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MAPPINGS IN ls-σ-FINITE PONOMAREV-SYSTEMS

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ABSTRACT. We introduce the notion of an ls-σ-finite Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ to give a consistent method to construct an s-mapping ($msss$-mapping, $mssc$-mapping, cs-mapping) f with covering-properties from a locally separable metric space M onto a space X. As applications, we systematically get characterizations of s-images ($msss$-images, $mssc$-images, cs-images) with covering-properties of locally separable metric spaces.

1. INTRODUCTION

Finding characterizations of nice images of metric spaces is one of the most important problems in theory of generalized metric spaces [11]. Various kinds of characterizations have been obtained by means of certain networks [16], [23]. Recently, many authors have been interested in finding characterizations of nice images of locally separable metric spaces under mappings with covering-properties (*covering-mapping* for short). The key to prove these results is to construct covering-mappings from a locally separable metric space onto a space. In [1], the authors introduced the notion of an ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_\lambda\})$ to give necessary and sufficient conditions such that the mapping f is an s-mapping with covering-properties from a locally separable metric space M onto a space X. As applications, characterizations of certain s-images of locally separable metric spaces were obtained by means of double covers.

In [15], S. Lin introduced $msss$-mappings and $mssc$-mappings to characterize spaces with σ-locally countable networks and spaces with σ-locally finite networks respectively. After that, Z. Qu and Z. Gao introduced cs-mappings to characterize spaces with compact-countable networks in [21]. These mappings have close relations with s-mappings and play important roles in finding characterizations for images of metric spaces [4], [9], [12], [13], [14]. However, for the ls-σ-finite Ponomarev-systems $(f, M, X, \{\mathcal{P}_\lambda\})$, we do not know what are the necessary and sufficient conditions such that the mapping f is an $msss$-mapping ($mssc$-mapping, cs-mapping) with covering-properties.

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We are interested in finding a consistent method to construct an s-mapping ($msss$-mapping, $mssc$-mapping, cs-mapping) f with covering-properties from a locally separable metric space M onto a space X.

In this paper, we use a special kind of the σ-Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ in [5], called a σ-finite Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$, to give a consistent method to construct a covering-mapping f from a separable metric space M onto a space X. Using a family of σ-finite Ponomarev-systems $\{(f_\lambda, M_\lambda, X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$, we introduce the notion of an ls-σ-finite Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ to give a consistent method to construct an s-mapping ($msss$-mapping, $mssc$-mapping, cs-mapping) f with covering-properties from a locally separable metric space M onto a space X. As applications, we systematically get characterizations of s-images ($msss$-images, $mssc$-images, cs-images) with covering-properties of locally separable metric spaces. These results make the study of images of locally separable metric spaces more complete.

The paper is organized as follows. In addition to the introduction, the paper contains two more sections. Section 2 presents definitions of networks and mappings, and lemmas which will be used throughout the paper. The main results are presented in Section 3.

2. Preliminaries

Throughout this paper, all spaces are T_1 and regular, all mappings are continuous and onto, a convergent sequence includes its limit point. We denote by $\mathbb{N}$ the set of all natural numbers and write $\omega = \mathbb{N} \cup \{0\}$. Let p_k denote the projection from $\prod_{n\in\mathbb{N}} X_n$ onto X_k. Let $f : X \longrightarrow Y$ be a mapping, $x \in X$, and $\mathcal{P}$ be a family of subsets of X, we define $\mathcal{P}_x = \{P \in \mathcal{P} : x \in P\}$, $f(\mathcal{P}) = \{f(P) : p \in \mathcal{P}\}$, $\bigcap \mathcal{P} = \bigcap\{P : P \in \mathcal{P}\}$ and $\bigcup \mathcal{P} = \bigcup\{P : p \in \mathcal{P}\}$. We say that a convergent sequence $\{x_n : n \in \mathbb{N}\} \cup \{x\}$, converging to x, is *eventually* in U if $\{x_n : n \geq n_0\} \cup \{x\} \subset U$ for some $n_0 \in \mathbb{N}$, and is *frequently* in U if $\{x_{n_k} : k \in \mathbb{N}\} \cup \{x\} \subset U$ for some subsequence $\{x_{n_k} : k \in \mathbb{N}\}$ of $\{x_n : n \in \mathbb{N}\}$.

Definition 2.1. Let $\mathcal{P}$ be a family of subsets of a space X and K be a subset of X.

(1) For each $x \in X$, $\mathcal{P}$ is called a *network at x in X* [19] if $x \in \bigcap \mathcal{P}$ and if $x \in U$ with U open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}$.

$\mathcal{P}$ is called a *network for X* [19] if $\mathcal{P}_x$ is a network at x in X for every $x \in X$.

(2) $\mathcal{P}$ is called a *cfp-network for K in X* [1] if for each compact subset H of K with $H \subset U$ and U open in X, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $H \subset \bigcup\{C_F : F \in \mathcal{F}\} \subset \bigcup \mathcal{F} \subset U$, where C_F is closed and $C_F \subset F$ for every $F \in \mathcal{F}$. If $K = X$, then a cfp-network for K in X is a cfp-network for X [24].

(3) $\mathcal{P}$ is called a *cs-network for K in X* (resp. *cs^*-network for K in X*) [1] if for each convergent sequence S in K converging to $x \in U$ with U open in X, S is eventually (resp. frequently) in $P \subset U$ for some $P \in \mathcal{P}$. If $K = X$, then a cs-network for K in X (resp. cs^*-network for K in X) is a *cs-network for X* [10] (resp. *cs^*-network for X* [7]).

(4) $\mathcal{P}$ is called a *wcs-network for K in X* [5] if for each convergent sequence S in K converging to $x \in U$ with U open in X, S is eventually in $\bigcup \mathcal{F} \subset U$ for some finite subfamily $\mathcal{F}$ of $\mathcal{P}_x$. If $K = X$, then a wcs-network for K in X is a *wcs-network for X*.

(5) $\mathcal{P}$ is called a *strong network* for X [5] if, for each $x \in X$, the exists a countable $\mathcal{P}(x) \subset \mathcal{P}$ such that $\mathcal{P}(x)$ forms a network at x in X.

Lemma 2.2 ([5], Lemma 2.6)**.** *If $\mathcal{P}$ is a cs-network for a convergent sequence $S \subset U$ with U open in a space X, then there exists $\mathcal{F} \subset \mathcal{P}$ satisfying the following:*

(1) *$\mathcal{F}$ is finite;*
(2) *For each $F \in \mathcal{F}$, $\emptyset \neq F \cap S \subset F \subset U$;*
(3) *For each $x \in S$, there exists a unique $F \in \mathcal{F}$ such that $x \in F$;*
(4) *If $F \in \mathcal{F}$ contains the limit point of S, then $S - F$ is finite.*

Such an $\mathcal{F}$ is said to have the property $cs(S, U)$.

Lemma 2.3 ([5], Lemma 2.7)**.** *If $\mathcal{P}$ is a cfp-network for a compact subset $K \subset U$ with U open in a space X, then there exists $\mathcal{F} \subset \mathcal{P}$ satisfying the following:*

(1) *$\mathcal{F}$ is finite;*
(2) *For each $F \in \mathcal{F}$, $\emptyset \neq F \cap K \subset F \subset U$;*
(3) *For each $F \in \mathcal{F}$, $\mathcal{F} - \{F\}$ is not a cover for K;*
(4) *For each $F \in \mathcal{F}$, $F \cap K$ is compact.*

Such an $\mathcal{F}$ is said to have the property $cfp(K, U)$.

Definition 2.4. A space X is called a *cosmic* space [20] (resp. $\aleph_0$*-space* [20]) if X has a countable network (resp. countable cs-network).

It is well-known that a space X is an $\aleph_0$-space if and only if X has a countable cfp-network (wcs-network, cs^*-network) [23].

Definition 2.5. Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a cover for a space X.

(1) $\mathcal{P}$ is called a *σ-network* for X if for each $x \in X$, there exists a countable network $\{P_{\alpha_n} : n \in \mathbb{N}\}$ at x in X such that $P_{\alpha_n} \in \mathcal{P}_n$ for every $n \in \mathbb{N}$.

(2) $\mathcal{P}$ is called a *σ-finite network* for X if $\mathcal{P}$ is a σ-network for X and every $\mathcal{P}_n$ is finite.

(3) A σ-finite network $\mathcal{P}$ is called a *σ-finite cs-network* (resp. *σ-finite cfp-network, σ-finite wcs-network, σ-finite cs*-network*) for X if $\mathcal{P}$ is a cs-network (resp. cfp-network, wcs-network, cs^*-network) for X.

Definition 2.6. Let $f : X \longrightarrow Y$ be a mapping.

(1) f is called a (Z)*-msss-mapping* or an *msss-mapping* [15] if X is a subspace of the product space $Z = \prod_{n\in\mathbb{N}} X_n$ of a family $\{X_n : n \in \mathbb{N}\}$ of metric spaces, and for each $y \in Y$, there exists a sequence $\{V_{y,n} : n \in \mathbb{N}\}$ of open neighborhoods of y in Y such that each $p_n(f^{-1}(V_{y,n}))$ is a separable subset of X_n.

(2) f is called a (Z)*-mssc-mapping* or an *mssc-mapping* [15] if X is a subspace of the product space $Z = \prod_{n\in\mathbb{N}} X_n$ of a family $\{X_n : n \in \mathbb{N}\}$ of metric spaces, and for each $y \in Y$, there exists a sequence $\{V_{y,n} : n \in \mathbb{N}\}$ of open neighborhoods of y in Y such that each $\overline{p_n(f^{-1}(V_{y,n}))}$ is a compact subset of X_n.

(3) f is called an *s-mapping* [2] if, for each $y \in Y$, $f^{-1}(y)$ is a separable subset of X.

(4) f is called a *cs-mapping* [21] if, for each compact subset K of Y, $f^{-1}(K)$ is a separable subset of X.

(5) f is called a *sequence-covering* mapping [22] if, for each convergent sequence S of Y, there exists a convergent sequence L of X such that $f(L) = S$. Note that a sequence-covering mapping is a *strong sequence-covering* mapping in the sense of [14].

(6) f is called a *subsequence-covering* mapping [17] if, for each convergent sequence S of Y, there exists a compact subset K of X such that $f(K)$ is a subsequence of S.

(7) f is called a *sequentially-quotient* mapping [3] if, for each convergent sequence S of Y, there exists a convergent sequence L of X such that $f(L)$ is a subsequence of S.

For terms which are not defined here, please refer to [6] and [16].

It is well-known that an image of a network (resp. *cs*-network, cfp-network, cfp-network, cs^*-network) under a mapping (resp. sequence-covering mapping, compact-covering mapping, pseudo-sequence-covering mapping, sequentially-quotient mapping) is a network (resp. *cs*-network, cfp-network, *wcs*-network, cs^*-network). From these facts and Definition 2.5, we have the following.

Lemma 2.7. *Let $f : X \longrightarrow Y$ be a mapping and $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-finite network for X. Then the following hold:*

(1) *$\bigcup\{f(\mathcal{P}_n) : n \in \mathbb{N}\}$ is a σ-finite network for Y.*

(2) *If $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a σ-finite cs-network (resp. cfp-network, cfp-network, cs^*-network) for X and f is a sequence-covering (resp. compact-covering, pseudo-sequence-covering, sequentially-quotient) mapping, then $\bigcup\{f(\mathcal{P}_n) : n \in \mathbb{N}\}$ is a σ-finite cs-network (resp. cfp-network, wcs-network, cs^*-network) for Y.*

Definition 2.8. Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-network for a space X. Assume that $X \in \mathcal{P}_n \subset \mathcal{P}_{n+1}$ and $\mathcal{P}_n$ is closed under finite intersections for every $n \in \mathbb{N}$. Let $\mathcal{P}_n = \{P_\alpha : \alpha \in A_n\}$, where each A_n is endowed with the discrete topology, then A_n is a metric space. We define

$$M = \Big\{a = (\alpha_n) \in \prod_{n\in\mathbb{N}} A_n : \{P_{\alpha_n} : n \in \mathbb{N}\} \text{ forms a network at some point } x_a \text{ in } X\Big\}.$$

Then, M is a metric space, and x_a is unique for each $a \in M$. Define $f : M \longrightarrow X$ by $f(a) = x_a$ for every $a \in M$. Then f is a mapping from the metric space M onto X. The system $(f, M, X, \{\mathcal{P}_n\})$ is a *σ-Ponomarev-system* [5].

If $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a σ-finite network for X, then a σ-Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ is a *σ-finite Ponomarev-system*.

Remark 2.9. For a σ-finite Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$, since A_n is finite for every $n \in \mathbb{N}$, the mapping f is a (Z)-*mssc*-mapping with $Z = \prod_{n\in\mathbb{N}} A_n$

and M is a separable metric space. From now on, the $mssc$-mapping (resp. $msss$-mapping) f in a σ-finite Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ is understood to mean a (Z)-$mssc$-mapping (resp. (Z)-$msss$-mapping) with $Z = \prod_{n\in\mathbb{N}} A_n$.

By [5, Lemma 2.11], we have the following.

Lemma 2.10. *Let $(f, M, X, \{\mathcal{P}_n\})$ be a σ-finite Ponomarev-system, $a = (\alpha_n) \in M$ where $\{P_{\alpha_n} : n \in \mathbb{N}\}$ is a network at some point x_a in X, and*

$$U_n = \{b = (\beta_i) \in M : \beta_i = \alpha_i \text{ if } i \leq n\},$$

for every $n \in \mathbb{N}$. Then the following hold:

(1) *$\{U_n : n \in \mathbb{N}\}$ is a base at a in M.*
(2) *$f(U_n) = \bigcap_{i=1}^{n} P_{\alpha_i}$ for every $n \in \mathbb{N}$.*

In [5, Theorem 3.2], necessary and sufficient conditions for a mapping f to be a covering-mapping in a σ-Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ have been obtained by means of strong networks. Next, we modify these conditions in a σ-finite Ponomarev-system.

Lemma 2.11. *Let $(f, M, X, \{\mathcal{P}_n\})$ be a σ-finite Ponomarev-system. Then the following hold:*

(1) *$\mathcal{P}$ is a cs-network for a convergent sequence S in X if and only if $Sf(L)$ for some convergent sequence L in M.*
(2) *$\mathcal{P}$ is a cfp-network for a compact subset K in X if and only if $K = f(L)$ for some compact subset L of M.*
(3) *$\mathcal{P}$ is a cs^*-network (wcs-network) for a convergent sequence S in X if and only if $S = f(L)$ for some compact subset L of M.*

Proof. (1). *Necessity.* Let $\mathcal{P}$ be a cs-network for a convergent sequence S in X. Suppose that $S = \{x_m : m \in \omega\}$ with the limit point x_0. We have that $\mathcal{F} = \{X\} \subset \mathcal{P}$ has the property $cs(S, X)$. Since $\mathcal{P}$ is countable, $\{\mathcal{F} \subset \mathcal{P} : \mathcal{F}$ has the property $cs(S, X)\}$ is countable. So we can write

$$\{\mathcal{F} \subset \mathcal{P} : \mathcal{F} \text{ has the property } cs(S, X)\} = \{\mathcal{F}_i : i \in \mathbb{N}\},$$

and put $\mathcal{F}_{n(1)} = \{X\} \subset \mathcal{P}_1$. For each $i \geq 2$ if there exists $j \in \mathbb{N}$ such that $\mathcal{F}_j \subset (\mathcal{P}_i - \{\mathcal{F}_{n(k)} : k = 1, \ldots, i-1\})$, then we define

$$n(i) = \min\{j \in \mathbb{N} : \mathcal{F}_j \subset (\mathcal{P}_i - \{\mathcal{F}_{n(k)} : k = 1, \ldots, i-1\})\};$$

otherwise, put $\mathcal{F}_{n(i)}\{X\}$. Then $\{\mathcal{F}_{n(i)} : i \in \mathbb{N}\} = \{\mathcal{F}_i : i \in \mathbb{N}\}$. Suppose that $\mathcal{F}_{n(i)}\{P_\alpha : \alpha \in B_i\}$, where B_i is a finite set. For every $m \in \omega$ and $i \in \mathbb{N}$, since $\mathcal{F}_{n(i)}$ has the property $cs(S, X)$, there exists a unique $\alpha_{im} \in B_i$ such that $x_m \in P_{\alpha_{im}} \in \mathcal{F}_{n(i)}$. Let us put $a_m = (\alpha_{im}) \in \prod_{i\in\mathbb{N}} B_i$ and $L = \{a_m : m \in \omega\}$. As in the sufficiency of the proof of (1) of [5, Theorem 3.2], we have that L is a convergent sequence in M and $f(L) = S$.

Sufficiency. Let S be a convergent sequence in X and $S = f(L)$ for some convergent sequence L in M. As in the necessity of the proof of (1) of [5, Theorem 3.2], we have that $\mathcal{P}$ is a cs-network for S in X.

(2). *Necessity.* Let $\mathcal{P}$ be a cfp-network for a compact subset K of X. We have that $\mathcal{F} = \{X\} \subset \mathcal{P}$ has the property $cfp(K, X)$. Since $\mathcal{P}$ is countable, $\{\mathcal{F} \subset \mathcal{P} : \mathcal{F}$ has the property $cfp(K, X)\}$ is countable. So we can write

$$\big\{\mathcal{F} \subset \mathcal{P} : \mathcal{F} \text{ has the property } cfp(K, X)\big\} = \{\mathcal{F}_i : i \in \mathbb{N}\},$$

and put $\mathcal{F}_{n(1)} = \{X\} \subset \mathcal{P}_1$. For each $i \geq 2$ if there exists $j \in \mathbb{N}$ such that $\mathcal{F}_j \subset \big(\mathcal{P}_i - \{\mathcal{F}_{n(k)} : k = 1, \ldots, i-1\}\big)$, then we can define

$$n(i) = \min\big\{j \in \mathbb{N} : \mathcal{F}_j \subset \big(\mathcal{P}_i - \{\mathcal{F}_{n(k)} : k = 1, \ldots, i-1\}\big)\big\};$$

otherwise, put $\mathcal{F}_{n(i)} = \{X\}$. Then $\{\mathcal{F}_{n(i)} : i \in \mathbb{N}\} = \{\mathcal{F}_i : i \in \mathbb{N}\}$. Suppose that $\mathcal{F}_{n(i)} = \{P_\alpha : \alpha \in B_i\}$, where B_i is a finite set. Let us define

$$L = \big\{a = (\alpha_n) \in \prod_{n \in \mathbb{N}} B_n : \bigcap_{n \in \mathbb{N}} (P_{\alpha_n} \cap K) \neq \emptyset\big\}.$$

As in the sufficiency part of the proof of (2) of [5, Theorem 3.2], L is a compact subset of M and $f(L) = K$.

Sufficiency. Let K be a compact subset of X and $K = f(L)$ for some compact subset L of M. As in the necessity part of the proof of (2) of [5, Theorem 3.2], $\mathcal{P}$ is a cfp-network for K in X.

(3). *Necessity.* Let $\mathcal{P}$ be a cs^*-network for a convergent sequence S in X. Since $\mathcal{P}$ is countable, $\mathcal{P}$ is precisely a *wcs*-network for S in X. This implies that $\mathcal{P}$ is a cfp-network for S in X. As in the necessity part of (2), there exists a compact subset L of M such that $f(L) = S$.

Sufficiency. Let S be a convergent sequence in X and $S = f(L)$ for some compact subset L of M. As in the sufficiency part of (2), $\mathcal{P}$ is a cfp-network for S in X. Since S is a convergent sequence, $\mathcal{P}$ is a cs^*-network (*wcs*-network) for S in X. □

By Lemma 2.11, we get the following.

Corollary 2.12. *Let $(f, M, X, \{\mathcal{P}_n\})$ be a σ-finite Ponomarev-system and $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$. Then the following hold:*

(1) *f is a sequence-covering mapping if and only if $\mathcal{P}$ is a cs-network for X.*
(2) *f is a compact-covering mapping if and only if $\mathcal{P}$ is a cfp-network for X.*
(3) *f is a sequentially-quotient (pseudo-sequence-covering) mapping if and only if $\mathcal{P}$ is a cs^*-network for X.*

3. Mappings in ls-σ-finite Ponomarev-systems

Definition 3.1. Let $\{X_\lambda : \lambda \in \Lambda\}$ be a cover for a space X such that each X_λ has a sequence of covers $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$.

(1) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is called a *double σ-finite cover* for X if for every $\lambda \in \Lambda$, $\bigcup\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a σ-finite network for X_λ.

(2) A double σ-finite cover $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is called a *point-countable* (resp. *locally countable, locally finite, compact-countable*) *double σ-finite cover* for X if $\mathcal{P}_n$ is point-countable (resp. locally countable, locally finite, compact-countable), where $\mathcal{P}_n = \bigcup\{\mathcal{P}_{\lambda,n} : \lambda \in \Lambda\}$ for every $n \in \mathbb{N}$. Because each

$\mathcal{P}_{\lambda,n}$ is finite, we have that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is point-countable (resp. compact-countable) if and only if $\{X_\lambda : \lambda \in \Lambda\}$ is point-countable (resp. compact-countable).

Definition 3.2. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double σ-finite cover for X and $\mathcal{P}_\lambda = \bigcup\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ for every $\lambda \in \Lambda$.

(1) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is called a *double σ-finite cs-cover* for X if, for each convergent sequence S converging to x in X, there exists $\lambda \in \Lambda$ such that S is eventually in X_λ and $\mathcal{P}_\lambda$ is a cs-network for $S \cap X_\lambda$ in X_λ.

(2) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is called a *double σ-finite cfp-cover* for X if for each compact subset K of X the following conditions hold:

i) There exists a finite subset Λ_K of Λ such that $K = \bigcup\{K_\lambda : \lambda \in \Lambda_K\}$.

ii) For each $\lambda \in \Lambda_K$, K_λ is compact.

iii) $\mathcal{P}_\lambda$ is a cfp-network for K_λ in X_λ.

(3) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is called a *double σ-finite wcs-cover* for X if for each convergent sequence S in X the following conditions hold:

i) There exists a finite subset Λ_S of Λ such that $S = \bigcup\{S_\lambda : \lambda \in \Lambda_S\}$.

ii) For each $\lambda \in \Lambda_S$, S_λ is a convergent sequence.

iii) $\mathcal{P}_\lambda$ is a wcs-network for S_λ in X_λ.

(4) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is called a *double σ-finite cs^*-cover* for X if for each convergent sequence S in X there exist $\lambda \in \Lambda$ and a subsequence S_λ of S such that $\mathcal{P}_\lambda$ is a cs^*-network for S_λ in X_λ.

Lemma 3.3. *Let $f : X \longrightarrow Y$ be a mapping and $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double σ-finite cover for X. Then the following hold:*

(1) $\{(f(X_\lambda), \{f(\mathcal{P}_{\lambda,n})\}) : \lambda \in \Lambda\}$ *is a double σ-finite cover for Y.*

(2) *If $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a σ-finite cs-cover (resp. cfp-cover, wcs-cover, cs^*-cover) for X and f is a sequence-covering (resp. compact-covering, pseudo-sequence-covering, sequentially-quotient) mapping, then $\{(f(X_\lambda), \{f(\mathcal{P}_{\lambda,n})\}) : \lambda \in \Lambda\}$ is a double σ-finite cs-cover (resp. cfp-cover, wcs-cover, cs^*-cover) for Y.*

(3) *If $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a point-countable (resp. compact-countable) double σ-finite cover for X and f is an s-mapping (resp. cs-mapping), then $\{(f(X_\lambda), \{f(\mathcal{P}_{\lambda,n})\}) : \lambda \in \Lambda\}$ is a point-countable (resp. compact-countable) double σ-finite cover for Y.*

Proof. (1). By Lemma 2.7.(1).

(2). By (1), $\{(f(X_\lambda), \{f(\mathcal{P}_{\lambda,n})\}) : \lambda \in \Lambda\}$ is a double σ-finite cover for X. It follows from [1, Lemma 2.13] that whenever f is a sequence-covering (resp. compact-covering, pseudo-sequence-covering, sequentially-quotient) mapping, then $\{(f(X_\lambda), \{f(\mathcal{P}_{\lambda,n})\}) : \lambda \in \Lambda\}$ is a double σ-finite cs-cover (resp. cfp-cover, wcs-cover, cs^*-cover) for Y.

(3). By (1), $\{(f(X_\lambda), \{f(\mathcal{P}_{\lambda,n})\}) : \lambda \in \Lambda\}$ is a double σ-finite cover for X. For the un-parenthetic part, see [4, Theorem 2.14]. Also, in view of the proof of [4, Theorem 2.14], we get the parenthetic part. □

Definition 3.4. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double σ-finite cover for a space X, and $(f_\lambda, M_\lambda, X_\lambda, \{\mathcal{P}_{\lambda,n}\})$ be the σ-finite Ponomarev-system for each $\lambda \in \Lambda$. We denote by $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, and $f = \bigoplus_{\lambda\in\Lambda} f_\lambda$. Then M is a locally separable metric space, and f is a mapping from a locally separable metric space M onto X. The system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ is called an *ls-σ-finite Ponomarev-system.*

Remark 3.5. For every $\lambda \in \Lambda$ and $n \in \mathbb{N}$, we write $\mathcal{P}_{\lambda,n} = \{P_\alpha : \alpha \in A_{\lambda,n}\}$, and $A_n = \bigcup\{A_{\lambda,n} : \lambda \in \Lambda\}$. Then the space M of an ls-σ-finite Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ is a subspace of $\prod_{n\in\mathbb{N}} A_n$. From now on, the $msss$-mapping (resp. $mssc$-mapping) f in an ls-σ-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ is understood to mean a (Z)-$msss$-mapping (resp. (Z)-$mssc$-mapping) with $Z = \prod_{n\in\mathbb{N}} A_n$.

In [5, Theorem 3.1], the necessary and sufficient conditions for f to be an s-mapping (resp. $msss$-mapping, $mssc$-mapping) from a metric space M onto a space X in a σ-Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ have been obtained. Next, we give necessary and sufficient conditions for f to be an s-mapping (resp. $msss$-mapping, $mssc$-mapping, cs-mapping) from a locally separable metric space M onto a space X in an ls-σ-finite Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$.

Theorem 3.6. *Let $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ be an ls-σ-finite Ponomarev-system. Then the following hold:*

(1) *f is an s-mapping if and only if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is point-countable.*
(2) *f is an msss-mapping if and only if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is locally countable.*
(3) *f is an mssc-mapping if and only if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is locally finite.*
(4) *f is a cs-mapping if and only if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is compact-countable.*

Proof. (1). *Necessity.* Let f be an s-mapping. For each $x \in X$, we have that $f^{-1}(x)$ is a separable subset of X. Then $\Lambda_x = \{\lambda \in \Lambda : f^{-1}(x) \cap M_\lambda \neq \emptyset\} = \{\lambda \in \Lambda : x \in X_\lambda\}$ is countable. Therefore $\{X_\lambda : \lambda \in \Lambda\}$ is point-countable. This implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is point-countable.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be point-countable. Then $\{X_\lambda : \lambda \in \Lambda\}$ is also point-countable. For each $x \in X$, we have that $\Lambda_x = \{\lambda \in \Lambda : x \in X_\lambda\} = \{\lambda \in \Lambda : f^{-1}(x) \cap M_\lambda \neq \emptyset\}$ is countable. Then $f^{-1}(x) = \bigcup\{f^{-1}(x) \cap M_\lambda : \lambda \in \Lambda_x\}$ is a separable subset of M. This proves that f is an s-mapping.

(2). *Necessity.* Let f be an $msss$-mapping. For each $x \in X$, there exists a sequence $\{V_{x,n} : n \in \mathbb{N}\}$ of open neighborhoods of x in X such that each $p_n(f^{-1}(V_{x,n}))$ is a separable subset of A_n. Then

$$\begin{aligned}\Lambda_{x,n} = \{\lambda \in \Lambda : p_n(f^{-1}(V_{x,n})) \cap A_{\lambda,n} \neq \emptyset\} \\ = \{\lambda \in \Lambda : f^{-1}(V_{x,n}) \cap M_\lambda \neq \emptyset\} = \{\lambda \in \Lambda : V_{x,n} \cap X_\lambda \neq \emptyset\}\end{aligned}$$

is countable. For each $\lambda \in \Lambda_{x,n}$, since $A_{\lambda,n}$ is finite, so is $A_{x,\lambda,n} = \{\alpha \in A_{\lambda,n} : V_{x,n} \cap P_\alpha \neq \emptyset\}$. Then

$$A_{x,n} = \{\alpha \in A_n : V_{x,n} \cap P_\alpha \neq \emptyset\} = \bigcup\{A_{x,\lambda,n} : \lambda \in \Lambda_{x,n}\}$$

is countable. This implies that $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is locally countable. Therefore, $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is locally countable.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be locally countable. For each $x \in X$, there exists a sequence $\{V_{x,n} : n \in \mathbb{N}\}$ of neighborhoods of x such that $A_{x,n} = \{\alpha \in A_n : V_{x,n} \cap P_\alpha \neq \emptyset\}$ is countable. Since $f^{-1}(V_{x,n}) \subset A_{x,n}$, $f^{-1}(V_{x,n})$ is a separable subset of A_n. This implies that f is an *msss*-mapping.

(3). *Necessity.* Let f be an *mssc*-mapping. For each $x \in X$, there exists a sequence $\{V_{x,n} : n \in \mathbb{N}\}$ of open neighborhoods of x in X such that each $\overline{p_n(f^{-1}(V_{x,n}))}$ is a compact subset of A_n. Then

$$\begin{aligned}\Lambda_{x,n} = \{\lambda \in \Lambda : \overline{p_n(f^{-1}(V_{x,n}))} \cap A_{\lambda,n} \neq \emptyset\}\\ = \{\lambda \in \Lambda : f^{-1}(V_{x,n}) \cap M_\lambda \neq \emptyset\} = \{\lambda \in \Lambda : V_{x,n} \cap X_\lambda \neq \emptyset\}\end{aligned}$$

is finite. For each $\lambda \in \Lambda_{x,n}$, since $A_{\lambda,n}$ is finite, so is $A_{x,\lambda,n} = \{\alpha \in A_{\lambda,n} : V_{x,n} \cap P_\alpha \neq \emptyset\}$. Then

$$A_{x,n} = \{\alpha \in A_n : V_{x,n} \cap P_\alpha \neq \emptyset\} = \bigcup\{A_{x,\lambda,n} : \lambda \in \Lambda_{x,n}\}$$

is finite. This implies that $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is locally finite. Therefore, $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is locally finite.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be locally finite. For each $x \in X$, there exists a sequence $\{V_{x,n} : n \in \mathbb{N}\}$ of neighborhoods of x such that $A_{x,n} = \{\alpha \in A_n : V_{x,n} \cap P_\alpha \neq \emptyset\}$ is finite. Since $f^{-1}(V_{x,n}) \subset A_{x,n}$, $\overline{f^{-1}(V_{x,n})}$ is a compact subset of A_n. This implies that f is an *mssc*-mapping.

(4). *Necessity.* Let f be a *cs*-mapping. For each compact subset K of X, we have that $f^{-1}(K)$ is a separable subset of X. Then $\Lambda_K = \{\lambda \in \Lambda : f^{-1}(K) \cap M_\lambda \neq \emptyset\} = \{\lambda \in \Lambda : K \cap X_\lambda \neq \emptyset\}$ is countable. Then $\{X_\lambda : \lambda \in \Lambda\}$ is compact-countable. This implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is also compact-countable.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be compact-countable. Then $\{X_\lambda : \lambda \in \Lambda\}$ is compact-countable. For each compact subset K of X, we have that $\Lambda_K = \{\lambda \in \Lambda : K \cap X_\lambda \neq \emptyset\} = \{\lambda \in \Lambda : f^{-1}(K) \cap M_\lambda \neq \emptyset\}$ is countable. Then $f^{-1}(K) = \bigcup\{f^{-1}(K) \cap M_\lambda : \lambda \in \Lambda_K\}$ is a separable subset of M. This proves that f is a *cs*-mapping. □

In [5, Theorem 3.2], the necessary and sufficient conditions for f to be a covering-mapping from a metric space M onto a space X in a σ-Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ have been obtained by means of strong networks. Next, we give the necessary and sufficient conditions for f to be a covering-mapping from a locally separable metric space M onto a space X in an ls-σ-finite Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ by means of double σ-finite cover.

Theorem 3.7. *Let $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ be an ls-σ-finite Ponomarev-system. Then the following hold:*

(1) *f is a sequence-covering mapping if and only if* $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ *is a double σ-finite cs-cover for X.*
(2) *f is a compact-covering mapping if and only if* $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ *is a double σ-finite cfp-cover for X.*
(3) *f is a pseudo-sequence-covering mapping if and only if* $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ *is a double σ-finite wcs-cover for X.*
(4) *f is a sequentially-quotient mapping if and only if* $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ *is a double σ-finite cs^*-cover for X.*

Proof. (1). *Necessity.* Let f be a sequence-covering mapping. For each convergent sequence S converging to x in X, there exists a convergent sequence L in M such that $f(L) = S$. We have that L is eventually in some M_λ. Then S is eventually in X_λ. Then $L_\lambda = L \cap M_\lambda$ is a convergent sequence in M_λ. It follows from Lemma 2.11 that $\bigcup\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a *cs*-network for $f(L_\lambda)$ in X_λ. Then $\bigcup\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a *cs*-network for $S \cap X_\lambda$ in X_λ. This implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double σ-finite *cs*-cover for X.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a σ-finite *cs*-cover for X. For each convergent sequence S in X, there exists $\lambda \in \Lambda$ such that S is eventually in X_λ and $\mathcal{P}_\lambda$ is a *cs*-network for $S \cap X_\lambda$ in X_λ. It follows from Lemma 2.11 that $S \cap X_\lambda = f_\lambda(L_\lambda)$ for some convergent sequence L_λ in M_λ. Since $S - X_\lambda$ is finite, $S - X_\lambda = f(F)$ for some finite subset F of M. Therefore $L = F \cup L_\lambda$ is a convergent sequence in M and $f(L) = S$. This implies that f is a sequence-covering mapping.

(2). *Necessity.* Let f be a compact-covering mapping. For each compact subset K of X, there exists a compact subset L of M such that $f(L) = K$. Since L is compact, $\Lambda_K = \{\lambda \in \Lambda : L \cap M_\lambda \neq \emptyset\}$ is finite and each $L \cap M_\lambda$ is compact. For each $\lambda \in \Lambda_K$, Let $K_\lambda = f_\lambda(L \cap M_\lambda)$. Then $K = \bigcup\{K_\lambda : \lambda \in \Lambda_K\}$, and for each $\lambda \in \Lambda_K$, K_λ is compact and $\mathcal{P}_\lambda$ is a cfp-network for K_λ in X_λ by Lemma 2.11. This proves that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double σ-finite cfp-cover for X.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double σ-finite cfp-cover for X. For each compact subset K of X, there exists a finite subset Λ_K of Λ such that $K = \bigcup\{K_\lambda : \lambda \in \Lambda_K\}$, and for each $\lambda \in \Lambda_K$, K_λ is compact and $\mathcal{P}_\lambda$ is a cfp-network for K_λ in X_λ. For each $\lambda \in \Lambda_K$, it follows from Lemma 2.11 that $K_\lambda = f_\lambda(L_\lambda)$ for some compact subset L_λ of M_λ. Then $L = \bigcup\{L_\lambda : \lambda \in \Lambda_K\}$ is a compact subset of M and $f(L) = K$. This implies that f is a compact-covering mapping.

(3). *Necessity.* Let f be a pseudo-sequence-covering mapping. For each convergent sequence S of X, there exists a compact subset L of M such that $f(L) = S$. Since L is compact, $\Lambda_S = \{\lambda \in \Lambda : L \cap M_\lambda \neq \emptyset\}$ is finite and each $L \cap M_\lambda$ is compact. For each $\lambda \in \Lambda_S$, we define $S_\lambda = f_\lambda(L \cap M_\lambda)$. Then $S = \bigcup\{S_\lambda : \lambda \in \Lambda_S\}$, and for each $\lambda \in \Lambda_S$, S_λ is compact. Since S_λ is a compact subset of S, S_λ is a convergent sequence for every $\lambda \in \Lambda_S$. Therefore, $\mathcal{P}_\lambda$ is a *wcs*-network for S_λ in X_λ by Lemma 2.11. This proves that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double σ-finite *wcs*-cover for X.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double σ-finite wcs-cover for X. For each convergent sequence S of X, there exists a finite subset Λ_S of Λ such that $S = \bigcup\{S_\lambda : \lambda \in \Lambda_S\}$, and for each $\lambda \in \Lambda_S$, S_λ is a convergent sequence and $\mathcal{P}_\lambda$ is a wcs-network for S_λ in X_λ. For each $\lambda \in \Lambda_S$, it follows from Lemma 2.11 that $S_\lambda = f_\lambda(L_\lambda)$ for some compact subset L_λ of M_λ. Then $L = \bigcup\{L_\lambda : \lambda \in \Lambda_S\}$ is a compact subset of M and $f(L) = S$. This proves that f is a pseudo-sequence-covering mapping.

(4). *Necessity.* Let f be a sequentially-quotient mapping. For each convergent sequence S of X, there exists a convergent sequence L of M such that $f(L)$ is a subsequence of S. As in the necessity of (1), $f(L)$ is eventually in some X_λ and $\mathcal{P}_\lambda$ is a cs-network for $f(L)$ in X_λ. This proves that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double σ-finite cs^*-cover for X.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double σ-finite cs^*-cover for X. For each convergent sequence S of X, there exist $\lambda \in \Lambda$ and a subsequence S_λ of S such that $\mathcal{P}_\lambda$ is a cs^*-network for S_λ in X_λ. It follows from Lemma 2.11 that $S_\lambda = f(L_\lambda)$ for some compact subset L_λ of M_λ. Then f is a subsequence-covering mapping. By [8, Proposition 2.1], f is a sequentially-quotient mapping. $\square$

By Theorem 3.6 and Theorem 3.7, we get the following.

Corollary 3.8. *Let $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ be an ls-σ-finite Ponomarev-system. Then the following are equivalent:*

(1) *f is a sequence-covering s-mapping.*
(2) *$\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a point-countable σ-finite cs-cover for X.*

The above statement still holds if we replace the pair ("sequence-covering", "cs-cover") by any of the following pairs: ("compact-covering", "cfp-cover"), ("pseudo-sequence-covering", "wcs-cover") and ("sequentially-quotient", "cs-cover") and replace the pair ("s-mapping", "point-countable") by any of the following pairs: ("msss-mapping", "locally countable") and ("mssc-mapping", "locally finite").*

Now, we systematically get characterizations of images of locally separable metric spaces under covering-mappings. For other characterizations of s-images (resp. cs-images) of locally separable metric spaces, see [1], [18] (resp. [12]). For a characterization of sequence-covering $msss$-images of locally separable metric spaces, see [4].

Corollary 3.9. *The following are equivalent for a space X.*

(1) *X is a sequence-covering s-image (resp. cs-image) of a locally separable metric space.*
(2) *X has a point-countable (resp. compact-countable) double σ-finite cs-cover.*

Moreover, the statement still holds if we replace the pair ("sequence-covering", "cs-cover") by any of the following pairs: ("compact-covering", "cfp-cover"), ("pseudo-sequence-covering", "wcs-cover") and ("sequentially-quotient", "cs-cover").*

Proof. (1) $\Rightarrow$ (2). Let X be a sequence-covering s-image (resp. cs-image) of a locally separable metric space. Then there exists a sequence-covering s-mapping (resp. cs-mapping) $f : M \longrightarrow X$ from a locally separable metric space M onto X. Since M is locally separable metric space, $M = \oplus_{\lambda\in\Lambda} M_\lambda$ where each M_λ is a separable metric space by [6, 4.4.F]. Since each M_λ is a separable metric space, M_λ has a countable base $\mathcal{B}_\lambda = \{B_{\lambda,n} : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $\mathcal{B}_{\lambda,n} = \{B_{\lambda,n}, M_\lambda\}$, then $\bigcup\{\mathcal{B}_{\lambda,n} : n \in \mathbb{N}\}$ is a σ-finite cs-network for M_λ. It is easy to see that $\{(M_\lambda, \{\mathcal{B}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double σ-finite cs-cover for M. By Lemma 3.3, we have that $\{(f(M_\lambda), \{f(\mathcal{B}_{\lambda,n})\}) : \lambda \in \Lambda\}$ is a point-countable (resp. compact-countable) double σ-finite cs-cover for X.

(2) $\Rightarrow$ (1). Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a point-countable (resp. compact-countable) double σ-finite cs-cover for X. Then the ls-σ-finite Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ exists. By Corollary 3.8, X is a sequence-covering s-image (resp. cs-image) of a locally separable metric space.

The remaining parts are similarly proved. □

Corollary 3.10. *The following are equivalent for a space X.*

1. (1) *X is a sequence-covering (resp. compact-covering, pseudo- sequence-covering, sequentially-quotient) msss-image of a locally separable metric space.*
2. (2) *X has a σ-locally countable cs-network (resp. cfp-network, wcs-network, cs^*-network) consisting of cosmic subspaces.*
3. (3) *X has a σ-locally countable cs-network (resp. cfp-network, wcs-network, cs^*-network) consisting of $\aleph_0$-subspaces.*
4. (4) *X has a locally countable double σ-finite cs-cover (resp. cfp-cover, wcs-cover, cs^*-cover).*

Proof. (1) $\Rightarrow$ (2). In view of the proof of [4, Theorem 2.8], we have that X has a σ-locally countable cs-network (resp. cfp-network, wcs-network, cs^*-network) consisting of cosmic subspaces.

(2) $\Rightarrow$ (3). Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-locally countable cs-network (resp. cfp-network, wcs-network, cs^*-network) for X consisting of cosmic subspaces. We shall prove that each $P \in \mathcal{P}$ is an $\aleph_0$-space.

For each $n \in \mathbb{N}$ and $x \in P$, since $\mathcal{P}_n$ is locally countable, there exists an open neighborhood $U_{x,n}$ of x such that $U_{x,n}$ meets only countably many members of $\mathcal{P}_n$. Since P is a cosmic space and $\{U_{x,n} : x \in P\}$ is an open cover for P, there exists a countable subset C of P such that $\{U_{x,n} : x \in C\}$ is a countable open cover for P. Then P meets only countably many members of $\mathcal{P}_n$. This implies that P meets only countably many members of $\mathcal{P}$. Therefore $\mathcal{Q} = \{Q\cap P : Q \in \mathcal{P}\}$ is countable. Since $\mathcal{P}$ is a cs-network (resp. cfp-network, wcs-network, cs^*-network) for X, $\mathcal{Q}$ is a cs-network (resp. cfp-network, wcs-network, cs^*-network) for P. Then $\mathcal{Q}$ is a countable cs-network (resp. cfp-network, wcs-network, cs^*-network) for P. This proves that P is an $\aleph_0$-space.

(3) $\Rightarrow$ (4). Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-locally countable cs-network (resp. cfp-network, wcs-network, cs^*-network) for X consisting of $\aleph_0$-subspaces, where each $\mathcal{P}_n = \{P_{\alpha_n} : \alpha_n \in A_n\}$ is a locally countable family. For each $n \in \mathbb{N}$, since

each P_{α_n} is an $\aleph_0$-space, P_{α_n} has a countable cs-network (resp. cfp-network, wcs-network, cs^*-network) $\mathcal{P}_{\alpha_n} = \{P_{\alpha_{n,i}} : i \in \mathbb{N}\}$ where $P_{\alpha_{n,i}} = P_{\alpha_n}$ for every $i < n$. For each $i \in \mathbb{N}$, let $\mathcal{Q}_{\alpha_{n,i}} = \{P_{\alpha_n}\} \cup \{P_{\alpha_{n,j}} : j \leq i\}$. Then $\bigcup\{\mathcal{Q}_{\alpha_{n,i}} : i \in \mathbb{N}\}$ is a σ-finite cs-network (resp. cfp-network, wcs-network, cs^*-network) for P_{α_n}. For each $n \in \mathbb{N}$, let $\mathcal{Q}_n = \bigcup\{\mathcal{Q}_{\alpha_{i,n}} : \alpha_i \in A_i, i \leq n\}$. Since each $\mathcal{P}_n$ is locally countable and each $\mathcal{Q}_{\alpha_{i,n}}$ is finite, $\mathcal{Q}_n$ is locally countable. This proves that $\{(P_{\alpha_n}, \{\mathcal{Q}_{\alpha_{n,i}}\}) : \alpha_n \in A_n, n \in \mathbb{N}\}$ is a locally countable σ-finite double cs-cover (resp. cfp-cover, wcs-cover, cs^*-cover) for X.

(4) $\Rightarrow$ (1). Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a locally countable double σ-finite cs-cover (resp. cfp-cover, wcs-cover, cs^*-cover) for X. Then the ls-σ-finite Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ exists. By Corollary 3.8, X is a sequence-covering (resp. compact-covering, pseudo-sequence-covering, sequentially-quotient) $msss$-image of a locally separable metric space. □

Corollary 3.11. *The following are equivalent for a space X.*

(1) *X is a sequence-covering (resp. compact-covering, pseudo-sequence-covering, sequentially-quotient) mssc-image of a locally separable metric space.*
(2) *X has a σ-locally finite cs-network (resp. cfp-network, wcs-network, cs^*-network) consisting of cosmic subspaces.*
(3) *X has a σ-locally finite cs-network (resp. cfp-network, wcs-network, cs^*-network) consisting of $\aleph_0$-subspaces.*
(4) *X has a locally finite double σ-finite cs-cover (resp. cfp-cover, wcs-cover, cs^*-cover).*

Proof. (1) $\Rightarrow$ (2). Let $f : M \longrightarrow X$ be a sequence-covering (resp. compact-covering, pseudo-sequence-covering, sequentially-quotient) $mssc$-mapping from a locally separable metric space M onto X, and let $\{X_n : n \in \mathbb{N}\}$ be the family of metric spaces such that M is a subspace of $\prod_{n\in\mathbb{N}} X_n$, and for each $x \in X$, there exists a sequence $\{V_{x,n} : n \in \mathbb{N}\}$ of open neighborhoods of x in X such that each $\overline{p_n(f^{-1}(V_{x,n}))}$ is a compact subset of X_n. Since M is a locally separable metric space, $M = \oplus_{\lambda\in\Lambda} M_\lambda$, where each M_λ is a separable metric space by [6, 4.4.F]. Since each X_n is a metric space, X_n has a σ-locally finite base $\mathcal{C}_n \bigcup\{\mathcal{C}_{n,i} : i \in \mathbb{N}\}$, where each $\mathcal{C}_{n,i}$ is locally finite. Assume, if necessary, that $\mathcal{C}_{n,i} \subset \mathcal{C}_{n,i+1}$ for every $i \in \mathbb{N}$. For each $n \in \mathbb{N}$, we define

$$\mathcal{B}_n = \{ \bigcap_{i\leq n} p_i^{-1}(C_i) : \bigcap_{i\leq n} p_i^{-1}(C_i) \subset M_\lambda, C_i \in \bigcup_{j\leq n} \mathcal{C}_{i,j}, i \leq n, \lambda \in \Lambda\},$$

and

$$\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \mathbb{N}\}, \mathcal{P}_n = f(\mathcal{B}_n), \mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}.$$

Then $\mathcal{B}$ is a base for M consisting of separable subsets. This implies that $\mathcal{B}$ is a cs-network (resp. cfp-network, wcs-network, cs^*-network) for M consisting of cosmic subspaces. Since f is a sequence-covering (resp. compact-covering, pseudo-sequence-covering, sequentially-quotient) mapping, $\mathcal{P}$ is a cs-network (resp. cfp-network, wcs-network, cs^*-network) for X consisting of cosmic subspaces.

For $x \in X$ and $n \in \mathbb{N}$, let $U_{x,n} = \bigcap_{i \leq n} V_{x,i}$. Then $U_{x,n}$ is an open neighborhood of x in X. For each $i \in \mathbb{N}$, since $\overline{p_i(f^{-1}(V_{x,i}))}$ is a compact subset of X_i and $\mathcal{C}_{i,j}$ is locally finite, $p_i(f^{-1}(V_{x,i}))$ meets only finitely many members of $\mathcal{C}_{i,j}$ for each $j \in \mathbb{N}$. So $f^{-1}(V_{x,i})$ meets only finitely many members of $\{p_i^{-1}(C_i) : C_i \in \bigcup_{j \leq n} \mathcal{C}_{i,j}\}$. This implies that $f^{-1}(U_{x,n})$ meets only finitely many members of $\mathcal{B}_n$. Then $\mathcal{P}_n$ is locally finite. This implies that $\mathcal{P}$ is σ-locally finite.

By the above inclusions, $\mathcal{P}$ is a σ-locally finite cs-network (resp. cfp-network, wcs-network, cs^*-network) for X consisting of cosmic subspaces.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1). In view of the proof (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) of Corollary 3.10. $\square$

Remark 3.12. Since subsequence-covering mappings and sequentially-quotient mappings are equivalent for metric domains, "sequentially-quotient" in the above results can be replaced by "subsequence-covering".

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