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MILD SOLUTIONS OF NONLINEAR EVOLUTION FUNCTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACE

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Abstract. The existence of a mild solution of the abstract functional differential inclusion

$$P(\varphi)\begin{cases} u'(t) + Au(t) - F(t, u_t) \ni 0, & t \in I = [0, T] \\ u(t) = \varphi(t), & t \in J = [-r, 0] \end{cases}$$

is obtained by a Filippov technique. Here A is an operator such that $A + \omega\mathcal{I}$ is m-accretive for some $\omega \in \mathbb{R}^+$, and the multimapping $F(t, .)$ is $h(t)$-Lipschitzian. The result is applied to a nonlinear functional control problem.

1. Introduction

Let $(E, |.|)$ be a real Banach space. Let $\mathcal{C} := C([-r, 0]; E)$ be the Banach space of continuous functions from $J := [-r, 0]$ to E with the usual supremum norm $\|.\|$. For any $u \in \mathcal{C}_T := C([-r, T]; E)$ and any $t \in I := [0, T]$ $(T > 0)$, we denote by u_t the element of $\mathcal{C}$ defined by $u_t(\theta) = u(t + \theta)$, $\theta \in J$.

We consider the nonlinear evolution functional differential inclusion

$$P(\varphi)\begin{cases} u'(t) + Au(t) - F(t, u_t) \ni 0, \ t \in I, \\ u_0 = \varphi, \ \varphi \in \mathcal{C} \end{cases}$$

where A is an operator such that $A + \omega\mathcal{I}$ is m-accretive for some $\omega \geq 0$ and $F : I \times \mathcal{C} \to \mathcal{F}(E)$ is a multimapping with $\mathcal{F}(E)$ being the family of all nonempty, closed and bounded subsets of E. Such a inclusion is a convenient tool to investigate for instance the control problem

$$\begin{cases} u'(t) + Au(t) - f(t, u_t, w(t)) \ni 0, \quad w(t) \in W(t), \\ u_0 = \varphi, \end{cases}$$

where W is a multimapping of controls. Setting

$$F(t, u_t) = \{f(t, u_t, w(t)) : \ w(t) \in W(t)\}$$

we reduce to above control problem to the above inclusion $P(\varphi)$.

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The study of evolution functional differential equations (or inclusions) has received much attention over the last forty years. Various conditions on A and F have been considered, and existence results for different spaces of initial functions have been obtained. When A and F are singlevalued, we mention here the work of Travis and Webb ([16]; A linear and $F(t,\psi)=F(\psi)$), Webb ([17]; $A+\omega\mathcal{I}$ accretive for some $\omega\in\mathbb{R}$ and $F(t,\psi)=F(\psi)$), Kartsatos and Parrott ([10, 11]; $A(t)$ nonlinear m-accretive operator and F is Lipschitz continuous in both variables)... When A is singlevalued and F a multimapping, we mention the work of Obukhovskii ([13]; A linear and F is a compact convex valued multimapping),... When A is a nonlinear m-accretive operator and $F(t,\psi)=F(\psi)$ is a Lipschitz continuous function, we mention the works of Ruess and Summers [15], and Ruess [14],... The purpose of this paper is to establish, under certain additional assumptions, the existence of a mild solution $u:[-r,T]\to E$ (to be defined precisely later) of $P(\varphi)$. Our technique for proving the existence of mild solution u of $P(\varphi)$ is by showing that the solution is the uniform limit of the sequence (u^n), where

$$u^n=\begin{cases}\varphi & \text{on } J\\ v^n & \text{on } I\end{cases}$$

and v^n are mild solutions of problems

$$(P_{f_n})\begin{cases}(v^n)'(t)+Av^n(t)\ni f_n(t)\\ v^n(0)=\varphi(0)\end{cases}$$

with $f_n(t)\in F(t,u_t^{n-1})$ a.e.

2. Preliminaries

Let (X,d) be a metric space, $\mathcal{F}(X)$ the family of all nonempty, closed and bounded subsets of X and δ the Hausdorff distance in $\mathcal{F}(X)$, i.e., for $A,B\in\mathcal{F}(X)$

$$\delta(A,B)=\max\big[\sup_{a\in A}d(a,B),\sup_{b\in B}d(b,A)\big]$$

where $d(a,B)=\inf\limits_{b\in B}d(a,b)$.

Let $G:I\to\mathcal{P}(E)$ (the family of all nonempty subsets of E) be a multimapping. A function $g:I\to E$ such that $g(s)\in G(s)$ for every $s\in I$ is called a selection of G. G is called measurable if for almost all $s\in I$, $G(s)\subset$ closure $\{g_n(s):n\in\mathbb{N}\}$ where g_n are measurable selections of G (see [18]). By the symbol of I_G^1 we will denote the set of all Bochner integrable selections of the multimapping G, i.e.

$$I_G^1=\{g\in L^1(I;E):g(s)\in G(s)\text{ a.e.}\}.$$

If $I_G^1\neq\emptyset$ then the measurable multimapping G is called integrable and

$$\int\limits_I G(s)ds=\Big\{\int\limits_I g(s)ds:g\in I_G^1\Big\}.$$

Lemma 2.1. [18] *Let $G : I \to \mathcal{P}(E)$ be a measurable multimapping and $u : I \to E$ a measurable function. Then for any measurable function $v : I \to \mathbb{R}^+$, there exists a measurable selection g of G such that*

$$|g(s) - u(s)|_E \leq d(u(s), G(s)) + v(s) \ a.e.$$

Definition 2.1 [4]. Let $A : D(A) \subset E \to \mathcal{P}(E)$ be an operator in E, where $D(A) = \{x \in E : Ax \neq \emptyset\}$ is the effective domain of A. A is accretive in E if

$$|u - \widehat{u} + \lambda(v - \widehat{v})| \geq |u - \widehat{u}|$$

whenever $\lambda \geq 0$ and (u, v), $(\widehat{u}, \widehat{v}) \in A$. A is m-accretive in E if

$$R(\mathcal{I} + \lambda A) := \bigcup_{u \in D(A)} (u + \lambda Au) = E \ \text{ for all } \lambda > 0.$$

At last, we present some basic concepts and results concerning the Cauchy problem $P(A, x, f)$

$$\begin{cases} u' + Au \ni f \\ u(0) = x \end{cases}$$

where A is an operator in E, $x \in E$ and $f \in L^1(I; E)$ (see [2, 3, 4]). We refer the reader to [7, 9, 12] for some informations and references about nonlinear evolution inclusions and their applications.

Definition 2.2 [4]. Let $\varepsilon > 0$. An ε-approximate solution of problem $P(A, x, f)$ on I is a piecewise constant function $v : I \to E$ such that there exist a partition $(0 = t_0 < t_1 < \cdots < t_N = T)$ of I and two finite sequences of E $(f_1, \ldots, f_N)$ and $(v_0, v_1, \ldots, v_N)$ satisfying the following properties

i) $\underset{1 \leq i \leq N}{\text{Max}} (t_i - t_{i-1}) < \varepsilon$;

ii) $v(0) = x = v_0$ and $v(t) = v(t_i) = v_i$ on $]t_{i-1}, t_i]$, $i = 1, \ldots, N$;

iii) $\dfrac{v_i - v_{i-1}}{t_i - t_{i-1}} + Av_i \ni f_i, \ \ i = 1, \ldots, N$;

iv) $\displaystyle\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} |f(s) - f_i| \, ds < \varepsilon$.

Definition 2.3 [4]. A mild solution of $P(A, x, f)$ on I is a function $u \in C(I; E)$ such that for each $\varepsilon > 0$ there is an ε-approximate solution v of problem $P(A, x, f)$ on I such that

$$|u(t) - v(t)| < \varepsilon \text{ for } t \in I.$$

Theorem 2.1. [4] *Let $\omega \in \mathbb{R}$ such that $A + \omega\mathcal{I}$ is m-accretive in E and $f \in L^1(0, T; E)$. Then for every $x \in \overline{D(A)}$, the initial value problem $P(A, x, f)$ has a* *unique mild solution on I. Moreover, if w is a mild solution on I of $P(A, y, g)$, then*

$$e^{-\omega t}|u(t)-w(t)| - e^{-\omega s}|u(s)-w(s)| \leq \int_s^t e^{-\omega\tau}|f(\tau)-g(\tau)|\,d\tau$$

for $0 \leq s \leq t \leq T$.

Definition 2.4 [4]. Let A_n be an operator in E for each positive integer n. Then $\liminf\limits_{n\to\infty} A_n$ is the operator defined by $(x,y) \in \liminf\limits_{n\to\infty} A_n$ if there are $(x_n, y_n) \in A_n$ such that $(x,y) = \lim\limits_{n\to\infty}(x_n, y_n)$. In particular, if $A_n = A$ for all $n \in \mathbb{N}$, then $\liminf\limits_{n\to\infty} A_n = \overline{A} =$ closure of A (i.e. graph of $\overline{A}$ is the closure of the graph of A in $E \times E$).

Theorem 2.2. [4] *Let $\omega \in \mathbb{R}$ such that $A_n + \omega\mathcal{I}$ is m-accretive in $E, x_n \in \overline{D(A_n)}$ and $f_n \in L^1(I;E)$ for $n = 1, 2, \ldots, \infty$. Let u_n be the mild solution of $P(A_n, x_n, f_n)$ on I for each n. If $\lim\limits_{n\to\infty} f_n = f$ in $L^1(I;E)$ and $\lim\limits_{n\to\infty} x_n = x$ and $A_\infty \subset \liminf\limits_{n\to\infty} A_n$, then $\lim\limits_{n\to\infty} u_n(t) = u(t)$ uniformly on I.*

3. Main result

Consider the nonlinear evolution functional differential inclusion

$$P(\varphi)\begin{cases} u'(t) + Au(t) - F(t, u_t) \ni 0, \ t \in I \\ u_0 = \varphi, \ \varphi \in \mathcal{C} \end{cases}$$

under the following assumptions:

(A) There exists $\omega \in \mathbb{R}^+$ such that $A + \omega\mathcal{I}$ is m-accretive, $\varphi(0) \in \overline{D(A)}$.

(F) The multimapping $F : I \times \mathcal{C} \to \mathcal{F}(E)$ satisfies the conditions:

(F_1) For every $\phi \in \mathcal{C}$, the multimapping $F(., \phi)$ is measurable on I.

(F_2) There is an integrable function $h : I \to \mathbb{R}^+$ such that for every $\phi, \xi \in \mathcal{C}$,

$$\delta(F(t,\phi), F(t,\xi)) \leq h(t)\|\phi - \xi\| \text{ a.e. in } I.$$

(F_3) The function $q : t \longmapsto d(0, F(t, 0))$ is integrable on I.

For $u \in C_T$ and $t \in I$, let $G_u : I \to \mathcal{F}(E)$ be the measurable multimapping defined by $G_u(s) = F(s, u_s)$ for every $s \in I$. We consider

$$I^1_{G_u} = \left\{ f \in L^1(I;E) : f(s) \in F(s, u_s) \text{ a.e. } s \in I \right\}.$$

We have that $I^1_{G_u}$ is nonempty.

Definition 3.1. A function $u \in C_T$ is called a mild solution of problem $P(\varphi)$ if $u(t) = \varphi(t)$ for $t \in J$ and u is a mild solution of the problem

$$(P_f) \quad \begin{cases} u'(t) + Au(t) \ni f(t) \text{ a.e. } t \in I \\ u(0) = \varphi(0) \end{cases}$$

where $f \in I^1_{G_u}$.

We are now ready to state our main result.

Theorem 3.1. *Assume that conditions* (A) *and* (F) *hold. Then there exists a mild solution of* $P(\varphi)$*.*

Proof. From the assumption (A) and the theorem of existence and uniqueness of mild solutions (see Theorem 2.1), it follows that there exists a unique solution v^0 of (P_0). Set

$$u^0(t) = \begin{cases} \varphi(t) & \text{if } t \in J, \\ v^0(t) & \text{if } t \in I. \end{cases}$$

Then by Lemma 2.1 there is a measurable selection f_1 of the multimapping $t \to F(t, u^0_t)$ such that, for almost all $t \in I$,

$$\begin{aligned} |f_1(t)| &\le d(0, F(t, u^0_t)) + q(t) \\ &\le d(0, F(t, 0)) + \delta(F(t, 0), F(t, u^0_t)) + q(t) \\ &\le 2q(t) + h(t) \sup_{\tau \in [-r,T]} \left| u^0(\tau) \right| \end{aligned}$$

and $f_1 \in L^1(I; E)$. By Theorem 2.1, let v^1 be the unique mild solution of the problem (P_{f_1}). Set

$$u^1(t) = \begin{cases} \varphi(t) & \text{if } t \in J, \\ v^1(t) & \text{if } t \in I. \end{cases}$$

We have for all $t \in I$, (see Theorem 2.1)

$$\begin{aligned} e^{-\omega t} \left| v^1(t) - v^0(t) \right| &\le \int_0^t e^{-\omega s} \left| f_1(s) \right| ds \\ &\le \int_0^t e^{-\omega s} (2q(s) + \sup_{\tau \in [-r,T]} \left| u^0(\tau) \right| h(s)) ds := m(t). \end{aligned}$$

By Lemma 2.1, there is a measurable selection f_2 of the multimapping $t \to F(t, u^1_t)$ such that, for almost all $t \in I$,

$$\begin{aligned} |f_2(t) - f_1(t)| &\le 2d(f_1(t), F(t, u^1_t)) \\ &\le 2\delta(F(t, u^0_t), F(t, u^1_t)) \\ &\le 2h(t) \left\| u^0_t - u^1_t \right\| \\ &\le 2h(t) \sup_{s \in [0,t]} \left| v^0(s) - v^1(s) \right| \\ &\le 2h(t) e^{\omega t} m(t) \end{aligned}$$

and then $f_2 \in L^1(I;E)$. Let v^2 be the unique mild solution of the problem (P_{f_2}). Set

$$u^2(t) = \begin{cases} \varphi(t) & \text{if } t \in J, \\ v^2(t) & \text{if } t \in I. \end{cases}$$

Thus, we can define by induction two sequences (u^n) and (f_n) with $u^n \in \mathcal{C}_T$ and $f_n \in L^1(I;E)$ such that:

(i) for all $n \in \mathbb{N}$,

$$u^n(t) = \begin{cases} \varphi(t) & \text{if } t \in J, \\ v^n(t) & \text{if } t \in I, \end{cases}$$

where v^n is the unique mild solution of the problem (P_{f_n});

(ii) $f_0 = 0$ and for all $n \geq 1$,

$$f_n(t) \in F(t, u_t^{n-1}) \text{ a.e. in } I;$$

(iii) for almost all $t \in I$ and $n \geq 1$,

$$|f_{n+1}(t) - f_n(t)| \leq 2h(t)\|u_t^n - u_t^{n-1}\|.$$

It follows from (iii) that

(iv) for all $t \in I$ and $n \geq 1$,

$$\begin{aligned} e^{-\omega t}\left|u^{n+1}(t) - u^n(t)\right| &\leq \int_0^t e^{-\omega t_1}|f_{n+1}(t_1) - f_n(t_1)|dt_1 \\ &\leq \int_0^t 2e^{-\omega t_1}h(t_1)\|u_{t_1}^n - u_{t_1}^{n-1}\|dt_1 \end{aligned}$$

$$\leq \int_0^t 2e^{-\omega t_1}h(t_1)e^{\omega t_1}\Big[\sup_{t_1+\theta\in[0,t_1]} e^{-\omega(t_1+\theta)}\left|u^n(t_1+\theta)\text{-}u^{n-1}(t_1+\theta)\right|\Big]dt_1$$

$$\leq \int_0^t 2h(t_1)\int_0^{t_1} e^{-\omega t_2}\left|f_n(t_2) - f_{n-1}(t_2)\right| dt_2 dt_1$$

$$\cdots \quad \cdots \quad \cdots$$

$$\leq \int_0^t 2h(t_1)\int_0^{t_1} 2h(t_2)\ldots\int_0^{t_{n-1}} 2h(t_n)e^{-\omega t_n}\|u_{t_n}^1\text{-}u_{t_n}^0\|dt_n\ldots dt_1$$

$$\leq m(t).\frac{\Big[\int_0^t 2h(s)ds\Big]^n}{n!}.$$

Then, for all $n \geq 1$,

$$\begin{aligned}\|u^{n+1} - u^n\|_\omega &= \sup_{t\in[-r,T]} e^{-\omega t}|u^{n+1}(t) - u^n(t)| \\ &= \sup_{t\in I} e^{-\omega t}|u^{n+1}(t) - u^n(t)| \\ &\leq m(T)\frac{\big[\int_0^T 2h(t)dt\big]^n}{n!}.\end{aligned}$$

We deduce that (u^n) is a Cauchy sequence of a continuous functions converging uniformly to a function $u \in \mathcal{C}_T$ and for almost all $t \in I$, $(f_n(t))$ is a Cauchy sequence in E. Hence $(f_n(.))$ converges pointwise almost everywhere to a measurable function $f(.)$ in E. Furthermore, there exists a function $\alpha \in L^1_+(I)$ such that $|f_n(t)| \leq \alpha(t)$ for almost all $t \in I$ and $n \in \mathbb{N}$. Thus (f_n) converges to f in $L^1(I;E)$ and by Theorem 2.2 $u_{|I}$ is a mild solution of problem (P_f). Moreover $f \in I^1_{G_u}$ since for almost all $t \in I$,

$$\begin{aligned}d(f(t), F(t,u_t)) &\leq |f(t) - f_n(t)| + d(f_n(t), F(t,u_t)) \\ &\leq |f(t) - f_n(t)| + h(t)e^{\omega t}\|u^{n-1} - u\|_\omega\end{aligned}$$

and the right hand side tends to zero almost everywhere on I as $n \to +\infty$. Consequently u is mild solution of $P(\varphi)$. □

Definition 3.2 (see [8]). Given $(\alpha,\beta) \in A$ and $f \in L^1(0,T;E)$, we call solution of

$$(PP)_f \quad \begin{cases}\dfrac{d}{dt}\big[u(t) + \lambda Au(t)\big] + Au(t) \ni f(t),\ t \in I,\\ u(0) = \alpha,\ \beta \in Au(0),\ \lambda > 0\end{cases}$$

a function $u \in C([0,T];E)$ such that

$$\begin{cases}\exists w \in C([0,T];E) \text{ with } u + \lambda w \in W^{1,1}(0,T;E) \text{ and}\\ u(0) = \alpha,\ w(0) = \beta,\ w(t) \in Au(t),\ \forall\, t \in I,\\ \dfrac{d}{dt}[u(t) + \lambda w(t)] + w(t) = f(t),\ \text{a.e. } t \in I\end{cases}$$

Definition 3.3. A function $u \in C_T$ is called solution for the abstract functional differential pseudoparabolic inclusion:

$$\begin{cases}\dfrac{d}{dt}[u(t) + \lambda Au(t)] + Au(t) - F(t,u_t) \ni 0,\ t \in I\\ u_0 = \varphi \in \mathcal{C},\ \beta \in A\varphi(0),\ \lambda > 0\end{cases}$$

if $u(t) = \varphi(t)$ for $t \in J$ and u is a solution of the problem $(PP)_f$ where $f \in I^1_{G_u}$.

Under the assumptions (A) and (F), using Propositions 1.1 and 1.3 from [8] and the same technique as above we obtain the existence of solutions for the abstract functional differential pseudoparabolic inclusion.

4. Application

We end this paper by giving an application of our results to a nonlinear evolution problem.

Let X be a complete metric space, $E = L^1(\Omega)$ (where Ω is a bounded open set in $\mathbb{R}^N$) and $f : \mathbb{R}^2 \times X \to \mathbb{R}$ generates the operator $\mathbf{f} : I \times E \times X \to E$ by the formula

$$\mathbf{f}(t, e, w)(x) = f(t, e(x), w).$$

We assume that for all $(e, w) \in E \times X$ the function $\mathbf{f}(., e, w)$ is measurable and for every $(t, e) \in I \times E$, $\mathbf{f}(t, e, .)$ is continuous.

Consider a measurable multimapping $W : I \to \mathcal{F}(X)$ and assume that

• there exists $h \in L^1(I)$ such that for almost every $t \in I$ and for all $w \in W(t), \mathbf{f}(t, ., w)$ is $h(t)$-Lipschitz;

• for almost all $t \in I$ and for all $e \in E$ the set $\mathbf{f}(t, e, W(t))$ is closed and $t \to q(t) = d(0, \mathbf{f}(t, 0, W(t))$ is integrable.

Set $\mathcal{W}_T = \{w : I \to X$ measurable and such that $w(t) \in W(t)\}$ and consider the following nonlinear functional control problem

$$(P) \begin{cases} \dfrac{\partial}{\partial t} v(t,x) - \triangle\gamma v(t,x) - f(t, v(t-r,x), w(t)) \ni 0, \\ \qquad\qquad t \in I,\ x \in \Omega,\ w \in \mathcal{W}_T, \\ 0 \in \gamma v(t,x),\ t \in I,\ x \in \partial\Omega, \\ v(\theta, x) = \varphi(\theta)(x),\ \theta \in J,\ x \in \Omega, \end{cases}$$

where γ is a maximal monotone operator in $\mathbb{R} \times \mathbb{R}$ with $0 \in \gamma(0)$ (see [1, 5]), and $\varphi \in C(J; E)$ such that $\varphi(0) \in \overline{D(A)}$ with $A = -\triangle\gamma$ in E, that is

$$A = \{(\zeta, \xi) \in E \times E : \vartheta \in W_0^{1,1}(\Omega) \text{ with } \xi = -\triangle\vartheta \text{ and } \vartheta(x) \in \gamma(\zeta(x)) \text{ a.e. } x \in \Omega\},$$

then A is m-accretive in E (see [6, 8]). Define the multimapping $F : I \times \mathcal{C} \to \mathcal{F}(E)$ by

$$F(t, \phi) = \mathbf{f}(t, \phi(-r), W(t)).$$

For this multimapping, the conditions $(F_1) - (F_3)$ are fulfilled. We can now write the problem (P) in the form $P(\varphi)$, and conclude, by virtue of Theorem 3.1, the existence of a mild solution.

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