^{q}k_iL_i^2\Big)\|x(t)\|^2.$$

From the last inequality it follows that the derivative $\dot V(t,x_t)$ is uniformly negative definite on R^+ provided

$$-\alpha+r+\sum_{i=1}^{q}k_i+\|Q(t)\|^2\sum_{i=1}^{r}\sup_{t\geq 0}\|A_i(t)\|^2+\|Q(t)\|^2\sum_{i=1}^{q}k_iL_i^2<0.$$

From Proposition 2.2 and condition (6) it follows that the derivative $\dot V(t,x_t)$ is uniformly negative definite on R^+ if

$$-\alpha+r+\sum_{i=1}^{q}k_i+\frac{\alpha^2K^4}{4\delta^2}\sum_{i=1}^{r}\sup_{t\geq 0}\|A_i(t)\|^2+\frac{\alpha^2K^4}{4\delta^2}\sum_{i=1}^{q}k_iL_i^2<0,$$

or

$$\sum_{i=1}^{r}\sup_{t\geq 0}\|A_i(t)\|^2+\sum_{i=1}^{q}k_iL_i^2<\frac{\alpha-\Big(r+\sum\limits_{i=1}^{q}k_i\Big)}{\alpha^2}\cdot\frac{4\delta^2}{K^4}=H(\alpha)\frac{4\delta^2}{K^4}, \tag{12}$$

where

$$H(\alpha):=\frac{\alpha-\Big(r+\sum\limits_{i=1}^{q}k_i\Big)}{\alpha^2}\cdot$$

Since the function $H(\alpha)$ attains its maximum value at $\alpha=2\Big(r+\sum\limits_{i=1}^{q}k_i\Big)$, the proof is now followed from Proposition 2.3 by choosing $A_i(t)$ and L_i such that

$$\sum_{i=1}^{r}\sup_{t\geq 0}\|A_i(t)\|^2+\sum_{i=1}^{q}k_iLi^2<\frac{\delta^2}{\Big(r+\sum\limits_{i=1}^{q}k_i\Big)K^4}\cdot$$

The proof is completed. □

As a special case, when $dG_i(t)=0$ $(i=1,2,...,q)$ or $A_i(t)=0$ $(i=1,2,...,r)$ for all $t\geq 0$, the following sufficient conditions for the stability of retarded system (7) can be derived.

Corollary 3.1. *For system* (7), *where* $dG_i(t) = 0$ *on* $[-k_i, 0]$ *for all* $i = 1, 2, ..., q$, *we assume the same conditions on* $A(t)$ *as in Theorem 3.1. Then* (7) *is asymptotically stable if*

$$\sum_{i=1}^{r} \sup_{t\geq 0} \|A_i(t)\|^2 < \frac{\delta^2}{rK^4}\,.$$

Corollary 3.2. *For system* (7), *where* $A_i(t) = 0, \ \ i = 1, 2, ..., r,$ *for all* $t \geq 0$, *we assume the same assumptions on* $A(t)$ *and* $G_i(t)$ *as in Theorem 3.1. Then* (7) *is asymptotically stable if*

$$\sum_{i=1}^{q} \frac{Li^2}{Ri^2} < 1,$$

where $R_i^2 := \dfrac{\delta^2}{K^4 k_i \sum\limits_{j=1}^{q} k_j}, \quad i = 1, 2, \ldots, q.$

Remark 3.1. By some arguments simillar to those in the proof of Theorem 3.1 we may extend the result for the following system

$$\dot{x}(t) = A(t)x(t) + \sum_{i=1}^{r} A_i(t)x(t - h_i) + \sum_{i=1}^{q} \int_{-k_i}^{-k_{i-1}} dG_i(s)x(t+s), \ t \geq 0. \tag{13}$$

Theorem 3.2. *Under the assumptions of Theorem 3.1, assume additionally that*

$$\sum_{i=1}^{r} \sup_{t\geq 0} \|A_i(t)\|^2 + \sum_{i=1}^{q} Li^2(k_i - k_{i-1}) < \frac{\delta^2}{(r+k)K^4}\,.$$

Then system (13) *is asymptotically stable.*

Example 3.1. Consider a linear retarded system of the form:

$$\dot{x}(t) = A(t)x(t) + \int_{-2}^{0} dG_1(s)x(t+s) + \int_{-3}^{0} dG_2(s)x(t+s), \ t \geq 0, \tag{14}$$

where

$$A(t) = \begin{pmatrix} -1 & 0 \\ e^{-t} & -1 \end{pmatrix}; \ G_1(s) = |0.02 \cdot s + 0.01| I_2; \ G_2(s) = |0.03 \cdot s + 0,01| I_2.$$

We may verify that $K = 2$, $\delta = 1$, $L_1 = 0.02$, $L_2 = 0.03$, $k_1 = 2$, $k_2 = 3$, $h = 3$. Hence

$$\frac{L_1^2}{R_1^2} + \frac{L_1^2}{R_2^2} = 0.28 < 1.$$

Applying Corollary 3.2 we can conclude that system (14) is asymptotically stable.

Now we consider the asymptotic stability of a class of nonlinear system of the form

$$\dot{x}(t) = A(t)x(t) + f(t, x(t-h_1), x(t-h_2), \ldots, x(t-h_r)), t \geq 0, \tag{15}$$

$$f(t, 0, \ldots, 0) = 0, \forall t \geq 0,$$

where $0 < h_1 < h_2 < ... < h_r := h$, $r \geq 1$, $A(t)$ is $n \times n$-matrix function on R^+; $f : R^+ \times \mathcal{C} \to R^n$. In the sequel, we will need the following condition: There are real bounded scalar functions $a_i(t) : R^+ \to R^+$, $i = 1, 2, \ldots, r$ such that

$$\|f(t, x_1, x_2, \ldots, x_r)\| \leq \sum_{i=1}^{r} a_i(t)\|x_i\|, \quad \forall t \in R^+, \forall x_i \in R^n \tag{16}$$

Let us denote

$$a^2 := \sum_{i=1}^{r} \sup_{t \geq 0} |a_i(t)|^2.$$

The following theorem gives a sufficient condition for the asymptotic stability of system (15).

Theorem 3.3. *Assume conditions* (16) *and* $A(t) \in BCAS(K, \delta)$. *Then system* (15) *is asymptotically stable if*

$$0 < a < \frac{\delta}{\sqrt{r}K^2} \cdot$$

Proof. By the same arguments used in the proof of Theorem 3.1, using Lyapunov functional

$$V(t, x_t) = \langle Q(t)x(t), x(t)\rangle + \sum_{i=1}^{r} \int_{-h_i}^{0} \|x_t(\tau)\|^2 d\tau,$$

where $Q(t)$ is a solution of matrix equation (4) with $P(t) = r\alpha I$, where $\alpha > 0$ will be chosen later, we arrive at the fact that

$$\begin{aligned} \dot{V}(t, x_t) &\leq -r(\alpha - 1)\|x(t)\|^2 + \sum_{i=1}^{r} a_i^2(t)\|Q(t)x(t)\|^2 \\ &= \Big\langle \Big[\Big(\sum_{i=1}^{r} |a_i(t)|^2\Big)I - r(\alpha - 1)Q^{-2}(t)\Big]Q(t)x(t), Q(t)x(t)\Big\rangle. \end{aligned}$$

By Proposition 2.1 we obtain

$$\dot{V}(t, x_t) \leq \|Q(t)x(t)\|^2 \big[a^2 - r(\alpha - 1)\lambda_{\min}([Q(t)]^{-2})\big].$$

Therefore, from Proposition 2.1 and (6) we can deduce the following estimations:

$$\begin{aligned} &\lambda_{\min}(Q^{-2}(t)) \geq \frac{4\delta^2}{r^2\alpha^2K^4}, \\ &\dot{V}(t, x_t) \leq \|Q(t)x(t)\|^2\Big[\alpha^2 - r(\alpha - 1)\frac{4\delta^2}{r^2\alpha^2K^4}\Big]. \end{aligned}$$

On the other hand, since $Q(t) \in LSUPD(R^n)$, the derivative of the functional $V(t, x_t)$ is uniformly negative definite on R^+ if

$$a^2 < \frac{\alpha - 1}{\alpha^2} \cdot \frac{4\delta^2}{rK^4} \cdot$$

Since the function $g(\alpha) = \dfrac{\alpha - 1}{\alpha^2}$ attains its maximum at $\alpha = 2$, it suffices to choose a satisfying

$$0 < a < \frac{\delta}{\sqrt{r}K^2} \cdot$$

The theorem is proved. □

Example 3.2. Consider the following system in R^2

$$\begin{cases} \dot{x}_1(t) = -(2 + \cos t)x_1(t) + a \sin x_2(t - 2), \\ \dot{x}_2(t) = -(2 - \cos t)x_2(t), \end{cases} \tag{17}$$

where $t \geq 0$, $a > 0$, $x_i(t) \in R^1$, $i = 1, 2$.

Note that $x = 0$ is a solution of system (17). Since

$$A(t) = \begin{pmatrix} -(2 + \cos t) & 0 \\ 0 & -(2 - \cos t) \end{pmatrix},$$

$$f(t, x_t) = [a \sin x_2(t - 2), 0]^T,$$

we can find the evolution matrix

$$U(t, s) = \begin{pmatrix} e^{-2(t-s)-(\sin t - \sin s)} & 0 \\ 0 & e^{-2(t-s)+(\sin t - \sin s)} \end{pmatrix}$$

and hence

$$\|U(t, s)\| \leq Ke^{-2(t-s)}, \quad \forall t \geq s \geq 0,$$

where $K = e^2$. On the other hand, we have

$$\|f(t, x(t - 2)\| \leq a\|x(t - 2)\|.$$

Therefore, the zero solution of system (17) is asymptotically stable when

$$a < \frac{\sqrt{2}}{e^2} \cdot$$

4. An application to stabilization

In this section we study the stabilizability problem of a class of control systems with delays using the results obtained in the previous section. Consider the following control system

$$\dot{x}(t) = A(t)x(t) + \int_{-h}^{0} dG(s)x(t + s) + B(t)u(t), \ t \geq 0, \tag{18}$$

where $h > 0$, $x(t) \in R^n$, $u(t) \in R^m$, $B(t) : R^m \to R^n$, $t \geq 0$; $A(t)$ and $G(s)$ satisfy the conditions stated in the previous section.

We recall that system (18) is stabilizable by a feedback controller $u(t) = K(t)x(t)$ if its closed-loop system is asymptotically stable in the Lyapunov sense. The feedback control for system (18) will be found in the form

$$u(t) = \mu \int_{-h}^{0} B^T(t)dG(t)x(t+s), \quad t \geq 0,\ s \in [-h, 0],\ \mu > 0. \tag{19}$$

Theorem 4.1. *Assume that $A(t) \in BCAS(K, \delta)$, $G(s)$ is L-Lipschitz on $[-h, 0]$ and $0 < \sup_{t \geq 0} \lambda_{max}\left[B(t)B^T(t)\right] < +\infty$. Then control system (18) is stabilizable by linear feedback controller (19) if the parameter μ satisfies the condition*

$$0 < \mu < \frac{\delta - K^2 Lh}{K^2 Lh \sup\limits_{t \geq 0} \lambda_{max}\left[B(t)B^T(t)\right]}. \tag{20}$$

Proof. By control (19), system (18) is reduced to the form

$$\dot{x}(t) = A(t)x(t) + \int_{-h}^{0} \left[I + \mu B(t)B^T(t)\right] dG(s)x(t+s). \tag{21}$$

Note that $x = 0$ is a solution of system (21). Let us take $P(t) = \alpha I$, where $\alpha > 0$ will be chosen later. According to Proposition 2.2, we can find a solution $Q(t) \in BSUPD(R^n)$ of the matrix equation (4). Consider a Lyapunov functional of the form

$$V(t, x_t) = \langle Q(t)x(t), x(t)\rangle + \int_{-h}^{0} ds \int_{s}^{0} \|x_t(\tau)\|^2 d\tau.$$

It is easy to verify that there are numbers $c_1 > 0$, $c_2 > 0$ such that

$$c_1\|x(t)\|^2 \leq V(t, x_t) \leq c_2\|x_t\|^2$$

Condition i) of Proposition 2.3 is fulfiled.

From Proposition 2.2 it follows that

$$\begin{aligned}
\dot{V}(t, x_t) &= \langle (h - P)Ix(t), x(t)\rangle - \int_{-h}^{0} \|x(t+s)\|^2 ds \\
&\quad + 2\Big\langle Q(t)x(t), \int_{-h}^{0} \left[I + \mu B(t)B^T(t)\right] dG(s)x(t+s)\Big\rangle \\
&\leq \Big(h - \alpha + \frac{\alpha^2 K^4}{4\delta^2} L^2 h \|I + \mu B(t)B^T(t)\|^2\Big)\|x(t)\|^2.
\end{aligned}$$

Since $\mu > 0$ and $B(t)B^T(t)$ is non-negative definite, we have

$$\|I + \mu B(t)B^T(t)\| = 1 + \mu\|B(t)B^T(t)\|, \quad \forall t \geq 0.$$

Therefore, the derivative $\dot{V}(t, x_t)$ is uniformly negative definite on R^+ if

$$h - \alpha + \frac{\alpha^2 K^4}{4\delta^2} L^2 h\big(1 + \mu B(t)B^T(t)\big)^2 < 0, \quad \forall t \geq 0$$

or

$$\big[1 + \mu \sup_{t\geq 0} \lambda_{max}\left[B(t)B^T(t)\right]\big]^2 < \frac{\alpha - h}{\alpha^2} \cdot \frac{4\delta^2}{K^4 L^2 h} \ .$$

Since $\max\limits_{\alpha>0}\left[\dfrac{\alpha - h}{\alpha^2}\right] = \dfrac{1}{4h}$, it is sufficient to choose μ such that

$$0 < \mu < \frac{\delta - K^2 Lh}{K^2 Lh \sup\limits_{t\geq 0} \lambda_{max}\left[B(t)B^T(t)\right]} \ .$$

In this case, Condition ii) of Proposition 2.3 is satisfied. The proof is completed. □

Example 4.1. Consider the following retarded control system in R^2

$$\dot{x}(t) = A(t)x(t) + \int_{-1}^{0} dG(s)x(t+s) + B(t)u(t), \ t \geq 0, \tag{22}$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}; \quad G(s) = \begin{pmatrix} 0.02\cos s & 0 \\ 0 & 0.02\cos s \end{pmatrix};$$
$$B(t) = \begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix}; \quad u \in R^1.$$

It is easy to verify that

$$\|U(t,s)\| \leq 8e^{-2(t-s)}, \quad t \geq s \geq 0;$$
$$\max_{-1\leq s\leq 0} \|G'(s)\| \leq 0.02\sin 1 \leq 0.018.$$

Taking $L = 0.018$ we have

$$\beta^2 := \lambda_{\max}[B(t)B^T(t)] = \|B(t)B^T(t)\| = 0.05$$

By Theorem 4.1 we can assert that system (22) is stabilizable by the linear feedback of form (19) provided

$$0 < \mu < \frac{\delta - hLK^2}{hLK^2\beta^2} = \frac{2 - 1 \cdot 0,018 \cdot 8^2}{1 \cdot 0.018 \cdot 0.05 \cdot 8^2} = 14.72...$$

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Department of Mathematics,
Hanoi University of Commerce

Department of Mathematics,
Hanoi University of Science

Hanoi Institute of Mathematics
P.O.Box 631 Bo Ho, 10000 Hanoi, Vietnam