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ON THE WEAK TAUTNESS AND THE LOCALLY WEAK TAUTNESS OF A DOMAIN IN A BANACH SPACE

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Abstract. It is shown that every Banach analytic manifold X which is weakly taut is hyperbolic in the sense of Kobayashi. A relation between the locally weak tautness and the weak tautness of a domain in a Banach space is also established.

1. Introduction

The main purpose of this paper is to establish a relation between the weak tautness and the hyperbolicity of a Banach analytic manifold as well as a relation between the locally weak tautness and the weak tautness of a unbounded domain in a Banach space.

One of the most essential problems of complex analysis is to derive global properties from the local ones. For example, in hyperbolic analysis it is proved that the global tautness and the local one are equivalent for bounded domains in $\mathbb{C}^n$. Is this equivalence is still true for unbounded domains in $\mathbb{C}^n$?

Recently, Gaussier [3] has solved this problem for unbounded domains in $\mathbb{C}^n$ for which there exist local peak and antipeak plurisubharmonic functions at infinity. We want to extend this result for unbounded domains in Banach spaces. However, the notion "taut" does not exist for the case of the domains in Banach spaces. Therefore, this notion should be changed in such a way that is suitable for this case. In this paper we first give the notion "weakly taut" and after that we extend a result of Gaussier to unbounded domains in Banach spaces.

The paper contains four sections. In Section 2 we give some definitions and notations. Section 3 is devoted to the relation between the hyperbolicity and the weak tautness. We prove that every Banach complex manifold which is weak taut is hyperbolic. The relation between the locally weak tautness and the weak tautness of an unbounded domain in a Banach space is considered in Section 4.

2. Basic notions

We shall make use of several properties of Banach analytic spaces presented in the books of Ramis [8] and Mazet [6].

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2.1. Let X be a Banach analytic space. As in the finite dimensional case we can define the Kobayashi pseudo-distance k_X on X. We say that X is hyperbolic if k_X is a distance defining the topology of X. It is known [2] that when $\dim X < \infty$ then k_X defines the topology of X if it is a distance. However this is not true in the case of infinite dimension [9].

2.2. As in the case of finite dimension, for $x \in X$ and $v \in T_x X$ the Kobayashi-Vesentini infinitesimal pseudo-metric $F(x, v)$ is defined by

$$F_X(x, v) = \inf\Big\{r > 0 : \exists f : \Delta \xrightarrow{hol} X,\ f(0) = x,\ f'(0) = \frac{v}{r}\Big\}.$$

For the case where X is a domain in a Banach space ,Vesentini [10] has proved that $F_X(x, v)$ is upper-semicontinuous on $T(X)$. Moreover, in this case he has shown that for two points p, q in X the following equality holds:

$$k_X(p, q) = \inf\Big\{ \int_0^1 F_X(\gamma(t), \gamma'(t))dt : \gamma \text{ is a piecewise } C^1\text{-curve joining } p \text{ with } q\Big\}.$$

For the case where X is a Banach manifold we don't know whether $F_X(x, v)$ is upper-semicontinuous on $T(X)$ or not. However it is easy to see that $F_X(x, v)$ is decreasing under holomorphic maps. Futhermore, for $x \in X$ we can choose a neighbourhood U_x of x which is isomorphic to a ball in a Banach space E. For all $x' \in U_x$ we can consider $T_{x'}X = E$. Then for $x' \in U_x$, and $v' \in T_{x'}X$ we have

$$F_X(x', v') \leq F_{U_x}(x', v').$$

Again using the above result of Vesentini we can conclude that $F_X(x, v)$ is locally upper bounded on $T(X)$. Hence we can define the upper-regulization of $F_X(x, v)$ by setting

$$F_X^*(x, v) = \limsup_{\substack{x' \to x \\ v' \to v}} F_X(x', v')$$

Then $F_X^*(x, v)$ is upper-semicontinuous on $T(X)$ and decreasing under holomorphic maps. Now, for two points $p, q \in X$ we define

$$k_X^*(p, q) = \inf\Big\{ \int_0^1 F_X^*(\gamma(t), \gamma'(t))dt : \gamma \text{ is a piecewise } C^1\text{-curve joining } p \text{ with } q\Big\}.$$

Then $k_X^*(p, q)$ is a pseudo-distance on X and $k_X^*(p, q)$ is decreasing under holomorphic maps. Hence

$$k_X^*(p, q) \leq k_X(p, q)$$

for all $p, q \in X$.

2.3. Since every infinite dimensional Banach space cannot be locally compact, we need to introduce a suitable change for the notion of tautness.

We say that X is weakly taut if for every sequence $\{f_n\} \subset Hol(\Delta, X)$, the space of holomorphic maps from the unit disc Δ in $\mathbb{C}$ to X equipped with the compact-open topology, one of the following two conditions holds:

(i) there exists a subsequence $\{f_{n_k}\}$ which is convergent in $Hol(\Delta, X)$,

(ii) there exists a discrete subset S of Δ and a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\big|_{\Delta\setminus S}\}$ is compactly divergent, i.e for every compact subset $K \subset \Delta \setminus S$ and $L \subset X$ there exists k_0 such that

$$f_{n_k}(K) \cap L = \emptyset$$

for $k \geq k_0$.

2.4. Let Ω be an unbounded domain in a Banach space E. A neighbourhood of infinity in Ω is a set containing the complement of a closed ball in Ω. If φ is a function defined on Ω and c a complex number, we set $\varphi(\infty) = c$ if

$$\lim \Big\{\varphi(z) : z \in \Omega,\ \|z\| \longrightarrow +\infty\Big\} = c.$$

2.5. Let Ω be a domain in a Banach space E. We say that Ω is locally weak taut if for each $p \in \partial\Omega$ there exists a neighbourhood U of p such that $U \cap \Omega$ is weakly taut.

The following definition is a minor modification of Gaussier's one.

2.6. A function φ is said to be a local peak plurisubharmonic function at a point $p \in \partial\Omega \cup \{\infty\}$ if there exists a neighbourhood V of p such that φ is plurisubharmonic on $\Omega \cap V$, continuous up to $\overline{\Omega} \cap V$, and satisfies

$$\begin{cases} \varphi(p) = 0, \\ \sup\big\{\varphi(z) : z \in \Omega \cap (V \setminus U)\big\} < 0 \end{cases}$$

for any neighbourhood U of p in V.

2.7. A function ψ is said to be a local antipeak plurisubharmonic function at $p \in \partial\Omega \cup \{\infty\}$ if there exists a neighbourhood V of p such that ψ is plurisubharmonic on $V \cap \Omega$, continuous up to $\overline{\Omega} \cap V$, and satisfies

$$\begin{cases} \psi(p) = -\infty, \\ \inf\big\{\psi(z) : z \in \Omega \cap (V \setminus U)\big\} > -\infty \end{cases}$$

for any neighbourhood U of p in V.

3. Weak tautness and hyperbolicity

In this section we establish the relation between the weak tautness and the hyperbolicity for a Banach analytic manifold.

Theorem 3.1. *Every Banach analytic manifold X which is weakly taut is hyperbolic.*

Proof. (i) First we show that for each $x_0 \in X$ there exist a neighbourhood U of x_0 and $c > 0$ such that $F_X(x, v) \geq c\|v\|$ for all $x \in U$ and $v \in T_x X$.

Indeed, in the converse case there exists $x_0 \in X$ and a sequence $\{x_n\} \subset X$, $x_n \longrightarrow x_0$ as $n \to \infty$ such that for each $n \geq 1$ $\exists\, f_n \in Hol(\Delta, X)$, $f_n(0) = x_n$ and $\|f_n'(0)\| \geq n$.

Let $\{\lambda_j\}$, $0 < |\lambda_j| < 1$ and $\lambda_j \to 0$. For each $n \geq 1$, put

$$\theta_n(\lambda) = \prod_{j=1}^{n} \frac{\lambda - \lambda_j}{1 - \overline{\lambda}_j \lambda} \quad \text{for } \lambda \in \Delta.$$

Then $\theta_n \in Hol(\Delta, \Delta)$ for $n \geq 1$ and

$$\begin{aligned} \theta_n(\lambda_j) &= 0 \quad \text{for} \quad 1 \leq j \leq n, \\ \theta_n'(\lambda_1) &\neq 0 \quad \text{for} \quad n \geq 1. \end{aligned}$$

Thus for each $k \geq 1$ we can find n_k such that

$$\left\| f_{n_k}'(\theta_k(\lambda_1))\theta_k'(\lambda_1) \right\| = \left\| f_{n_k}'(0) \right\| \, |\theta_k'(\lambda_1)| \geq k$$

for $k \geq 1$.

For $g_k = f_{n_k} \cdot \theta_k \in Hol(\Delta, X)$ we have

$$\lim_k g_k(\lambda_j) = \lim_k f_{n_k}(\theta_k(\lambda_j)) = \lim_k f_{n_k}(0) = \lim_k x_{n_k} = x_0$$

for all $j \geq 1$. By the hypothesis there exists a subsequence $\left\{g_{k_p}\right\} = \left\{f_{n_{k_\rho}} \cdot \theta_{k_p}\right\}$ of $\{g_k\}$ which is convergent in $Hol(\Delta, X)$. Hence

$$\sup_{p \geq 1} \left\| g_{k_p}'(\lambda_1) \right\| < \infty.$$

However

$$\sup_{p \geq 1} \left\| g_{k_p}'(\lambda_1) \right\| = \sup_p \left\| f_{n_{k_p}}'(0)\theta_{k_p}'(\lambda_1) \right\| = \infty.$$

(ii) Now we show that k_X is a metric on X. Let p, q be two different points on X. Take a neighbouhood U_q of q such that $p \notin \overline{U}_q$. By (i) we can choose a neighbourhood V_q of q, $V_q \subset U_q$, and $c > 0$ such that

$$F_X(x, v) \geq c\|v\|$$

for all $x \in V_q$ and $v \in T_x X$.

Take a ball $B(q, r) \subset V_q$. Let $\gamma : [0, 1] \to X$ be a piecewise C^1-curve joining q to p, $\gamma(0) = q$, $\gamma(1) = p$. Let $0 < s < 1$ such that $\gamma([0, s)) \subset B(q, r)$ and $\gamma(s) = y \in \partial B(q, r)$. Then

$$\begin{aligned} \int_0^1 F_X^*(\gamma(t), \gamma'(t))dt &\geq \int_0^s F_X^*(\gamma(t), \gamma'(t))dt \\ &\geq c \int_0^s \|\gamma'(t)\|dt \geq c\|\gamma(s) - \gamma(0)\| = cr > 0. \end{aligned}$$

Hence

$$k_X(p,q) \geq k_X^*(p,q) \geq cr > 0.$$

Thus k_X is a metric on X.

(iii) We show that k_X defines the topology of X. First we assume that $k_X(x_n, x_0) \to 0$ but $x_n \not\to x_0$ as $n \to \infty$. We choose a connected neighbourhood U of x_0 and $c > 0$ such that

$$F_X(x,v) \geq c\|v\| \tag{1}$$

holds for all $x \in U$ and $v \in T_xX$. Next we choose a neighbourhood V of x_0 such that $V \subset U$. Without loss of generality we may assume that

$$V = B(x_0, r) = \big\{x \in E : \|x - x_0\| < r\big\} \subset U.$$

Since $\{x_n\} \not\to x_0$ there exists a subsequence $\{y_n\} \subset \{x_n\}$ such that $\{y_n\} \notin B(x_0, r)$ for $n \geq 1$. We have

$$k_X^*(y_n, x_0) = \inf\Big\{ \int_0^1 F_X^*(\gamma(t), \gamma'(t))dt : \gamma \in \Omega_{y_n,x_0}\Big\} \tag{2}$$

where Ω_{y_n,x_0} denotes the set of piecewise C^1-curves $\gamma : [0,1] \to X$, $\gamma(0) = x_0$, $\gamma(1) = y_n$. According to (2) , we can find $\gamma_n \in \Omega_{y_n,x_0}$ such that

$$k_X^*(y_n, x_0) \geq \int_0^1 F_X^*(\gamma_n(t), \gamma_n'(t))dt - \frac{1}{n}\cdot$$

Let $0 < t_n < 1$ such that

$$\gamma_n([0,t_n)) \subset B(x_0, r) \quad \text{and} \quad y_n' = \gamma_n(t_n) \in \partial B(x_0, r).$$

Then

$$k_X^*(y_n', x_0) \leq \int_0^{t_n} F_X^*(\gamma_n(t), \gamma_n'(t))dt \leq \int_0^1 F_X^*(\gamma_n(t), \gamma_n'(t))dt$$
$$< k_X^*(y_n, x_0) + \frac{1}{n} \leq k_X(y_n, x_0) + \frac{1}{n}\cdot$$

Thus

$$k_X^*(y_n', x_0) \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

It is easy to see that x_0, y_n' belong to U. Choose $\beta_n \in \Omega_{y_n',x_0} \subset U$ such that

$$k_X^*(y_n', x_0) \geq \int_0^1 F_X^*(\beta_n(t), \beta_n'(t))dt - \frac{1}{n}$$
$$\geq \int_o^{s_n} F_X^*(\beta_n(t), \beta_n'(t))dt - \frac{1}{n}\,,$$

where $0 < s_n < 1$ is chosen such that $\beta_n([0, s_n)) \subset B(x_0, r)$ and $\beta_n(s_n) \in \partial B(x_0, r)$.

Using (1) and the definition of F^*_X we have

$$\int_0^{s_n} F^*_X(\beta_n(t), \beta'_n(t))dt \geq c \int_0^{s_n} \|\beta'_n(t)\|dt \geq c\|\beta_n(s_n) - \beta_n(0)\| = cr \quad \text{for all } n \geq 1.$$

This is impossible because

$$\int_0^{s_n} F^*_X(\beta_n(t), \beta'_n(t))dt \leq k^*_X(y'_n, x_0) + \frac{1}{n} \longrightarrow 0$$

as $n \to \infty$.

On the other hand, since X is a Banach manifold, by using a similar argument as in [5, Proposition 1.7] we can show that if $y_n \to y_0$ on X then $k_X(y_n, y_0) \longrightarrow 0$. Theorem 3.1 is proved. □

4. Weak tautness of domains in Banach spaces

Similarly as in [3], we will prove the following

Theorem 4.1. *Let Ω be a domain in a Banach space E. Suppose that there exist local peak and antipeak plurisubharmonic functions at infinity. Then Ω is hyperbolic.*

Similarly as in [3], we first prove the following

Lemma 4.1. *Let p be a point in $\partial\Omega \cup \{\infty\}$. Assume that there are local peak and antipeak plurisubharmonic functions φ and ψ at p, both defined on a neighbourhood V_p of p. Then for every neighbourhood U of p there exists a neighbourhood U' of p such that every holomorphic map $f : \Delta \to \Omega$ satisfies*

$$f(0) \in U' \Rightarrow f\big(\Delta_{\frac{1}{2}}\big) \subset U,$$

where $\Delta_{\frac{1}{2}} = \big\{z \in \Delta : |z| < \dfrac{1}{2}\big\}$.

Proof. Since φ is a local peak plurisubharmonic function at p, there exist neighbourhoods U and V of p, $\overline{U} \subset V \subset V_p$ and two positive constants c, c' $(c > c')$ such that

$$\begin{cases} \inf\limits_{z \in \overline{\Omega} \cap \partial U} \varphi(z) &= -c', \\ \sup\limits_{z \in \overline{\Omega} \cap \partial V} \varphi(z) &= -c. \end{cases}$$

Then the function $\widetilde{\varphi}$ defined on $\overline{\Omega}$ by the formula

$$\begin{cases} \widetilde{\varphi}(z) = \varphi(z) & \text{if } z \in \overline{\Omega} \cap U, \\ \widetilde{\varphi}(z) = \sup\Big(\varphi(z), -\dfrac{c+c'}{2}\Big) & \text{if } z \in \overline{\Omega} \cap (V \setminus \overline{U}), \\ \widetilde{\varphi}(z) = -\dfrac{c+c'}{2} & \text{if } \quad z \in \overline{\Omega} \setminus V \end{cases}$$

is a global negative peak plurisubharmonic function at p.

Let $f \in Hol(\Delta, \Omega)$. Since $\widetilde{\varphi} \cdot f$ is subharmonic and $(\widetilde{\varphi} \cdot f)(e^{i\theta}) \leq 0$ for all $\theta \in [0, 2\pi]$, for every negative α such that $(\widetilde{\varphi} \cdot f)(0) > \alpha$ we have $mes(E_\alpha) \geq \pi$ where

$$E_\alpha = \big\{\theta \in [0, 2\pi] : (\widetilde{\varphi} \cdot f)(e^{i\theta}) \geq 2\alpha\big\}.$$

Indeed,

$$\begin{aligned} \alpha < (\widetilde{\varphi} \cdot f)(0) &\leq \frac{1}{2\pi} \int_0^{2\pi} (\widetilde{\varphi} \cdot f)(e^{i\theta}) d\theta \\ &= \frac{1}{2\pi} \int_{E_\alpha} \big(\widetilde{\varphi} \cdot f\big)(e^{i\theta}) d\theta + \frac{1}{2\pi} \int_{\partial\Delta \setminus E_\alpha} \big(\widetilde{\varphi} \cdot f\big)(e^{i\theta}) d\theta \\ &\leq \frac{1}{2\pi} \int_{\partial\Delta \setminus E_\alpha} (\widetilde{\varphi} \cdot f)(e^{i\theta}) d\theta < \frac{1}{2\pi} \cdot 2\alpha \; mes(\partial\Delta \setminus E_\alpha) \\ &= \frac{\alpha}{\pi}(2\pi - mes E_\alpha). \end{aligned}$$

Hence

$$mes(E_\alpha) \geq \pi. \tag{1}$$

Choose now a sufficiently small positive number ε such that

$$\begin{cases} \inf\limits_{\overline{\Omega} \cap \partial U} (\varphi + \varepsilon\psi) & = -c_1 < 0, \\ \sup\limits_{\overline{\Omega} \cap \partial V} (\varphi + \varepsilon\psi) & = -c_2 < -c_1. \end{cases}$$

Define a function ρ on $\overline{\Omega}$ by setting

$$\rho(z) = \begin{cases} (\varphi + \varepsilon\psi)(z) & \text{if } z \in \overline{\Omega} \cap U, \\ \sup\Big((\varphi + \varepsilon\psi)(z), -\dfrac{c_1 + c_2}{2}\Big) & \text{if } z \in \overline{\Omega} \cap (V \setminus \overline{U}), \\ -\dfrac{c_1 + c_2}{2} & \text{if } z \in \overline{\Omega} \setminus V. \end{cases}$$

Then $\rho(z)$ is a continuous negative plurisubharmonic function on Ω and $\rho^{-1}(-\infty) = \{p\}$. Using the Poisson integral we have

$$\rho f(\lambda) \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} \rho f(e^{i\theta}) d\theta$$

for $|\lambda| = r < \dfrac{1}{2}$.

Since $0 \le r \le \dfrac{1}{2}$, we have

$$\frac{1-r^2}{1-2r\cos(\theta - t)+r^2} \ge \frac{1}{3}$$

for $0 \le \theta \le 2\pi$. Hence

$$\rho f(\lambda) \le \frac{1}{6\pi}\int_0^{2\pi}(\rho\cdot f)(e^{i\theta})d\theta \quad \text{for} \quad |\lambda| < \frac{1}{2}. \tag{2}$$

Since $\widetilde{\varphi}$ is a peak function at p and ρ satisfies $\rho(p) = -\infty$, for each positive constant L there exists a negative constant α such that for any point z in $\overline{\Omega}$ the inequality $\widetilde{\varphi}(z) \ge 2\alpha$ implies $\rho(z) < -L$. Using inequalities (1), (2) and the fact that ρ is a negative function we can show that for every $f \in Hol(\Delta, \Omega)$ and $\lambda \in \Delta_{\frac{1}{2}}$,

$$\widetilde{\varphi}(f(0)) > \alpha \Rightarrow (\rho\cdot f)(\lambda) \le -\frac{L}{6}\cdot \tag{3}$$

Since $\rho^{-1}(-\infty) = \{p\}$, the family

$$U_n = \Big\{z \in \overline{\Omega} : \rho(z) < \Big(-\frac{1}{6}\Big)n\Big\}$$

is a neighbourhood basis of p in $\overline{\Omega}$. For each positive integer n there exists a negative constant α_n such that

$$\widetilde{\varphi}(z) \ge 2\alpha_n \Rightarrow \rho(z) < -n.$$

Let U_n' be the neighbourhood of p in $\overline{\Omega}$ defined by

$$U_n' = \big\{z \in \overline{\Omega} : \widetilde{\varphi}(z) > \alpha_n\big\}.$$

Then from (3) it follows that if $f \in Hol(\Delta, \Omega)$ and $f(0) \in U_n'$ then $f\big(\Delta_{\frac{1}{2}}\big) \subset U_n$. Lemma 4.1 is proved. □

Proof of Theorem 4.1. As in Theorem 3.1, it suffices to show the following: $\forall a \in \Omega$ $\exists$ a neighbourhood U of a and $c > 0$ such that

$$F_\Omega(z, u) \ge c\|u\|$$

for $z \in U$ and $u \in E$.

To obtain a contradiction suppose that we can find $z_0 \in \Omega$ and a sequence $\{z_n\} \subset \Omega$, $z_n \to z_0$ as $n \to \infty$, $\{u_n\} \subset E$, $\|u_n\| = 1$ such that

$$F_\Omega(z_n, u_n) \le \frac{1}{n}. \tag{4}$$

Choose a sequence $\{f_n\} \subset Hol(\Delta, \Omega)$ such that

$$f_n(0) = z_n, \quad \|f_n'(0)\| \ge n. \tag{5}$$

Assume that there exists $\varepsilon > 0$ such that

$$M = \sup\Big\{\|f_n(\lambda)\| : |\lambda| \le \varepsilon, n \ge 1\Big\} < \infty.$$

Then

$$\|f_n'(0)\| = \Big\| \frac{1}{2\pi i} \int\limits_{|t|=\varepsilon} \frac{f_n(t)dt}{t^2} \Big\| \le \frac{M}{\varepsilon}$$

for all $n \ge 1$. This contradicts (5). Hence, we can find $\{\lambda_k\}$, $\lambda_k \to 0$ such that $\big\|f_{n_k}(\lambda_k)\big\| \to \infty$. Composing every f_{n_k} with a Moebius transform we get a family $\big\{\tilde{f}_{n_k}\big\} \subset Hol(\Delta, \Omega)$ such that $\tilde{f}_{n_k}(0) = f_{n_k}(\lambda_k)$ and $\tilde{f}_{n_k}(\lambda_k) = z_{n_k}$. This contradicts Lemma 4.1. $\square$

We now establish the weak tautness of an unbounded domain in a Banach space through the locally weak tautness.

Theorem 4.2. *Let Ω be a domain in a Banach space E. Assume that Ω is locally weak taut at each point in $\partial\Omega$ and that there are local peak and antipeak plurisubharmonic functions at infinity. Then Ω is a weakly taut domain.*

Proof. Given $\{f_n\} \subset Hol(\Delta, \Omega)$.

First case. Suppose that there exists a point λ in Δ and a subsequence $\big\{g_n\big\}_{n\in A} \subset \{f_n\}$ such that

$$\lim_{n\in A} \|g_n(\lambda)\| = \infty.$$

Let

$$C = \Big\{z \in \Delta : \lim_{n\in A} \|g_n(z)\| = \infty\Big\}.$$

By Lemma 4.1, for any point $\lambda \in C$, we have

$$\lim_{n\in A} \big\|g_n \cdot h_{\lambda,\theta}\big\| = +\infty$$

uniformly on $\Delta_{\frac{1}{2}}$, where

$$h_{\lambda,\theta}(z) = \frac{\lambda - e^{i\theta}z}{1 - e^{i\theta}\overline{\lambda}z} \tag{6}$$

is an automorphism of Δ.

According to Lemma 2.2.1 in [3], for each $\lambda \in C$ there exists a positive real number r_λ such that

$$\lim_{n\in A} \big\|g_n\big(\overline{\Delta(\lambda, r_\lambda)}\big)\big\| = +\infty,$$

where

$$\Delta(\lambda, r_\lambda) = \big\{z \in \Delta : |z - \lambda| < r_\lambda\big\}.$$

Thus C is open in Δ. Moreover, if $\big(\lambda_k\big)_k$ is a sequence of points in Δ converging to a point λ in Δ, the compactness of the set $\overline{\{\lambda_k,\ k \ge 1\} \cup \{\lambda\}}$ implies by [3, Lemma 2.2.1] that there exists a positive real constant r such that for every positive integer k $\lim_{n\in A}\|g_n\| = +\infty$ uniformly on $\Delta(\lambda_k, r)$. Hence $\lim_{n\in A}\|g_n(\lambda)\| = +\infty$. Therefore, the set C is closed in Δ. This implies $C = \Delta$. Hence for each $\lambda \in \Delta$ there exists a positive real number r_λ such that $\lim_{n\in A}\|g_n\| = +\infty$ uniformly on $\Delta(\lambda, r_\lambda)$. In particular, the sequence $\big(g_n\big)_{n\in A}$ diverges to infinity uniformly on compact subsets of Δ. Hence the sequence $\big(g_n\big)_{n\in A}$ is compactly divergent.

Second case. Assume that for every point $\lambda \in \Delta$ and every subsequence $\{f_{n_k}\} \subset \{f_n\}$ the subsequence $\{f_{n_k}(\lambda)\}$ is bounded. In this case we prove that $\{f_n\}$ is locally bounded. Indeed, assume to the contrary that $\{f_n\}$ is not locally bounded. Then there is $\lambda_0 \in \Delta$ and a sequence $\{\lambda_k\}$, $\lambda_k \to \lambda_0$ as $k \to \infty$ and a subsequence $\{f_{n_k}\}$ of the sequence $\{f_n\}$ such that $\lim_k \|f_{n_k}(\lambda_k)\| = +\infty$. Lemma 2.1.1 of [3] implies that

$$\lim_k \|f_{n_k} \cdot h_{\lambda_k,\theta}\| = +\infty$$

uniformly on $\Delta_{\frac{1}{2}}$ where $h_{\lambda_k,\theta}$ is defined as in (6). Using Lemma 2.2.1 of [3] for the compact set $\{\lambda_k,\ k \geq 1\} \cup \{\lambda_0\}$ we conclude that there is a positive constant $r > 0$ such that

$$\lim_k \|f_{n_k}\big(\Delta(\lambda_k, r)\big)\| = +\infty.$$

Hence

$$\lim_k \|f_{n_k}(\lambda_0)\| = +\infty.$$

This contradicts the boundedness of the subsequence $\{f_{n_k}(\lambda_0)\}$.

Now we suppose that $\{f_n\}$ is not compactly divergent. By Theorem 4.1, Ω is hyperbolic. Since k_Ω is a distance decreasing in holomorphic maps we deduce that there is $\lambda_0 \in \Delta$ and subsequence $\{g_n\}_{n\in A} \subset \{f_n\}$ such that $\{g_n(\lambda_0)\}_{n\in A}$ converges to $x_0 \in \Omega$. Let

$$C = \Big\{\lambda \in \Delta : \exists \lim_{n\in A} g_n(\lambda)\Big\}.$$

By C' we denote the set of limit points of C in Δ.

(a) Assume that $C' \neq \emptyset$. Since $\{g_n\}_{n\in A}$ is locally bounded, the sequence $\{g_n\}_{n\in A}$ converges to g in $Hol(\Delta, E)$ by [1]. If $g(\Delta) \subset \Omega$ then Ω is weakly taut. In the converse case, there exists $\lambda_1 \in \Delta$ such that $g(\lambda_1) \in \partial\Omega$.

Put

$$D = \big\{\lambda \in \Delta : g(\lambda) \in \partial\Omega\big\}.$$

By the above argument, D is nonempty. Moreover, it is easy to see that D is closed in Δ. Let λ be a point in D. Then $p = g(\lambda) \in \partial\Omega$. Since Ω is locally weak taut at p, there exists a neighbourhood U of p such that $U \cap \Omega$ is weakly taut. Since $\{g_n\}_{n\in A}$ converges to g in $Hol(\Delta, E)$, there exists $\delta > 0$ such that $g_n(\Delta(\lambda, \delta)) \subset U \cap \Omega$ for $n \in A$, $n \geq n_0$. Since $U \cap \Omega$ is weakly taut and $g(\lambda) \in \partial\Omega$, there exists a discrete subset $S \subset \Delta(\lambda, \delta)$ such that $g\big(\Delta(\lambda, \delta) \setminus S\big) \subset \partial\Omega$. We may choose a sufficiently small $0 < \delta_1 < \delta$ such that $g(\Delta(\lambda, \delta_1)) \subset \partial\Omega$. Hence D

is open in Δ and, therefore, $D = \Delta$. Thus $g(\Delta) \subset \partial\Omega$. This is imposible because $g(\lambda_0) = x_0 \in \Omega$.

(b) Assume that $C' = \emptyset$. By $\mathcal{A}$ we denote the set

$$\mathcal{A} = \Big\{U : U \text{ is open in } \Delta \text{ and there exists a subsequence}$$
$$\big\{g_n^U\big\} \subset \big\{g_n\big\}_{n\in A} \text{ such that } \big\{g_n^U\big\} \text{ is compactly divergent on } U\Big\}$$

We endow $\mathcal{A}$ with the ordered relation defined as follows: write $U_1 < U_2$ if $U_1 \subset U_2$ and every $\big\{g_n^{U_1}\big\} \subset \{g_n\}$ which is compactly divergent on U_1 contains $\big\{g_n^{U_2}\big\} \subset \big\{g_n^{U_1}\big\}$ such that $\big\{g_n^{U_2}\big\}$ is compactly divergent on U_2.

First we check that $\mathcal{A} \neq \emptyset$. To do this, we shall prove that $\exists \varepsilon > 0 \;\exists \big\{g_n\big\}_{n\in B} \subset \big\{g_n\big\}_{n\in A}$ such that $\big\{g_n\big\}_{n\in B}$ is compactly divergent on

$$\Delta^*(\lambda_0, \varepsilon) = \big\{\lambda \in \Delta : 0 < |\lambda - \lambda_0| < \varepsilon\big\}.$$

Consequently, $\Delta^*(\lambda_0, \varepsilon) \in \mathcal{A}$. Vice-versa, for each k there exists a subsequence $B_k \subset A$ and $\lambda_k : 0 < |\lambda_k - \lambda_0| < \dfrac{1}{k}$ such that the subsequence $\big\{g_n(\lambda_k)\big\}_{n\in B_k}$ is convergent. Moreover, we may asume that B_{k+1} is a subsequence of B_k. Then by the diagonal process we can find a subsequence $\big\{g_n\big\}_{n\in B} \subset \big\{g_n\big\}_{n\in A}$ such that λ_0 is a limit point of the set $\widetilde{C} = \Big\{\lambda \in \Delta : \exists\ \lim_{n\in B} g_n(\lambda)\Big\}$. Hence the above argument shows that $\big\{g_n\big\}_{n\in B}$ is convergent in $Hol(\Delta, \Omega)$ and, therefore, Ω is weakly taut.

Now let $\big\{U_\alpha\big\}_{\alpha\in I}$ be a linearly ordered subset of $\mathcal{A}$. The Lindelofness of $\mathbb{C}$ implies that there exists

$$U_{\alpha_1} < U_{\alpha_2} < \cdots < U_{\alpha_n} < \dots$$

such that

$$U = \bigcup_{\alpha\in I} U_\alpha = \bigcup_{j=1}^{\infty} U_{\alpha_j}.$$

By the diagonal process we can find a subsequence $\big\{g_n\big\}_{n\in B} \subset \big\{g_n\big\}_{n\in A}$ such that it is compactly divergent on U. Hence $U \in \mathcal{A}$. Moreover, by the same argument we can show that for every sequence $\big\{g_n^{U_{\alpha_j}}\big\} \subset \{g_n\}$, which is compactly divergent on U_{α_j}, there exists a subsequence $\big\{g_n\big\}_{n\in J} \subset \big\{g_n^{U_{\alpha_j}}\big\}$ which is compactly divergent on U. We now prove that the family $\big\{U_\alpha\big\}_{\alpha\in I}$ has a supremum. Let $\alpha_0 \in I$. If we can find j_0 such that $U_{\alpha_0} < U_{\alpha_{j_0}}$ then the above argument shows that $U_{\alpha_0} < U$. Hence U is a supremum of the family $\big\{U_\alpha\big\}_{\alpha\in I}$. Vice-versa, let $U_{\alpha_j} < U_{\alpha_0}$ for all α_j. Then $U_{\alpha_0} = U$. On the other hand, since $\big\{U_\alpha\big\}_{\alpha\in I}$ is a linearly ordered subset, for each $\beta \in I$ either $U_\beta < U_{\alpha_0}$ or $U_{\alpha_0} < U_\beta$. As $U_{\alpha_0} = U$, from the definition of $\mathcal{A}$ together with the ordered relation over it we deduce that $U_\beta < U_{\alpha_0}$ for all $\beta \in I$. Thus, U_{α_0} is a supremum of the family $\big\{U_\alpha\big\}_{\alpha\in I}$.

By the Zorn lemma, $\mathcal{A}$ has a maximal element Ω. Assume that $\{g_n^\Omega\}_{n\in J}$ is a compactly divergent subsequence associated to Ω. Let

$$T = \Big\{\lambda \in \Delta \setminus \Omega : \text{there exists a subsequence } \{g_n\}_{n\in\widetilde{J}} \subset \{g_n^\Omega\}_{n\in A} \text{ such that } \{g_n(\lambda)\}_{n\in\widetilde{J}} \text{ is convergent in } \Omega\Big\}.$$

For each $\lambda \in T$, by using the argument in the preceding part of this proof we can find $\varepsilon > 0$ and a subsequence $\{h_n\}_{n\in J_1} \subset \{g_n\}_{n\in\widetilde{J}}$ such that $\{h_n\}_{n\in J_1}$ is compactly divergent on $\Delta^*(\lambda,\varepsilon)$. Then $\{h_n\}_{n\in J_1}$ is compactly divergent on $\Delta^*(\lambda,\varepsilon)\cup\Omega$. The maximality of Ω implies that $\Delta^*(\lambda,\varepsilon)\subset\Omega$. This shows that T is discrete and $\{g_n^\Omega\}_{n\in J}$ is compactly divergent on $\Omega\setminus T$.

Consequencely, from the definition of the weakly taut notion we can conclude that Ω is weakly taut. Theorem 4.3 is proved. □

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