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ON THE STEIN COVERINGS FOR 1-CONVEX SPACES

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Dedicated to the memory of Le Van Thiem

Introduction

Unless the contrary is explicitly stated, all $\mathbb{C}$-analytic spaces considered here are assumed to be non compact, finite dimensional and of $\mathbb{C}$-dimension $n \geq 1$. Furthermore, 3-dimensional (resp. 2-dimensional) connected C-analytic manifolds, will be referred to simply as threefolds (resp. surface).

Let X be a 1-convex space with its exceptional set S. Then it is known [V10] that X is q-complete with $q = \dim S + 1$; also this bound is sharp; hence, the natural generalization of such a result could be stated as follows:

Question 0.1. Is it always possible to convex X by q Stein open subsets ?

On the other hand, let M be a compact C-analytic space carrying a normally ample, effective divisor D. Then $X := M \setminus D$ is actually a 1-convex space. If its exceptional set S consists of only finitely many points, then X is an affine variety. So one would like to raise the following.

Question 0.2. Can such an X be covered by $q := \dim S + 1$ open affine subsets of M ?

This paper is organized as follows. In Section 1, various notions of positivity of line bundles over 1-convex spaces will be established. In Section 2, an affirmative answer to Question 0.1 is given, provided X is embeddable. In Section 3, in the spirit of Chevalley's criterion, a positive answer to Question 0.2 is given, provided M is projective algebraic. Finally, in Section 4 various related problems will be discussed.

1. The positive line bundles

Definition 1.1. (see [G], [V1]) A $\mathbb{C}$-analytic space X is said to be *1-convex space* (or *strongly pseudoconvex*) if there exist:

a) a Stein space Y and finitely many points $T \subset Y$,

b) a surjective, proper and holomorphic map $\pi : X \to Y$ inducing a biholomorphism $X \setminus S \cong Y \setminus T$ where $S := \pi^{-1}(T)$, and

c) $\pi * O_X = O_Y$.

Henceforth, S will be referred to as the *exceptional set* of X. Also, in the special case where $\dim S = 0$, 1-convex spaces are exactly Stein spaces.

Furthermore, a 1-convex space X is said to be *embeddable* if it can be realized as a closed $\mathbb{C}$-analytic subvariety in some ambient space $\mathbb{C}^k \times \mathbb{P}^\nu$.

Example 1.1 (see [G]). Let E be a rank q holomorphic vector bundle on some compact C-analytic manifolds S. Assume that E^* is ample in the sense of [H]. Let X be the total space of the bundle E. By compactifying each fibre $\mathbb{C}^q$ to $\mathbb{P}^q$ one obtains a $\mathbb{P}^q$ bundle M over S, and an effective divisor $D \subset M$ such that $L|D$ is ample since E^* is ample, where L is the holomorphic line bundle on M determined by D. One can check that $X \equiv M \setminus D$ is an embeddable 1-convex manifold with exceptional by D. One can check that $X \cong M \setminus D$ is an embeddable 1-convex manifold with exceptional set S.

On the other hand, non embeddable 1-convex space are explicitly exhibited in [V1, 6, 9].

Definition 1.2. Let X be a $\mathbb{C}$-analytic space, let L be a holomorphic line bundle on X and *let* $\{U_i, e_{ij}\}$ be a system of 1-cocycles determining L.

Then L is said to be

a) *weakly positive* if there exists a system $\{h_i\}$ of positive, smooth and real valued functions on U_i such that on $U_i \cap U_j$

$$h_j = |e_{ij}|^2 h_j$$

and such that the functions $g_i(z) := -\log h_i(z)$ are smooth and strongly plurisubharmonic on U_i

b) *cohomologically positive* if $H^i(X, L^N \otimes F) = 0$ for any $N \gg 0$, any $i \geq 1$, and any $F \in Coh(X) :=$ the category of analytic coherent sheaves on X.

c) *ample* if each stalk $(L^N \otimes F)_X$ is generated by its global sections for any $x \in X$, any $N \gg 0$ and any $F \in Coh(X)$.

It is well known that these 3 notions are equivalent when X is compact [V3]. It is our main purpose here to establish the following generalization.

Theorem 1.1. *Let X be a 1-convex space with its exceptional set S and let L be a holomorphic line bundle on X. Then the following conditions are equivalent:*

i) *L is weakly positive,*

ii) *L is cohomologically positive,*

iii) *L is ample.*

Proof. The implication i) $\Rightarrow$ ii) follows from a result in [AT] (Theorem 1). Meanwhile, the implication ii) $\Rightarrow$ iii) is trivial. Assume that iii) holds. Then, in view of a result in [VI] (Extension lemma), by suitably modifying the metric of L, one can find an integer $k(S)$ such that for any integer $n > k(S)$ the linear system $|nL|$ will give rise to an embedding morphism $\Phi_{|nL|} : X \to \mathbb{C}^k \times \mathbb{P}^\nu$ and that

$\Phi^*_{|\Lambda|} \cong L^n$, where Λ is the line bundle on $\mathbb{C}^k \times \mathbb{P}^\nu$ determined by the linear hypersurface $\mathbb{C}^k \times \mathbf{E}$, with $\mathbf{E}$ being the hyperplane in $\mathbb{P}^\nu$. Since Λ is weakly positive (see Theorem 2.1 below), our desired conclusion follows. □

Definition 1.3. Let L be a holomorphic line bundle on a 1-convex space X. Then L is said to be *positive* if it satisfies one of the equivalent conditions in Theorem 1.1.

Remark 1.1. For arbitrary $\mathbb{C}$-analytic spaces X, the above conditions of positivity are not equivalent, in general. In fact, let $X := \mathbb{C}^k \times \mathbb{P}^\nu$ and let L be the line bundle on X determined by $D := \mathbb{C}^k \times \mathbf{E}$, where $\mathbf{E}$ is the hyperplane bundle on $\mathbb{P}^\nu$. Then L is weakly positive (cf. Theorem 2.1 below). However, one can check that L is not cohomologically positive.

2. The Stein coverings

We are now in a position to provide an affirmative answer to Question 0.1 within the framework of embeddable 1-convex spaces.

Theorem 2.1. *Let X be a 1-convex space with its exceptional set S. Assume that X is embeddable. Then X can be covered by $q := dimS + 1$ Stein open subsets of X.*

Proof. As notice earlier, X is Stein $\dim S = 0$. So by induction. we will assume that our theorem holds for embeddable 1-convex spaces with exceptional sets of $\mathbb{C}$-dimension $< \dim S$. Now, by hypothesis, X can be realized as a closed analytic subvariety in some $\mathbb{C}^k \times \mathbb{P}^\nu$. Let Λ be the line bundle on $\mathbb{C}^k \times \mathbb{P}^\nu$ determined by $Y := \mathbb{C}^k \times \mathbf{E}$ where $\mathbf{E}$ is the hyperplane divisor in $\mathbb{P}^\nu$

Claim. Λ is a weakly positive line bundle.

Indeed, let $(z_1, \dots, z_k) \in \mathbb{C}^k$ and let $(\zeta_0, \dots, \zeta_\nu)$ be homogeneous coordinates in $\mathbb{P}^\nu$. Let

$$U_i := \{(\zeta_0, \dots, \zeta_\nu) \in \mathbb{P}^\nu : \zeta_i \neq 0\} \quad \text{with } 0 \leq i \leq \nu.$$

Certainly $\Lambda \big| U_i$ is trivial transition function $\zeta_j \zeta_i^{-1}$. Let

$$h_i := \Big(\sum |\xi_i|^2\Big)^{-1} \exp\Big(\sum -|z_i|^2\Big).$$

Then one can easily check that $h_i = |\xi_j \xi_i^{-1}|^2 h_j$ and that $-\partial\overline{\partial} \log h_i > 0$, and our claim is proved.

Now let $L : \Lambda|X$. Since X is 1-convex, the vanishing of $H^1(X, L^\nu \otimes \mathbb{F}) = 0$ shows that for a fixed integer $\nu \gg 0$, one can find, for any point $y \in D := X \cap Y$, an element $\tau \in H^0(X, L^\nu)$ such that $\tau(y) \neq 0$. Let $\sigma \in \Gamma(X, L)$ be the canonical section determined by D. Then the function $\phi := \tau/\sigma^\nu$ is meromorphic on X, holomorphic on $Z := X \setminus D$ and has a pole of order ν at y. Since X is holomorphically convex, so is Z. Since $L|S$ is ample, it follows readily that $S \setminus (S \cap Y)$ is affine; hence Z is free of compact analytic subvarieties of positive dimension. Then Z is Stein. On the other hand, D is 1-convex with exceptional set of $\mathbb{C}$-dimension $< \dim S$. By induction hypothesis, D can be covered by $q-1$ Stein open subsets U_j of D with $1 \leq j \leq q-1$. In view of a main result in [S], one can find $q-1$ Stein open subsets V_j in X which cover D. Therefore Z and V_j are the desired Stein coverings of X. □

Corollary 2.1. *Any* 2*-dimensional 1-convex space without* 2*-dimensional compact, irreducible components, can be covered by* 2 *Stein open subsets.*

Proof. In fact, it follows from [V2] that such a 1-convex space is embeddable. Hence the conclusion follows. □

Corollary 2.2. [B-d] (Theorem (1)) *Let X be a relative compact subset of some surface M. Assume that X is 1-convex. Then X can be covered by 2 Stein open subsets.*

In fact, their proof shows even more; namely those 2 open Stein subsets are actually Zariski open which will be our motivation for the next section.

Question 2.1. Is the converse of Theorem 2.1 true?

Unfortunately, the answer is No; in fact, there exist normal 2-dimensional Moihezon spaces M, see e.g. [V1], which

i) have exactly 1 isolated singular point

ii) are non projective

and iii) can be covered by 3 Stein open subsets.

Notice that the exceptional S of any 1-convex space is Moishezon [V1]. On the other hand, a main result in [A], (Theorem 7.3) tells us that there is an equivalence between the categories of Moishezon spaces and proper $\mathbb{C}$-algebraic spaces. So from now on we shall adhere to the following notation.

Notations. Let X be a $\mathbb{C}$-algebraic space in the sense of Artin [A] and let $\mathbb{F} = Coh(\mathbb{X}) :=$ the category of algebraic coherent sheaves on X. Then we shall denote by X_{an} (resp. $\mathbb{F}_{an}$) the underlying $\mathbb{C}$-analytic space of X (resp. the associated analytic coherent sheaf to $\mathbb{F}$). Furthermore, if $\pi : X \to Y$ is an algebraic morphism of $\mathbb{C}$-algebraic spaces, then $\pi_{an} : X_{an} \to Y_{an}$ denotes the associated analytic morphism of $\mathbb{C}$-analytic spaces.

Remark 2.1. Let X be a $\mathbb{C}$-algebraic space. It is known that, for complete $\mathbb{C}$-algebraic space X, $\dim_{\mathbb{C}} H^i(X, \mathbb{F}) < \infty$ for all $\mathbb{F} \in Coh(X)$ and any integer $i \geq 0$. Hence we are ready to establish an analogue of GAGA comparison theorem of Serre and Grothendieck in this framework.

Theorem 2.2. *Let X be a $\mathbb{C}$-algebraic space such that $\dim_{\mathbb{C}} H^1(X, \mathbb{F}) < \infty$ for all $\mathbb{F} \in Coh(X)$.*

Let $A \in Coh(X)$ and let $\mathbb{H} := \mathbb{A}_{an}$. Then

i) *$X := X_{an}$ is a 1-convex space, and*

ii) *the natural maps of cohomology groups*

$$\alpha_i : H^i(X, \mathbb{A}) \longrightarrow H^i(X, \mathbb{H})$$

are bijective for any $i \geq 1$.

Proof. i) It follows from [GH] that X is a proper modification of some $\mathbb{C}$-algebraic space Y which is affine; precisely, one has a proper morphism $\pi : X \to Y$ such that

a) $\pi * OX = OY$, and

b) the set $T := \{y \in Y | \dim\pi^{-1}(y) > 0\}$ is finite.

Consequently, we obtain the following commutative diagram

$$(*) \qquad \begin{array}{ccc} Y_{an} & \longrightarrow & Y \\ \pi_{an}\Big\uparrow & & \Big\uparrow\pi \\ X_{an} & \longrightarrow & X \end{array}$$

Since Y_{an} is Stein, it follows readily that X is 1-convex with exceptional set $S := S_{an}$ where $S := \pi^{-1}(T)$.

ii) Now let $\mathbb{R}$ be the extension to zero on $X \setminus S$ of the (set) restriction of $\mathbb{A}$ to S and let $\mathbb{G} := \mathbb{R}_{an}$. Then one has the following commutative diagram

$$(**) \qquad \begin{array}{ccc} H^i(X, \mathbb{R}) - \beta_i & \longrightarrow & H^i(X, \mathbb{G}) \\ \cong\Big\uparrow & & \Big\uparrow\cong \\ H^i(X, \mathbb{A}) - \alpha_i & \longrightarrow & H^i(X, \mathbb{H}) \end{array}$$

By virtue of $(*)$, the isomorphisms of the vertical arrows, for any $i > 0$, follows from a main result in [N] (Theorem V). In view of $(**)$, it suffices to show that the induced maps β_i are isomorphisms for any i. Notice that $\mathbb{R}$ is not, in general, a coherent sheaf of O_S-module.

Claim. There exists a decreasing sequence of algebraic coherent sheaves

$$0 = R_k \subset \cdots \subset R_i \subset \cdots \subset R_0 = R$$

such that R_i/R_{i+1} are algebraic coherent sheaves of O_S-module for any $i \geq 0$.

In fact, let $a(R)$ be the ideal sheaf of annihilator of R, namely

$$a(R) := \big\{\text{Kernel of the canonical sheaf morphism } O_X \to Hom(R, R)\big\}$$

and let I be the ideal sheaf in O_X determined by S. Then, in view of the definition of $\mathbf{R}$, one has $V(a(R)) \subset V(I)$ where $V(\ )$ is the algebraic subvariety of X determined by the ideal sheaf ( ). From the compactness of S, we infer the existence of an integer $\kappa \geq 0$ such that $I^k.R = 0$. Let $R_i := I^i R$. Certainly R_i/R_{i+1} are algebraic coherent sheaves of O_S-modules, since $I.\ R_i/R_{i+1} = 0$ for any $i \geq 0$. Hence our claim is proved.

Now let $G_i := (R_i)_{an}$. Following Artin [A], one obtains the isomorphisms

$$(\#) \qquad H^\nu(S, R_i/R_{i+1}) \longrightarrow H^\nu(S, G_i/G_{i+1})$$

for any $\nu \geq 0$ and any $i \geq 0$. From the following exact sequence

$$0 \longrightarrow R_i/R_{i+1} \longrightarrow R/R_{i+1} \longrightarrow R/R_i \longrightarrow 0$$

one deduces, for any fixed integer $k \geq 0$ and any $i \geq 0$, the following commutative diagram of cohomology groups, with exact rows

$$\begin{array}{ccccccc} \to H^k(X, G_i/G_{i+1}) \cong H^k(S, G_i/G_{i+1}) & \to & H^k(X, G/G_{i+1} & \to & H^k(X, G/G_i) & \to \\ \cong\uparrow & & \uparrow & & \uparrow & \\ \to H^k(X, R_i/R_{i+1}) \cong H^k(S, R_i/R_{i+1}) & \to & H^k(X, R/R_{i+1}) & \to & H^k(X, R/R_i) & \to \end{array}$$

By using the ascending induction on i, $(\#)$ and the five-lemma will provide us the desired conclusion. □

Remark 2.2. a) By using the same notation as in the previous proof, one can check easily that the commutative diagram $(*)$ will give rise to the following natural isomorphisms of direct image of analytic coherent sheaves

$$R^k\pi *_{an} \mathbf{F}_{an} \cong R^k\pi * (\mathbf{F})_{an}$$

for any integer $k \geq 1$.

b) When X is non singular, a slightly weaker version than Theorem 2.2 has been established in [H] by using different techniques. Our next goal is to find necessary and sufficient conditions for a given 1-convex space to be an underlying $\mathbb{C}$-analytic space of some $\mathbb{C}$-algebraic space.

3. The affine covers

As a first step toward that goal, Example 1.1 is generalized as follows.

Proposition 3.1. (see [V8], Theorem 3.3) *Let M be a $\mathbb{C}$-analytic space and let D be an effective divisor which is normally ample, i.e. $L \setminus D$ is ample where L is the holomorphic line bundle on M determined by D. Then*

i) *$X := M \setminus D$ is 1-convex, and*

ii) *M is Moishezon.*

In view of this result, we introduce the following

Definition 3.1. Let X be a $\mathbb{C}$-algebraic space (resp. a non singular $\mathbb{C}$-algebraic space). Then X is said to an algebraic 1-convex (resp. a non singular *algebraic 1-convex*) *model* if X is algebraically isomorphic to $M \setminus D$ where M is some complete $\mathbb{C}$-algebraic space (resp. some non singular complete $\mathbb{C}$-algebraic space), and $D \subset M$ a normally ample effective divisor. If furthermore M can be selected to be projective algebraic, then we say that X is *projectivizable.*

The motivation for the terminology in Definition 3.1 stems from the following results.

Proposition 3.2. *Let X be a $\mathbb{C}$-algebraic space. Then the following conditions are equivalent:*

(a) *X is an algebraic 1-convex model,*

(b) *X is a modification in the sense of* [GH] *of some affine $\mathbb{C}$-algebraic space A,*

(c) $\dim_{\mathbb{C}} H^1(X, \mathbf{J}) < \infty$ *for any* $\mathbf{J} \in Coh(X)$.

Proof. (a) $\Rightarrow$ (b) Let M be a complete $\mathbb{C}$-algebraic space with a normally ample divisor $D \subset M$ such that X is algebraically isomorphic to $M \setminus D$. In view of Proposition 3.1, $X := X_{an}$ is a 1-convex $\mathbb{C}$-analytic space with an exceptional set $S \subset X$. Since S can be contracted to finite points, one obtains a compact $\mathbb{C}$-analytic space Y, a finite set $T \subset Y$, and a proper, surjective holomorphic morphism $\Pi : M \to Y$ which induces a biholomorphism $M \setminus S \cong Y \setminus T$. Let $D := \Pi(D)$. Since X is 1-convex, $Y \setminus D$ is free of compact subvarieties of positive dimensions. Therefore the normally ample divisor D is actually the support of an ample divisor [G]. Consequently, Y is in fact projective algebraic and $A := Y \setminus D$ is affine. We infer that Π is actually an algebraic morphism and $\pi := \Pi | X : X \to A$ is the desired algebraic modification.

(b) $\Rightarrow$ (a) Assume that $\pi : X \to A$ a modification of some affine variety A; let $T \subset A$ be a finite subset, such that $U := X \setminus S \cong A \setminus T$ where $S := \pi^{-1}(T)$. Then X is an algebraic 1-convex model. Indeed, since A is affine one can find a complete $\mathbb{C}$-algebraic space Z and an effective ample divisor $E \subset Z$, such that $Z \setminus E \cong A$. By gluing Z and X along U, one obtains a complete $\mathbb{C}$-algebraic space M, a surjective and algebraic morphism $\pi : M \to Z$. Now let $D := \pi^{-1}(E) \subset M$. It is obvious that D is an effective normally ample divisor on M such that X is algebraically isomorphic to $M \setminus D$. Finally, the equivalence of (b) and (c) follows from the results in $[GH]$. □

Definition 3.2. a) A $\mathbb{C}$-analytic space (resp. a $\mathbb{C}$-analytic manifold) X is said to be *algebraically 1-convex* if there exists a algebraic 1-convex (resp. a non singular) model X such that X is biholomorphic to X_{an}. Furthermore, X is said to be *projectivizable* if X is projectivizable in the sense of Definition 3.1.

b) An 1-convex space X is said to be *quasi-algebraic* if there exists a C-algebraic space X such that X is biholomorphic to X_{an}.

Remark 3.1. a) Any 2-dimensional algebraically 1-convex surface X is always projectivizable.

b) From the proof of Proposition 3.2, one notices that M is birational to some projective variety Y; yet M itself is not so in general, as shown by the following example.

Example 3.1. (see [V9]) There exist Moishezon 3-folds M which is a small resolution of some normal 3-dimensional projective variety Y with only one isolated singular point, and a normally ample divisor $D \subset M$ such that $X := M \setminus D$ is 1-convex. Yet X *is not even Kahlerian.*

Confronted with this state of affairs, we have the following theorem.

Theorem 3.1. (Chevalley's criterion) *Let M be a Moishezon space. Then M is projective algebraic iff any finite set of points in M is contained in some open affine subsets of M.*

This result was first established by Kleiman for complete varieties with only factorial singularities, see e.g. [H]. This version was established by C. Horst in her thesis [Ho]. With this in mind, Question 0.2 can be answered as follows.

Theorem 3.2. *Let X be an irreducible algebraic 1-convex space with its exceptional set S. Then X is projectivizable if and only if X can be covered by $q := \dim S + 1$ (Zariski) open affine subsets.*

Proof. i) Let M be an irreducible compact $\mathbb{C}$-analytic space with a normally ample divisor D such that $X \cong M \setminus D$.

Step 1. Assume that $\dim S = 0$. Then D is the support of some ample divisor say D. Consequently, for some integer $n \gg 0$, the linear system $|nD|$ will embed M into some P_m such that $D = H_0 \cap M$, where H_0 is some hyperplane divisor in P_m. Hence X can be covered by the affine open set $P_m \setminus H_0$.

Step 2. Assume that X is projectivizable and $\dim S > 0$. By hypothesis, one can assume that M is embedded in some P_m. Let us select $q - 1 := \dim S$ hyperplane divisor H_i, $0 \leq i \leq q-1$, which intersect S transversally. Let X (resp. M, S and D) be the intersection of X (resp. M, S and D) with $\bigcap_i H_i$ $(1 \leq i \leq q-1)$. Obviously, the exceptional set S of the algebraic 1-convex space $X \cong M \setminus D$ consists of only finite many points. Hence we infer from Step 1 that X can be covered by the affine open set $P_m \setminus H_q$ for some suitable hyperplane divisor H_q. Consequently, X can be covered by q affine subsets $V_j := P_m \setminus H_j$ with $1 \leq j \leq q$.

ii) Assume that $C - \dim S = 0$. It follows readily that X is projectivizable. Now, by a tedious inductive argument, one can infer from Theorem 3.1 that X is indeed projectivizable. □

4. Epilogue

From the prototypes of the previous sections, we would like to discuss for a given 1-convex space X, the connection between its various algebraic structures. First of all, a new basic definitions are in order.

Definition 4.1. A $\mathbb{C}$-analytic space (resp. $\mathbb{C}$-analytic manifold) X is said to be *compactifiable* if there exists a compact $\mathbb{C}$-analytic space (resp. compact $\mathbb{C}$-analytic manifold) M and a compact analytic subvariety $\Gamma \subset M$ such that $X \cong M \setminus \Gamma$. Furthermore, if M is projective algebraic, then X is said to be *quasi projective.*

We have the following characterization.

Proposition 4.1. (see [Gi], Theorem 3.1) *Let X be an algebraic 1-convex manifold. Then X is projectivizable if and only if X is quasi projective.*

Problem 4.1. Is Proposition 4.1 still true for arbitrary algebraic 1-convex spaces ?

On the other hand, one of our main concerns would be

Problem 4.2. Are quasi-algebraic 1-convex spaces always algebraic 1-convex ?

We are ready to formulate the central theme for further investigations.

Problem 4.3. Let X be a compatifiable 1-convex space.

i) Is X always quasi-algebraic ?

ii) Is X always algebraically 1-convex ?

The motivation behind this problem stems from

Theorem 4.1. (see [V4, 5]) *Let X be a compactifable 1-convex surface. Then X is quasi projective.*

On the basis of this result, one would like to implement Proposition 4.1 as follows.

Problem 4.4. Let X be a compactifiable 1-convex surface. Is X always algebraically 1-convex ?

In fact, one can prove the following result.

Proposition 4.2. (see also [V5]) *Let X be a compactifiable Stein space. Then X is algebraically 1-convex if and only if X carries some affine structure.*

On the other hand, we have the following example.

Example 4.1. (see [V11]) There exists (minimal) ruled surface M of genus $g > 1$ and a section $D \subset M$ with $D^2 = 0$ such that $X := M \setminus D$ is a Stein surface and X does not carry any affine structures so X is not algebraically 1-convex.

However, as noted in [V4, 5] there are significant differences between the structure of compactifiable Stein surfaces and those of compactifiable 1-convex surfaces. Thus Problem 4.4 still remains open for (minimal) 1-convex surfaces which are not Stein; in which case, algebraic 1-convexity is equivalent for X to acquire some pseudo-affine structure in the sense of [V4]. As far as the compactifiable 1-convex and Stein 3-folds are concerned. The situation is getting more complicated, due to the lack of classification of compact 3-folds, so we refer to [V6, 7] for further discussions. To round off this discussion, for the reader's convenience, we would like to present the following scheme to summarize the current status of our investigation.

Let X be a given 1-convex space. Then, upon which, the potential additional structures introduced in this paper are distributed as follows:

Projectivizable $\underset{\neq \text{Ex. } 3.1}{\subset}$ Algebraic $\subset$ Quasi-algebraic

$\underset{\neq \text{Ex. } 4.1}{\bigcap}$ $\qquad\qquad\qquad\qquad$ $\bigcap$

Quasi projective $\underset{\neq \text{ Ex. } 3.1}{\subset}$ Compactifiable

(= for 1-convex surfaces, Theorem 4.1)

The symbol $\subset$ with $\neq$ means strict inclusion, in view of the existence of counterexamples. Otherwise, the problem remains open.

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