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PLURIPOTENTIAL THEORY ON ANALYTIC SETS AND APPLICATIONS TO ALGEBRAICITY

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Dedicated to the memory of Le Van Thiem

1. Introduction

Our aim here is first to investigate some aspects of pluripotential theory on analytic sets and then to give, as an application, a quantitative version of Sadullaev's criterion of algebraicity for complex analytic sets ([Sa]) in terms of their local behaviour in the spirit of our earlier work on the subject (see [Z2]). By an *analytic subset* in $\mathbb{C}^N$ we mean a local irreducible analytic subset of $\mathbb{C}^N$ and *a piece of an algebraic set* will be a local irreducible analytic subset of an irreducible algebraic subvariety of $\mathbb{C}^N$ of the same dimension. Then given an analytic subset X in $\mathbb{C}^N$, the fundamental criterion of algebraicity of A. Sadullaev [Sa] states that X is a piece of an algebraic set if and only if there exists a compact subset $K \subset X$ such that for any subdomain $U \Subset X$ there exists a constant $R = R(K;U) > 0$ such that the following polynomial inequalities known as Bernstein-Walsh inequalities hold:

$$\|f\|_U \leq \|f\|_K \; R^d, \quad \forall f \in \mathcal{A}_d(X), \quad \forall d \in \mathbb{N}, \tag{1.1}$$

where $\mathcal{A}_d(X) := \{P|X; P \in \mathcal{P}_d(\mathbb{C}^N)\}$ and $\mathcal{P}_d(\mathbb{C}^N)$ is the space of polynomials in N complex variables of degree at most d. Moreover if X is a piece of an algebraic set then the Bernstein-Walsh inequalities (1.1) hold for any non pluripolar compact subset $K \subset X$ and any subdomain $U \Subset X$. Observe that the best constant $R(K;U) > 0$ for which (1.1) holds is given by the following formula

$$R(K;U)^{-1} = \tau(K;U) := \inf_{d \geq 1} \tau_d(K;U) \tag{1.2}$$

where for each $d \in \mathbb{N}^*$

$$\tau_d(K;U) := \inf \{\|f\|_K^{1/d}; f \in \mathcal{A}_d(X), \|f\|_U = 1\} \tag{1.3}$$

is the *Chebyshev constant* of order d of the compact set K with respect to the set U. Then it is possible to formulate Sadullaev's criterion of algebraicity saying that X is a piece of an algebraic set if and only if for some compact set K and some subdomain $U \Subset X$ containing K, the Chebyshev constant $\tau(K;U) > 0$.

Our main goal is to give more precise statement in terms of the behaviour of the Chebyshev constant defined by (1.3). First we show that these constants can be localised and expressed as the algebraic multiplicities of the analytic subset

X at its regular point and then we use these multiplicities in order to estimate the Hilbert function of X defined as follows

$$h_X(d) := \dim_{\mathbb{C}} \mathcal{A}_d(X), \quad d \in \mathbb{N}^*. \tag{1.4}$$

It turns out that the Hilbert function of X coincides with the Hilbert function of the following algebraic set

$$Z(X) := \operatorname{loc}(\mathcal{I}(X)) \tag{1.5}$$

which is the set locus of the polynomial ideal $\mathcal{I}(X)$ of the set X defined as follows

$$\mathcal{I}(X) := \big\{P \in \mathbb{C}[z_1, \dots, z_N]; P|X \equiv 0\big\}. \tag{1.6}$$

In deed, from the Nullstellensatz of Hilbert (see [Ha]) if follows that $\mathcal{I}(Z) \equiv \operatorname{Rad}(\mathcal{I}(X)) = \mathcal{I}(X)$, since $Z = Z(X) := \operatorname{loc}(\mathcal{I}(X))$. Then the following identity holds

$$h_Z(d) = h_X(d), \quad \forall d \in \mathbb{N}. \tag{1.7}$$

On the other hand, it is well known in Algebraic Geometry that for d large enough, the Hilbert function of Z is a polynomial in d of degree $p = p(Z) := \dim_{\mathbb{C}} Z$ whose leading coefficient is $\delta(Z) \cdot d^p / p!$, where $\delta(Z)$ is the degree of algebraicity of Z that is the number of points of intersection of Z with a generic $(N-p)$-plane in $\mathbb{C}^N$ (see [Ha]). Thus the asymptotic behaviour of the Hilbert function $h_X(d)$ will give us useful informations about the algebraic subvariety $Z(X)$. In particular we obtain an estimate of the degree of algebraicity of $Z(X)$ in terms of its minimal graded multiplicity. Some of the results obtained here are similar to our earlier results on the subject (see [Z2]), but to make the paper selfcontained we have presented direct proofs of the main results in the spirit of our paper [Z2]. Furthermore we show that the criterion of algebraicity obtained here can be easily extended to real analytic subsets.

2. Chebyshev constants and graded multiplicities

2.1. Chebyshev constants

From our point of view, it is also natural to consider the following induced class of plurisubharmonic functions on X:

$$\mathcal{L}_X := PSH(X) \cap \big\{v|X; v \in \mathcal{L}(\mathbb{C}^N)\big\}. \tag{2.1}$$

Here we denote by $PSH(X)$ the cone of plurisubharmonic functions on X in the weak sense, which are not identically $-\infty$ on X. Then $PSH(X) \subset L^1_{loc}(X)$ and it is well known that $PSH(X)$ endowed with the L^1_{loc} topology is a closed subset (see [Hö 2]).

Define $\mathcal{L}(X)$ to be the class of plurisubharmonic functions of *restricted logarithmic growth* on X, which is the closure of $\mathcal{L}_X$ in $PSH(X)$. Then the class $\mathcal{L}(X)$ is a convex translation-invariant class on X which contains the real constants (see [Z2]). To each subset $E \Subset X$ we can associate its $\mathcal{L}(X)$-extremal function defined by the following formula

$$L_E(x) := \sup\big\{v(x); v \in \mathcal{L}(X), v|E \leq 0\big\}, \quad x \in X \tag{2.2}$$

A compact subset $B \subset X$ will be called *pluriregular* in X if for any plurisubharmonic function u in a neighbourhood of B the inequality $u \leq 0$ quasi-everywhere on B implies the inequality $u \leq 0$ everywhere on B (see [Sic2] for the case of $\mathbb{C}^N$ and [Z1] for the general case).

Let $B \subset X$ be a fixed pluriregular compact subset of X. Then for any compact subset $K \subset X$, the $\mathcal{L}(X)$-capacity of K with respect to B is defined as follows

$$\mathrm{cap}_{\mathcal{L}}(K;B) := \exp\big(-\sup_{x\in B} L_K(x)\big). \tag{2.3}$$

On the other hand, for each integer $d \in \mathbb{N}^*$ and each compact subset $K \subset X$, we can define the d^{th} *Chebyshev constant* of the compact set K with respect to B as follows

$$\tau_d(K;B) := \inf\big\{\|f\|_K^{1/d}; f \in \mathcal{A}_d(X), \|f\|_B = 1\big\}. \tag{2.4}$$

The relationship between capacity and Chebyshev constants is given by the following result, which can be proved using a classical argument (see [Sic2]).

Lemma 2.1. *Let B be a pluriregular compact subset of X, then for each compact subset $K \subset X$, the following limit exists*

$$\tau(K;B) := \inf_{d\geq 1} \tau_d(K;B) = \lim_{d\to+\infty} \tau_d(K;B). \tag{2.5}$$

Moreover the following identity holds:

$$\tau(K;B) = cap_{\mathcal{L}}(K;B). \tag{2.6}$$

Proof. We proceed as in [Sic2]. For each $d \in \mathbb{N}^*$, we define the following function

$$\Phi_d(x) := \sup\big\{|f(x)|^{1/d}; f \in \mathcal{A}_d(X), \|f\|_K \leq 1\big\}, \quad x \in X. \tag{2.7}$$

Then it is clear from the definitions (2.4) and (2.7) that the following equation holds

$$\tau_d(K;B) = \frac{1}{\sup_B \Phi_d}, \quad \forall d \in \mathbb{N}^*. \tag{2.8}$$

Now it is clear from the definition (2.7) that for each $x \in B$, the sequence of real numbers $d \longmapsto (\Phi_d(x))^d$ is logarithmically superadditive and then the following limit exists

$$\Phi_K(x) := \lim_{d\to+\infty} \Phi_d(x) = \sup_{d\geq 1} \Phi_d(x) \tag{2.9}$$

in $\mathbb{R}^+ \cup \{+\infty\}$. Clearly, the equations (2.9) imply that $\sup\limits_B \Phi_K = \sup\limits_{d\geq 1}\big(\sup\limits_B \Phi_d\big)$. Moreover, for any $x \in B$ the following inequalities holds

$$\Phi_K(x) \leq \liminf_d \big(\sup_B \Phi_d\big) \leq \limsup_d \big(\sup_B \Phi_d\big) \leq \sup_d \big(\sup_B \Phi_d\big) = \sup_B \Phi_K$$

and then $\lim\limits_{d\to+\infty}\big(\sup\limits_B \Phi_d\big) = \sup\limits_B \Phi_K$, which proves the identities (2.5). The equation (2.6) follows from the fact that $L_K = \log \Phi_K$ on X (see [Sic1], [Z1]. □

We can also use a normalization with respect to a probability measure ϑ on B as for Alexander's projective capacity (see [A]). The measures which will be interesting for us are the so called *admissible measures* defined as follows.

Definition 2.1. A probability measure ϑ on a compact subset $B \subset X$ is called an admissible measure on B if any $u \in PSH(X)$ is ϑ-integrable on B.

This is a variant of the definition introduced by Ciciak $\mathbb{C}^N$ (see [Sic2]).

Let us now give some examples.

Example. 1) Let $U \Subset X_{reg}$ be a coordinate neighbourhood in X such that there exists a biholomorphic mapping ϕ from U onto the open unit polydisc in $\mathbb{C}^n$ which extends continuously to $\overline{U}$. Then the push-forward by ϕ of the normalized Lebesgue measure on the closed unit polydisc in $\mathbb{C}^n$ (resp. on the torus $\mathbb{T}^n$) is an admissible measure on $\overline{U}$ (resp. on $\partial_0 U := \phi^{-1}(\mathbb{T}^n)$).

2) Let K be a non pluripolar compact subset of X which has a Stein neighborhood D. Then it is known that there exists a locally bounded plurisubharmonic function w of D such that the Monge-Ampère measure $(dd^c w)^n$ puts a positive mass on K: for example the relative equilibrium measure of the condenser (K, D) always has this property (see B-T], [B]). Then it is also known that any plurisubharmonic function u on X is locally integrable with respect to the Monge-Ampère measure $(dd^c w)^n$ (see [D]). Therefore the restriction of the measure $\vartheta_w := (dd^c w)^n$, up to a normalization, gives raise to an admissible measure on K.

Let

$$\tau_d(K; B; \vartheta) := \inf \left\{ \|f\|_K^{1/d}; f \in \dot{\mathcal{A}}_d(X; B; \vartheta) \right\}, \quad d \in \mathbb{N}^*, \tag{2.10}$$

where

$$\dot{A}_d(X; B; \vartheta) := \left\{ f \in \mathcal{A}_d(X); \int_B \log |f| dv = 0 \right\}. \tag{2.11}$$

It is clear that $\tau_d(K; B) \leq \tau_d(K; B; \vartheta)$ for any $d \in \mathbb{N}^*$. The main advantage of this normalization is explained by the following result.

Lemma 2.2. *The sequence of real numbers defined by the formula*

$$T_d(K; B; \vartheta) := \big(\tau_d(K; B; \vartheta)\big)^d, \quad d \in \mathbb{N}^* \tag{2.12}$$

satisfies the following logarithmic subadditivity property

$$T_{d+d'}(K; B; \vartheta) \leq T_d(K; B; \vartheta) \cdot T_{d'}(K; B; \vartheta) \tag{2.13}$$

for any $d \in \mathbb{N}^*$ and $d' \in N^*$. Thus the following limit exists

$$\tau(K; B; \vartheta) := \lim_{d \to +\infty} \tau_d(K; B; \vartheta) = \inf_{d \geq 1} \tau_d(K; B; \vartheta). \tag{2.14}$$

The proof of such results is standard and will be be omitted (see [Sic2], [T]). The constant defined by the formula (2.14) will be called the Chebyshev constant of K with respect to (B,ϑ).

In order to compare the Chebyshev constants defined by (2.4) and (2.10), we will need the next theorem, which is an analogue of a result of Siciak (see [Sic 2]).

Lemma 2.3. *Let X be an analytic subset and $B\subset X$ a pluriregular compact subset. If ϑ is an admissible measure on B, then for any open subset $\omega \Subset X$ such that $B\subset\omega$, there exists a constant $C>0$ such that the following inequality holds:*

$$\Big|\int_B u\,d\vartheta\Big| \le C\cdot\int_\omega |u|\,d\lambda, \quad \forall u\in PSH(X). \tag{2.15}$$

where λ is the induced Lebesgue measure on ω.

Proof. Assume that the inequality (2.15) is not satisfied, then there exists a sequence (u_j) of functions in $PSH(X)$ such that

$$\int_\omega |u_j|\,d\lambda = 1 \quad \text{for any } j\in\mathbb{N}^* \text{ and } \quad \lim_{j\to+\infty}\Big|\int_B u_j\,d\vartheta\Big| = +\infty$$

From the submean-value inequality, we easily deduce that for any compact subset $K\subset\omega$, there exists a constant $A>0$ such that

$$\max_K u_j \le A\int_\omega |u_j|\,d\vartheta = A \quad \text{for any } j\in\mathbb{N}^*.$$

Hence the sequence (u_j) is locally bounded above on ω and then

$$\lim_{j\to+\infty}\int_B u_j\,d\vartheta = -\infty.$$

Now fix an arbitrary sequence (ε_j) of positive numbers such that $\sum_j \varepsilon_j = 1$. Then the function $u := \sum_j \varepsilon_j u_j$ is plurisubharmonic on X and since the series converge in $L^1(\omega)$, we conclude that $u\in PSH(X)$. Therefore we have

$$\sum_j \varepsilon_j \int_B u_j\,d\vartheta = \int_B u\,d\vartheta > -\infty.$$

On the other hand, since

$$\lim_{j\to+\infty}\int_B u_j\,d\vartheta = -\infty,$$

we can choose the sequence (ε_j) such that

$$\sum_j \varepsilon_j \cdot \Big(\int_B u_j d\vartheta \Big) = -\infty,$$

which gives a contradiction. □

Now we can prove the following result which is a generalisation of a result in [A] (see also [Sic2]).

Theorem 2.1. *Let B be a non pluripolar compact subset of X and ϑ an admissible measure on B. Then for each integer $d \geq 1$, there exists a constant $\gamma_d > 0$ such that*

$$\frac{1}{d} \log \|f\|_B \leq \frac{1}{d} \int_B \log |f| d\vartheta + \gamma_d, \quad \forall f \in \mathcal{A}_d(X). \tag{2.16}$$

In particular, for any compact subset $K \subset X$, the following estimates holds:

$$\log \tau_d(K;B) \leq \log \tau_d(K;B;\vartheta) \leq \log \tau_d(K;B) + \gamma_d \tag{2.17}$$

Moreover if X is a piece of an algebraic set and $B \subset X$ is a pluriregular compact subset then the best constants γ_d in (2.16) satisfy the condition $\gamma := \sup_d \gamma_d < +\infty$ and the following inequality holds

$$\max_B u \leq \int_B u d\vartheta + \gamma, \quad \forall u \in \mathcal{L}(X) \tag{2.18}$$

Proof. Since B is pluripolar, it follows that $\|.\|_B$ is a norm on the finite dimensional space $\mathcal{A}_d(X)$. On the other hand, since an admissible measure cannot have a pluripolar carrier, the maping

$$f \longmapsto \int_B \log |f| d\vartheta$$

is well defined and is continuous on the sphere $\dot{\mathcal{A}}_d := \big\{ f \in \mathcal{A}_d(X); \|f\|_B = 1 \big\}$, which is a compact subset of $\mathcal{A}_d(X)$. Therefore the positive constant

$$\gamma_d := -\inf \Big\{ (1/d) \int_B \log |f| d\vartheta; f \in \dot{\mathcal{A}}_d \Big\}$$

is finite and satisfies the inequality (2.17). The estimates (2.18) follow from (2.17) and the definitions (2.4) and (2.10).

Now assume that X is a piece of an algebraic set in $\mathbb{C}^N$, then we know from [Sa] (see also [Z2]) that the following extremal function

$$L_B(x) := \sup_{d \geq 1} \Big\{ \frac{1}{d} \log |f(x)|; f \in \mathcal{A}_d(X), \|f\|_B = 1 \Big\}, \quad x \in X \tag{2.19}$$

is locally bounded on X. Moreover, it follows from [Z2] that the following set

$$\mathcal{K} := \{u \in \mathcal{L}(X); \max_B u = 0\}$$

is compact in $PSH(X)$ and then the right hand side of (2.15) is bounded on $\mathcal{K}$, which implies that the left hand side is also bounded on $\mathcal{K}$ and proves that $\gamma := \sup_d \gamma_d < +\infty$.

Let us now prove the estimate (2.18). For every $u \in \mathcal{L}(X)$ there exists a sequence (d_j) of positive integers and a sequence of holomorphic functions (f_j) with $f_j \in \mathcal{A}_{d_j}(X)$ for any $j \in \mathbb{N}$ such that

$$u = \Big(\limsup_j (1/d_j) \log |f_j|\Big)^* \quad \text{on } X.$$

Then using the estimate (2.16), the Hartogs lemma and Fatou's lemma, we obtain the estimate (2.18) for u. □

Corollary 2.1. *Let X be a piece of an algebraic set, $B \subset X$ a pluriregular compact subset and ϑ an admissible measure on B. Then there exists a constant $C > 0$ such that for any compact subset $K \subset X$ the following estimates hold*

$$\tau(K; B) \leq \tau(K; B; \vartheta) \leq C\tau(K; B). \tag{2.20}$$

This means that in the algebraic case, the two capacities are equivalent and we know that their nul sets are precisely the pluripolar sets in X (see [Z1]).

2.2. Graded multiplicities

Now using the Chebyshev constants defined in the last subsection, we are going to define for each $d \in \mathbb{N}^*$, a d^{th} graded multiplicity at each regular point $a \in X_{reg}$ and use it to give an upper bound of the degree of the Hilbert function $h_X(d)$ of the analytic subset X.

Most of the results presented in this section are contained in our previous work (see [Z2]), but here we give a direct proof. As before, let $a \in X_{reg}$ be a regular point and $\phi := \overline{U} \to \overline{\Delta}^n$ a regular coordinate system at the point a. Let Δ_s^n be the open polydisc of radius $s > 0$ centered at the origin in $\mathbb{C}^n$ and consider the sets $B := \overline{U}$ and $B_s := \phi^{-1}(\overline{\Delta}_s^n)$, $0 < s < 1$. Then the following result will be important.

Lemma 2.4. *Let $a \in X_{reg}$ be a fixed point and (U, ϕ) a regular coordinate system at a. Then the following properties hold:*

1) *For each integer $d \geq 1$, the following limit exists and is finite*

$$\kappa_d(a) := \lim_{s \to 0} \frac{\log \tau_d(B_s; B)}{\log s}\,; \tag{2.21}$$

2) *For any admissible measure ϑ on $B := \overline{U}$, the following formula holds*

$$\kappa_d(a) = \lim_{s \to 0^+} \frac{\log \tau_d(B_s; B; \vartheta)}{\log s}\,, \quad \forall d \geq 1; \tag{2.22}$$

3) *The sequence of integers* $d \longmapsto \lambda_d(a) := d \cdot \kappa_d(a)$ *satisfies the following superadditivity property*

$$\lambda_{d+d'}(a) \geq \lambda_d(a) + \lambda_{d'}(a), \quad \forall d \in N^*, \ \forall d' \in \mathbb{N}^*. \tag{2.23}$$

So that the following limit exists

$$\kappa_X(a) := \lim_{d \to +\infty} \kappa_d(a) = \sup_{d \geq 1} \kappa_d(a) \tag{2.24}$$

in $\mathbb{R}^+ \cup \{+\infty\}$.

Proof. From the definition of the d^{th} Chebyshev constant we get the following formula

$$\frac{\log \tau_d(B_s; B)}{\log s} = \sup \Big\{ \frac{\log \|f\|_{B_s}}{d \cdot \log s}; f \in \mathcal{A}_d(X), \|f\|_B = 1 \Big\} \tag{2.25}$$

for any $s \in]0,1[$ and any $d \in \mathbb{N}^*$. It is well known that for each $f \in \mathcal{A}_d(X)$ with $\|f\|_B = 1$, the function

$$s \longmapsto \log \|f\|_{B_s} = \sup \big\{ \log |f \circ \phi^{-1}(z)|; \|z\| \leq s \big\}$$

is a convex (increasing) function of $\log s$ on the real interval $]0,1[$. Hence the following function

$$s \in]0,1[\longmapsto \frac{\log \tau_d(B_s; B)}{\log s}$$

is an increasing function with positive real values. Thus the limit in (2.21) exists and the following formula holds

$$\kappa_d(a) = \inf_{s>0} \Big(\sup \Big\{ \frac{\log \|f\|_{B_s}}{d \cdot \log s}; \ f \in \mathcal{A}_d(X), \|f\|_B = 1 \Big\} \Big) \tag{2.26}$$

for any $d \in \mathbb{N}^*$.

To prove the formula (2.22), one can apply Corollary 2.1 on admissible measures to obtain the following estimates:

$$\frac{\log \tau_d(B_s; B)}{\log s} + \frac{c_d}{\log s} \leq \frac{\log \tau_d(B_s; B; \vartheta)}{\log s} \leq \frac{\log \tau_d(B_s; B)}{\log s} \tag{2.27}$$

for any $s \in]0,1[$ and any $d \geq 1$. Taking the limit in (2.27) when $s \to 0^+$, we obtain the formula (2.22).

Now to prove the subadditivity property (2.23) of the lemma, apply the formula (2.22) to the normalized Lebesgue measure ϑ on $B := \overline{U}$ which is an admissible measure on B as it was seen in the examples above. Then we obtain the following formula

$$\kappa_d(a) = \lim_{s \to 0^+} \frac{\log \tau_d(B_s; B; \vartheta)}{\log s}, \quad \forall d \geq 1. \tag{2.28}$$

From the formula (2.28) and the subadditivity of the sequence

$$d \longmapsto \log T_d(B_s; B; \vartheta) = d \cdot \log \tau_d(B_s; B; \vartheta)$$

it follows that the sequence $d \longmapsto \lambda_d(a)$ is superadditive and then the limit

$$\kappa_X(a) := \lim_{d\to+\infty} \kappa_d(a) = \sup_{d\geq 1} \kappa_d(a)$$

exists and may be infinite. □

Now we can prove the following fundamental result.

Theorem 2.2. *Let $a \in X_{reg}$ be a regular point and (U, ϕ) a regular coordinate system at the point a. Then the following properties hold:*

1) *The number defined by* (2.21) *satisfy the following identity*

$$\kappa_d(a) = \sup\big\{(1/d)\cdot m_f(a); f \in \mathcal{A}_d(X), f \not\equiv 0\big\}, \quad \forall d \in \mathbb{N}^* \tag{2.29}$$

where $m_f(a)$ is the order of vanishing of the holomorphic function f at the regular point a. In particular, the numbers defined by the formula (2.21) *are rational numbers which do not dependant on the regular coordinate system at the point a.*

2) *For each $d \geq 1$, the function κ_d is upper semi-continuous on X_{reg}.*

Proof. From the proof of Lemma 2.4, we get the following formula

$$\kappa_d(a) = \inf_{s>0}\Big(\sup\big\{\frac{\log\|f\|_{B_s}}{d\cdot\log s}; f \in \mathcal{A}_d(X), \|f\|_B = 1\big\}\Big) \tag{2.30}$$

for any $d \in \mathbb{N}^*$. On the other hand, we know that for any $f \in \mathcal{A}_d(X)$ such that $\|f\|_B = 1$, the number defined by the following formula:

$$\nu(\log|f|; a) := \inf_{s>0} \frac{\log\|f\|_{B_s}}{\log s} \tag{2.31}$$

is equal to the Lelong number of the plurisubharmonic function $\log|f|$ at the point a (see [Lel], [Ki]) and coincide with the order of vanishing of the holomorphic function f at the point $a \in X_{reg}$ (see [Hö 1]), that is

$$\nu(\log|f|; a) = m_f(a). \tag{2.32}$$

Let us denote by $\mu_d(a)$ the right hand side of the identity (2.29) and by $\mathcal{A}_d(X; B)$ the set of all polynomials $f \in \mathcal{A}_d(X)$ normalized by the condition $\|f\|_B = 1$. Then from the expression (2.31) and the equation (2.32), we deduce the following formula

$$\mu_d(a) = \sup_{f\in\mathcal{A}_d(X;B)} \Big\{\inf_{s>0} \frac{\log\|f\|_{B_s}}{d\cdot\log s}\Big\} = \frac{1}{d} \sup_{f\in\mathcal{A}_d(X;B)} \nu(\log|f|; a) \tag{2.33}$$

for any $d \in \mathbb{N}^*$. Therefore, from the formulas (2.30) and (2.33) it follows that $\mu_d(a) \leq \kappa_d(a)$.

On the other hand, fix an integer $d \in \mathbb{N}^*$ and let $\kappa < \kappa_d(a)$. Then by the formula (2.30). it follows that for every $s \in]0, 1[$, there exists $f_s \in \dot{\mathcal{A}}_d(X; B)$ such that

$$\frac{\log\|f\|_{B_s}}{\log s} > \kappa. \tag{2.34}$$

Taking a decreasing sequence $(s_j)_{j\geq 0}$ of numbers in $]0,1[$ converging to 0, we obtain a sequence $(f_{s_j})_{j\geq 0}$ from the set $\dot{\mathcal{A}}_d(X;B)$ satisfying the estimate (2.34) for $s = s_j$ and $f = f_{s_j}$ with $j \geq 1$. Since $\mathcal{A}_d(X)$ is of finite dimension, the set $\dot{\mathcal{A}}_d(X;B)$ is compact and then, taking a subsequence if necessary, we can assume that the sequence $(f_{s_j})_{j\geq 0}$ converges uniformly on B to a polynomial $f \in \dot{\mathcal{A}}_d(X;B)$. Since the function $s \longmapsto \log \|f\|_{B_s}/\log s$ is increasing on $[0,1]$ for any holomorphic function f, from the formula (2.34) it follows that for any $t \in]0,1[$ and any j large enough so that $0 < s_j < t$, the following inequalities hold

$$\frac{\log \|f_{s_j}\|_{B_t}}{\log t} \geq \frac{\log \|f_{s_j}\|_{B_{s_j}}}{\log s_j} > \kappa. \tag{2.35}$$

Since $\log \|f\|_{B_t} = \lim_{j\to+\infty} \log \|f_{s_j}\|_{B_t}$ for any $t \in]0,1[$, it follows from (2.35) that the following inequality holds:

$$\frac{\log \|f\|_{K_t}}{\log t} \geq \kappa, \quad \forall t \in]0,1[\tag{2.36}$$

From the formula (2.33) and the inequality (2.36), we get the inequality $\mu_d(a) \geq \kappa$, which implies that $\mu_d(a) \geq \kappa_d(a)$. Thus we have proved that $\kappa_d(a) = \mu_d(a)$. Therefore we obtain (2.29).

The upper semicontinuity of the function κ_d on X_{reg} follows by a standard argument from the upper semi-continuity of the mapping $(f,a) \longmapsto \nu(\log|f|;a)$ on $\dot{\mathcal{A}}_d(X) \times X_{reg}$ and the compactness of the set $\dot{\mathcal{A}}_d(X;B)$. □

The number $\mu_d(a)$ is called the *graded multiplicity* of order d of X at the point a and the number $\mu_X(a) := \sup_d \mu_d(a)$ is called the *graded multiplicity* of X at the point a. By the last theorem, we have $\kappa_d(a) = \mu_d(a)$ for any d.

3. A criterion of algebraicity for analytic subsets

Our goal here is to use the estimate on the Hilbert function of an analytic subset X in terms of its graded multiplicities at its regular points and to deduce a version of our previous criterion of algebraicity [Z2] which will be extended to the real case.

3.1. Asymptotic estimate on the Hilbert function

Let us recall here the main estimate obtained in [Z2] on the Hilbert function of an analytic subset in terms of the sequence of graded multiplicities of X at a regular point $a \in X_{reg}$ defined by the following formula

$$\mu_d(a) := \sup\left\{(1/d)m_f(a); f \in \mathcal{A}_d(X)\right\}, \quad d \in \mathbb{N}^*. \tag{3.1}$$

Theorem 3.1. *Let X be an analytic subset of dimension n in $\mathbb{C}^N$. Assume that the minimal graded multiplicity of X defined by $\mu(X) := \inf_{a\in X_{reg}} \mu_X(a)$ is finite.*

Then the Hilbert function of X defined by the formula (1.4) satisfy the following asymptotic estimate

$$\limsup_{d\to+\infty}\frac{h_X(d)}{d^n}\le\mu(X)^n. \tag{3.2}$$

Proof. For convenience, let us recall the main steps of the proof of this theorem. Fix a regular point $a\in X_{reg}$, an open neighbourhood $U\subset X$ of a and a biholomorphic mapping ϕ from U onto the open unit polydisc Δ^n in $\mathbb{C}^n$, which extends continuously to $B:=\overline{U}$. For each $f\in\mathcal{A}_d(X)$, define $\tilde f:=f\circ\phi^{-1}$, which is holomorphic on Δ^n and continuous up to the boundary. Let us denote by

$$\mathcal{Q}_d:=\{\tilde f; f\in\mathcal{A}_d(X)\}\quad\text{for } d\in\mathbb{N}^*.$$

The main idea of the proof, which is the same as in [Z2], is to compare the dimension of the complex linear space $\mathcal{Q}_d$ with the dimension of the well known complex linear space $\mathcal{P}_m(\mathbb{C}^n)$ for a suitable value of the integer $m:=m(d)$ and d large enough. To this end we will consider these spaces as subspaces of the Banach space $C_s:=C(\overline{\Delta}_s^n;\mathbb{C})$ of complex valued continuous functions on the compact polydisc $\overline{\Delta}_s^n$ endowed with the norm $\|.\|_s$ of uniform convergence on $\overline{\Delta}_s^n$. In this way, we will estimate the distance of any element of $\mathcal{Q}_d$ to the finite dimensional subspace $\mathcal{P}_m(\mathbb{C}^n)$ for a fixed integer $m\ge 1$. Indeed, each $F\in\mathcal{Q}_d$ is holomorphic on Δ^n and continuous on $\overline{\Delta}^n$, so it can be expanded into an entire series on the polydisc Δ^n as follows

$$F(z)=\sum_{\alpha\in\mathbb{N}^n}c_\alpha z^\alpha,\quad z\in\Delta^n \tag{3.3}$$

with uniform convergence on compacts subsets of Δ^n. Let us consider the Taylor polynomials of the function F given by the formula $T_m(z):=\sum_{|\alpha|\le m}c_\alpha z^\alpha$, for $m\in\mathbb{N}$.

Fix a real number θ such that $0<\theta<1/2$, take a real number $0<s<\theta$ and put $t:=s/\theta<1$. Then an easy computation using Cauchy's inequalities shows that there exists a constant c_n depending only on the dimension n such that the following estimates hold:

$$\|F-T_m\|_s\le c_n(n+m)^{n-1}\theta^{m+1}\|F\|_t,\quad\forall F\in\mathcal{Q}_d,\ \forall m\in\mathbb{N} \tag{3.4}$$

which imply immediately the following estimates:

$$\mathrm{dist}_{C_s}\big(F;\mathcal{P}_m(\mathbb{C}^n)\big)\le c_n(n+m)^{n-1}\theta^{m+1}\|F\|_t \tag{3.5}$$

for any $F\in\mathcal{Q}_d$, $m\in\mathbb{N}$ and $d\in\mathbb{N}$, the distance being calculated in the Banach space C_s. On the other hand, let us consider the numbers

$$\alpha_d(s):=\sup_{F\in\mathcal{Q}_d}\left\{\frac{\log\|F\|_s-\log\|F\|_1}{\log s}\right\}=\frac{d\cdot\log\tau_d(U_s;U)}{\log s} \tag{3.6}$$

where the last identity follows immediately from (2.4). Since for each $F\in\mathcal{Q}_d$, the function $r\longmapsto\log\|F\|_r$ is a convex function of the variable $\log r$ for $r\in]0,1]$, it is easy to derive from (3.6) the following fundamental inequality

$$\|F\|_t \leq \|F\|_s \theta^{-\alpha_d(s)}, \quad \forall F \in \mathcal{Q}_d. \tag{3.7}$$

Combining (3.5) with (3.7) we obtain the following fundamental estimates

$$\mathrm{dist}_{C_s}(F; \mathcal{P}_m(\mathbb{C}^n)) \leq c_n (n+m)^{n-1} \|F\|_s \theta^{m+1-\alpha_d(s)} \tag{3.8}$$

for any $F \in \mathcal{Q}_d$ and $m \in \mathbb{N}^*$.

Now take a real number $\mu > \mu(X)$. According to the definition of $\mu(X)$, we can choose the regular point $a \in X_{reg}$ so that $\mu_X(a) < \mu$. Fix $\varepsilon > 0$ and take a large integer d_0 such that

$$\eta_d := c_n (n + \mu d + \varepsilon d)^{n-1} \theta^{\varepsilon d} < 1, \quad \forall d \geq d_0. \tag{3.9}$$

Let $d \geq d_0$ and let m_d be the unique integer satisfying the inequalities $m_d \leq (\mu + \varepsilon) \cdot d < m_d + 1$. Observe that

$$\lim_{s \to 0^+} \alpha_d(s) = d \cdot \kappa_d(a) \leq d \cdot \mu_X(a) < d \cdot \mu,$$

thanks to (3.6) and (2.1). Then it is possible to choose s so small that $0 < s < \theta$ and $\alpha_d < d \cdot \mu$, which implies that $m_d + 1 - \alpha_d(s) \geq \varepsilon d$. From (3.8) and (3.9) we deduce the following estimates:

$$\mathrm{dist}_{C_s}(F; \mathcal{P}_{m_d}(\mathbb{C}^n)) \leq \eta_d \cdot \|F\|_s \leq \|F\|_s, \quad \forall F \in \mathcal{F}_d \setminus \{0\}. \tag{3.10}$$

Using (3.10) we want to show that $\dim \mathcal{A}_d \leq \dim \mathcal{P}_{m_d}(\mathbb{C}^n)$. Assume that the converse is true, that is $\dim \mathcal{Q}_d > \dim \mathcal{P}_{m_d}(\mathbb{C}^n)$. Since $\mathcal{P}_{m_d}(\mathbb{C}^n)$ is a subspace of finite dimension of Banach space C_s, we can apply the "projection theorem" in Banach spaces, known as the Krein-Krasnoselski-Milman theorem (see [Sin]), to obtain a function $F_0 \in \mathcal{F}_d \setminus \{0\}$ which is "orthogonal" to the subspace $\mathcal{P}_{m_d}(\mathbb{C}^n)$ in the Banach space C_s in the sense that

$$\big\|F_0\big\|_s = \mathrm{dist}_{C_s}(F_0; \mathcal{P}_{m_d}(\mathbb{C}^n)).$$

This contradicts (3.10) and proves the following inequality

$$\dim \mathcal{A}_d = \dim \mathcal{Q}_d \leq \dim \mathcal{P}_{m_d}(\mathbb{C}^n) = \binom{m_d + n}{n}, \quad \forall d \geq d_0. \tag{3.11}$$

Since $m_d \sim (\mu + \varepsilon)^n d^n$ as $d \to +\infty$ and $\mu > \mu(X)$ and $\varepsilon > 0$ are arbitrary, the estimate (3.11) implies clearly the estimate (3.1), which prove the theorem. $\square$

The use of the Krein-Krasnoselski-Milman theorem is originated to W. Plesniak [P]. This theorem has also been used in our earlier papers on the subject ([Z1], [Z2]).

3.2. A criterion of algebraicity for complex analytic subsets

From the main theorem of the last section, we will derive some general information about the Zariski closure $Z(X)$ of an analytic set X.

The main result of this section will be the following version of our earlier criterion of algebraicity [Z2].

Theorem 3.2. *Let X be an analytic subset of dimension n in $\mathbb{C}^N$. Then the following statements are equivalent:*

(i) *X is a piece of an algebraic set;*

(ii) *For any regular point $a \in X_{reg}$, the graded multiplicity of X at a is finite, i.e. $\mu_X(a) := \sup_{d\geq 1} \mu_d(a) > +\infty$;*

(iii) *For some regular point $a \in X_{reg}$, the graded multiplicity of X at a is finite, i.e. $\mu_X(a) := \sup_{d\geq 1} \mu_d(a) < +\infty$.*

Moreover, if one of these conditions is satisfied then X is a piece of an algebraic set whose degree of algebraicity $\delta(Z)$ satisfies the inequalities $\mu(X) \leq \delta(Z) \leq \mu(X)^n$, where $\mu(X) := \inf_{a\in X_{reg}} \mu_X(a) < +\infty$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from our earlier criterion (see [Z2]) but let us give here a direct proof using the same ideas as in [Z2]. Assume first that X is a piece of an algebraic set and let $Z := Z(X)$. By [Ch], Corollary 11.3.1), there exists an $(N-n)$-plane Γ in $\mathbb{C}^N$ such that the projection $\pi : Z \to Z^{\perp}$ in a δ-sheeted analytic cover, where $\delta := \delta(Z)$ is the degree of algebraicity of Z. Moreover, after a unitary change of variables in $\mathbb{C}^N$, we can assume that for some constant $c > 0$ the following inclusion holds

$$Z \subset \big\{\zeta = (\zeta', \zeta'') \in \mathbb{C}^n \times C^{N-n}; |\zeta''| \leq c(1 + |\zeta'|)\big\}, \tag{3.12}$$

where $\zeta' := \pi(\zeta) = (\zeta_1, \ldots, \zeta_n)$ and $\zeta'' := (\zeta_{n+1}, \ldots, \zeta_N)$.

Let S be the critical set of the projection π. We claim that, for any $a \in Z \setminus S$ and any $w \in \mathcal{L}(Z)$, $\nu(w, a) \leq \delta$. Indeed given any $w \in \mathcal{L}(Z)$ we consider the functions

$$\pi_* w(z) := \sum_{\pi(\zeta)=z} w(\zeta), \quad z \in \mathbb{C}^n. \tag{3.13}$$

Since $\pi : Z \to \mathbb{C}^n$ is a δ-sheeted analytic cover which satisfied (3.12), it follows that $v := \pi_* w \in \mathcal{L}_\delta(\mathbb{C}^n)$ in the sense that v is plusisubharmonic on $\mathbb{C}^n$ and satisfies the estimate

$$v(z) \leq \delta \log(1 + |z|) + c, \quad \forall z \in \mathbb{C}^n, \tag{3.14}$$

where C is a constant depending only on v. Let $a \in Z \setminus S$ and $a' := \pi(a)$. Then there exists an open neighbourhood U' of a' in $\mathbb{C}^n$ and an open neighbouhood $U \Subset Z$ of a such that the restriction $\pi_U : U \to U'$ is biholomorphic. To estimate the Lelong number $\nu_w(a)$, we can assume that $w \leq 0$ on U. Then estimate the Lelong number $\nu_w(a)$, we can assume that $w \leq 0$ on U. Then it follows from (3.13) that $v = \pi_* w \leq w \circ \pi_U^{-1}$ on U', which implies the following estimates:

$$\nu_v(a') \geq \nu_w(a). \tag{3.15}$$

It is known that the function $r \to \max_{|z|=r} v(z)$ is a convex function of $\log r$ for $r > 0$. Then the function

$$r \to \frac{\max_{|z|=r} v(z) - \max_{|z|=s} v(z)}{\log r - \log s}$$

is increasing for $r > s$ for any fixed $s > 0$. By (3.14), this implies that

$$\frac{\max_{|z|=r} v(z) - \max_{|z|=s} v(z)}{\log r - \log s} < \lim_{r \to +\infty} \frac{\max_{|z|=r} v(z)}{\log r} \leq \delta \tag{3.16}$$

for any $r > s > 0$. Letting $s \to 0$ in (3.16) we obtain the inequality $\nu(v; a') \leq \delta$. By (3.15) we have $\nu_w(a) \leq \delta$. This proved our claim. Now if we apply this inequality to $w := (1/d) \log |f| \in (Z)$, where $f \in \mathcal{A}_d(Z) \setminus \{0\}$ we obtain the inequality $\kappa_d(a) \leq \delta$ for any $d \in \mathbb{N}^*$, which proves that X has a finite graded multiplicity at a which is less or equal to δ. This proves (ii) and the estimate $\mu(X) \leq \delta$. Since the implication (ii) $\Longrightarrow$ (iii) is obvious, it remain to prove that (iii) $\longrightarrow$ (i). Indeed let $p := p(Z)$ be the dimension of the algebraic set $Z := Z(X)$. Then it is well known that $h_Z(d) \sim \delta \cdot d^p/p!$ as d tends to $+\infty$, where $\delta = \delta(Z)$ is the degree of algebraicity of Z (see [Ha]). On the other hand, from the identity (1.7) and Theorem 3.1 it follows that

$$h_Z(d) = h_X(d) \leq \binom{d \cdot \mu + n}{n} \sim d^n \mu^n d^n / n!,$$

where $\mu = \mu(X)$. From these estimates it follows that $p \leq n$ and $\delta(Z) \leq \mu^n/n!$. Since $X \subset Z$ we also have $p \leq n$ which proves that $p = n$ and $\delta(Z) \leq \mu^n$. The proof of the theorem is complete. □

It is clear from the definitions that if X is a piece of a complex linear space then $\mu_d(a) = 1$ for any $d \in \mathbb{N}^*$ and then $\mu_X(a) = 1$ for any $a \in X$. Surprisingly, the converse is also true.

Corollary 3.1. *Let X be an analytic subset of dimension n in $\mathbb{C}^n$. Suppose that for some $a \in X_{reg}$ the graded multiplicity of X at a is equal to 1, i.e. $\mu_X(a) = 1$, then X is a piece of a complex linear subspace of $\mathbb{C}^N$, i.e. the Zariski closure $Z(X)$ of X is a complex linear subspace of dimension $n = \dim X$ in $\mathbb{C}^N$.*

Proof. Indeed, from the last theorem it follows that X is a piece of an algebraic set Z of dimension n whose degree of algebraicity satisfies the inequality $\delta(Z) \leq \mu^n$. Since $\mu \leq \mu_X(a) = 1$, this means that Z is a complex linear space of dimension $n = \dim X$. □

From the last theorem, it follows that if X is a transcendental analytic subset then the sequence of graded multiplities $(\mu_d(a))$ is unbounded for any $a \in X_{reg}$. It is an interesting problem to study the relationship between the transcendency of an algebraic set X and the growth of the sequence of its graded multiplicities.

3.3. A criterion an algebraicity for real analytic subsets

Let $M \subset \mathbb{R}^N$ be a real analytic subset of dimension n, i.e. a local irreducible real analytic subset of dimension n in $\mathbb{R}^N$. Let us denotes by $\mathcal{I}_{\mathbb{R}}(M)$ the real polynomial ideal of M in $\mathbb{R}^N$. Then we want to study the following real algebraic subvariety:

$$Z_{\mathbb{R}}(M) := \operatorname{loc} \mathcal{I}_{\mathbb{R}}(M)$$

in terms of the semi-local behaviour of M. More precisely, we want to give a necessary and sufficient (semi-local) condition on M in order that the real algebraic subvariety $Z_{\mathbb{R}}$ be of real dimension n in $\mathbb{R}^N$ in the same spirit as in the complex case. Let us first define the graded multiplicity of M. Let $f : M \to \mathbb{C}$ be an analytic function with $f \notin 0$ and $a \in M$. If $f(a) = 0$, then we will denote by $m_{f,M}(a)$ the order of vanishing of f at the point $a \in M$. If $f(a) \neq 0$ then set $m_f(a) = 0$. We can define the following number

$$\mu_d(M; a) := \sup \left\{ m_{f,M}(a)/d; f \in \mathcal{A}_d(M; \mathbb{R}), f \not\equiv 0 \right\}, \quad d \in \mathbb{N}^*, \tag{3.17}$$

where $A_d(M; \mathbb{R})$ is the space of restrictions to M of real polynomials of degree at most d. For each $d \in \mathbb{N}^*$, the number $\mu_d(M; a)$ will be called the d^{th} real graded multiplicity of M at the point a and the following (possibly infinite) number

$$\mu(M; a) := \sup \left\{ \mu_d(M; a); a \in M \right\}$$

will be called the real graded multiplicity of M at the point a.

Now we can state the main result of this section

Theorem 3.3. *Let $M \subset \mathbb{R}^n$ be a real analytic subset of dimension n in $\mathbb{R}^N$. Then M is a piece of a real algebraic set of dimension n in $\mathbb{R}^N$ if and only if M is of finite real graded multiplicity, i.e.*

$$\mu(M) := \inf \left\{ \mu(M; a); a \in M \right\} < +\infty.$$

In this case, $Z_{\mathbb{R}}(M)$ is an irreducible algebraic subvariety of dimension n, whose degree of algebraicity satisfies the inequality $\delta(Z_{\mathbb{R}}(M)) \leq \mu(M)^n$.

A natural idea is to complexity M in a neighbourhood of a fixed regular point and to apply our previous criterion to this complexification. To do so we then need to compare the real graded multiplicity of M at a fixed point $a \in M$ with the graded multiplicity of its local complexification at the same point a. Before going into the proof of our theorem, we need some preliminaries.

Let Ω be a connected complex manifold of dimension n and M a real analytic submanifold of Ω. We say that M is totally real if for any $a \in M$ $T_aM \cap JT_aM = \{0\}$, where J is the complex structure of Ω. If M is a totally real submanifold of Ω of (real) dimension n, then it is generic in the sense that, for each fixed point $a \in M$ we have $T_aM + JT_aM = T_a\Omega$. Moreover, there exists an open neighbourhood U of a in Ω, a holomorphic isomorphism $\phi : U \to \Delta^n$ from U

onto the unit polydisc in $\mathbb{C}^n$ such that $\phi(a) = 0$ and a real analytic mapping $\psi : I^n = \Delta^n \cap \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\psi(0) = 0, \quad d\psi(0) = 0 \quad \text{and} \quad \phi(M \cap U) = \big\{x + i\psi(x); \|x\| < 1\big\}.$$

Such a coordinate system (U, ϕ) will be called an *adapted coordinate system* to M at the point a and we will write

$$M_s := \big\{\zeta \in M \cap U; \psi(\zeta) = x + i\psi(x), \|x\| \le s\big\}$$

for $0 < s < 1$. Then it is easy to deduced that if f is complex-valued holomorphic function on an open subset of X such that $f\big|(M \cap U) \equiv 0$ then $f \equiv 0$ on U. Therefore if we assume that $f \not\equiv 0$ and $f(a) = 0$, then $f\big|(M \cap N) \not\equiv 0$ so that we can define two integers, the multiplicity $m_f(a)$ of f at the point a as a holomorphic function f in a neighbourhood of the point a in X and the multiplicity $m_{f,M}(a)$ of the function $f\big|(M \cap U)$ at the point a as a real analytic function in a neighbourhood of a in M. It is clear that $m_f(a) \le m_{f,M}(a)$. It is quite clear that there two numbers are equal since M is generic. However, we will state and prove this result, since we do not know any explicit reference to it.

Lemma 3.1. *Let Ω be a complex manifold of dimension n and M a totally real analytic submanifold of real dimension n. Let f be a complex-valued holomorphic function on an open subset $U \subset \Omega$ such that $f \not\equiv 0$ on U. Let $a \in M \cap U$ such that $f(a) = 0$. Then we have the following identity:*

$$m_f(a) = m_{f,M}(a).$$

Moreover, if (U, ϕ) is an adapted coordinate system to M at the point a then the following formula holds

$$m_f(a) = \lim_{s \to 0^+} \frac{\max_{M_s} \log |f(z)|}{\log s} \cdot \tag{3.18}$$

Proof. Since the problem is local, using an adapted coordinate system at the point a, we can assume that $U = \Delta^n$, $a = 0$ is the origin in $\mathbb{C}^n$ and

$$M = \big\{z = x + i\psi(x) \in \mathbb{C}^n; \|x\| < 1\big\},$$

where ψ is a real analytic mapping from the unit cube in $\mathbb{R}^n$ into $\mathbb{R}^n$ vanishing at order at least 2 at the origin. Then we can expand f into homogenuous polynomials and get the following approximation:

$$f(z) = P_m(z) + O(\|z\|^{m+1}) \tag{3.19}$$

uniformly in a neighbourhood of the origin, where $m := m_f(a)$ and P_m is a homogenuous polynomial of degree m. From this it follows that if $P_m(z) \ne 0$ then

$$m = \lim_{s \to 0^+} \frac{\log |f(sz)|}{\log s} = \lim_{s \to 0^+} \frac{\max_{\|z\| \le s} \log |f(z)|}{\log s} \cdot \tag{3.20}$$

Now let us take $z \in M$ then

$$P_m(z) = P_m(x + i\psi(x)) = P_m(x) + O(\|x\|^{m+1}) \quad \text{for } \|x\| \ll 1,$$

since ψ vanishes at order at least 2 at the origin. Pluging this in the formula (3.19), we get that

$$f(x + i\psi(x)) = P_m(x) + O(\|x\|^{m+1}) \quad \text{for } \|x\| \ll 1,$$

which proves that $m = m_{f,M}(0)$ and implies as before that if $P_m(x) \neq 0$, then the following identities hold

$$m = \lim_{s \to 0^+} \frac{\log |f(sx + i\psi(sx))|}{\log s} = \lim_{s \to 0^+} \frac{\max_{\|x\| \leq x} \log |f(x + i\psi(x))|}{\log s} \cdot \tag{3.21}$$

This yield (3.13). □

Now let X be a complex analytic subset of dimension n in $\mathbb{C}^N$ and $M \subset X$ a totally real analytic submanifold of dimension n in some open subset of X_{reg}. We will consider M as a real analytic subset of $\mathbb{C}^N$ by the usual embedding of $\mathbb{R}^N$ into $\mathbb{C}^N$ and denote by $\mathcal{I}(M)$ the polynomial ideal of M in $\mathbb{C}^N$. Let us denote by $Z(M) := \mathrm{loc}\mathcal{I}(M)$ the Zariski closure of M in $\mathbb{C}^N$.

Observe that since M is generic, we have $\mathcal{I}(M) = \mathcal{I}(X)$ and then $Z(M) = Z(X)$. From the lemma above, we can easily deduce the following result.

Proposition 3.1. *Let X be a complex analytic subset of dimension n in $\mathbb{C}^N$ and $M \subset X$ a totally real analytic submanifold of dimension n in some open set in X_{reg} such that*

$$\mu(M) := \inf \{\mu(M; a); a \in M\} < +\infty.$$

Then the Zariski closure $Z := Z(M) = Z(X)$ is an algebraic subvariety of $\mathbb{C}^N$ of dimension $n = \dim M$ and of degree of algebraicity $\delta(Z) \leq \mu(M)^n$.

Proof. Using the lemma, we get $\mu_d(X; a) = \mu_d(M; a)$ for any $a \in M$. Then we deduce that $\mu(X) \leq \mu(M) < +\infty$. Therefore, the proposition follows from Theorem 3.2. □

Now let us deduce Theorem 3.3 from the last proposition. Indeed, let X be a local complexification of M in a neighbourhood ω of a fixed regular point of M in $\mathbb{C}^N$. Then $M' := M \cap \omega$ is a totally real analytic submanifold of real dimension n of the complex manifold $X' := X \cap \omega$. Assume that $\mu(M) < +\infty$. Since $\mu(M') \leq \mu(M)$, if follows from Proposition 3.1, that $Z(M) = Z(M')$ is a complex algebraic subvariety of dimension n. It is easy to see that $Z(M)$ is the complexification of the real algebraic subvariety $Z_{\mathbb{R}}(M) := \mathrm{loc}\mathcal{I}_{\mathbb{R}}(M)$, where $\mathcal{I}_{\mathbb{R}}(M)$ is the real polynomial ideal of M in $\mathbb{R}^N$. Therefore $Z_{\mathbb{R}}(M)$ is an irreducible real algebraic subvariety of dimension n. The converse follows from Theorem 3.2, since, by Lemma 3.1, we have $\mu(M; a) = \mu(X; a)$ for any regular point of M.

Let us mention the following open problem.

Problem. Let $M \subset \mathbb{R}^N$ be a real analytic subset. Assume that there exists a compact subset $K \subset M$ and a subdomain $U \Subset M$ containing K, a real constant

$R = R(K;U) \geq 1$ such that the following "real Bernstein-Walsh inequalities" hold:

$$\|f\|_U \leq \|f\|_K R^d, \quad \forall f \in \mathcal{A}_d(M), \quad \forall d \in \mathbb{N}.$$

Is M a piece of a real algebraic set ?

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