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# ONE DIMENSIONAL MODELS FOR QUANTUM OSCILLATORS

FRÉDÉRIC PHAM

*Dedicated to the memory of Le Van Thiem*

The subject I will review here has been the "organizing center" of my research during the last ten years or so. It takes its roots as far as the middle of the $19^{th}$ century (Airy, Stokes), and is characterized by strong interaction between mathematics and physics.

## 1. The objects of study: some more or less "special" functions

The seminal example is the *Airy function*, a remarkable solution of the differential equation

$$\text{(Airy eq.)} \qquad \Big(\frac{d^2}{dX^2} - X\Big)U = 0.$$

It was originally introduced by the British astronomer G. Airy in 1837 for modelizing the *rainbow* phenomenon, and more generally for modelizing light waves near a generic point of a *caustic*.

Second after the Airy model (in increasing complexity) comes the *Weber* model, given by the differential equation

$$\text{(Weber eq.)} \qquad \Big(\frac{d^2}{dX^2} - X^2 + E\Big)U = 0.$$

As in the Airy case, remarkable solutions of this equation can be written under simple integral form (the Weber integral): these special functions are the so-called "parabolic cylinder functions" (cf. e.g. [14], Chap. XVI). A remarkable feature of this equation is the fact that there is a discrete set of values of the parameter $E$ ($E = 1, 3, 5, 7, \dots, 2n+1$) for which the equation has "bound states", i.e. solutions which tend to zero rapidly for $X \to \infty$ and also for $X \to -\infty$ along the real axis. This is one of the simplest examples of the "quantization of energy" in wave mechanics (the Weber equation is the Schrödinger equation for the so-called *harmonic oscillator*, and the parameter $E$ is the energy of the oscillator).

My main efforts during the last ten years have been directed towards studying "higher order analogs" of the Airy and Weber models, all of which given by

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differential equations of the form

$$\left(\frac{d^2}{dX^2} - F(X)\right)U = 0 \tag{1}$$

with $F$ a polynomial function, the degree $m$ of which I shall call the *order* of the model (in particular, the sine or exponential function may be called a model of order zero !).

Famous among physicists is the *anharmonic oscillator*, corresponding to $F(X) = X^2 + \lambda X^4 - E$, which physicists like to consider as a "toy model" for the more sophisticated "perturbation expansion" problems of QED (Quantum Electrodynamics). For positive $\lambda$ there is again a discrete sequence of "bound state energies" $E_0(\lambda),\ E_1(\lambda), \dots, E_n(\lambda), \dots$ which physicists like to expand in powers of the "perturbation parameter" $\lambda$:

$$E_n(\lambda) = (2n+1) + E_n^{(1)}\lambda + E_n^{(2)}\lambda^2 + \dots \tag{2}$$

(the Rayleigh Schrödinger perturbation series, as physicists call them). These expansions are known to be *divergent*, and physicists have spent much effort analysing the nature of that divergence. It is not difficult to show that the $E_n$'s are analytic functions of $\lambda$ for $\lambda > 0$, and can be analytically continued in large sectors of the complex $\lambda$-plane. In 1969 and 1973 Bender and Wu ([2], [3]) tried to analyze in detail the singular structure near $\lambda = 0$, suggesting by various heuristic arguments that $E_0, E_2, E_4, \dots$ on the one hand, $E_1, E_3, E_5, \dots$ on the other hand, are just branches of *one* multivalued analytic function of $\lambda$ with a discrete set of square-root branch points accumulating at the origin of the complex $\lambda$-plane in imaginary directions. These conjectures were made more precise and proved in 1997 by E. Delabaere and myself ([5])[1]. Notice that the anharmonic oscillator corresponds to choosing for $F$ the generic *even* polynomial of degree 4 (an obvious rescaling allows one to rewrite $F$ as $X^4 + \alpha X^2 - E$). In [5] we also studied, although less extensively, the case when $F$ is the *generic polynomial of degree* 4 ($F = X^4 + \alpha X^2 + \beta X - E$), analysing in particular the so-called *avoided crossing* phenomenon, which can be interpreted in terms of square-root branch points close to the real $\beta$-axis, for fixed negative real $\alpha$.

More recently we started investigating the *cubic oscillator*, where $F$ is the generic polynomial of degree 3. In that case bound states occur only for *complex F*. Of special interest for physicists are "PT-symmetric" bound states, corresponding to the case where $F(-X) = \overline{F(\overline{X})}$. For instance Zinn-Justin and Bessis conjectured a long time ago (on the basis of numerical evidence) that for $F = iX^3 + X^2 - E$ bound states occur only a discrete sequence $(E_n)$ of *real* values of $E$. In [6] we proved an analogous statement with $iX^3$ replaced by $i\lambda X^3$, $\lambda$ real and *small*. Actually we have good reasons to believe that the statement holds true for arbitrary real $\lambda$. Our program of proof involves precise conjectures (similar to Bender and Wu's) on the ramified structure of the bound state energies $E_n$ as functions of the *complex* parameter $\lambda$. Our thesis student Trinh Duc Tai (from Dalat) has started working on this program.

[1] Actually we only proved them for small enough $|\lambda|$

## 2. The tools: resurgence theory

In order to study the differential equation (1), we rewrite it in apparently more general forms, depending on a "scale parameter" $\hbar$ (the notation comes from physics: $\hbar$ is Planck's constant): we write $\hbar^2 \dfrac{d^2}{dx^2}$ instead of $\dfrac{d^2}{dX^2}$, and also make the coefficients of the polynomial $F$ depend (possibly) on $\hbar$ in various "natural" ways, thus yielding various "rescalings" of the differential equation (1) (which yield again (1) for $\hbar = 1$). Actually the gain of generality is only apparent: inasmuch as the coefficients of the polynomial $F$ are considered as free parameters, the rescaled equation can be obtained from (1) by multiplying $X$ with a suitable fractional power of $\hbar$, and making the coefficients of $F$ depend on $\hbar$ in a suitable way. Intuitively, every such way of "rescaling" (1) can be thought of as way of "looking at the solutions of (1) through a magnifying glass", the parameter $\hbar$ being the inverse of the amplitude of magnification ($\hbar$ can be thought of as "small").

In order to study the solutions of the rescaled equation, one tries to expand them in formal power series of $\hbar$. Such expansions (which are *divergent*) are known to physicists under the name of WKB *expansions* (WKB stands for Wentzel, Kramers and Brillouin), and their study is called *semi-classical asymptotics.* After the works of Voros [13] and Ecalle [8], we know that these expansions belong to the class of so-called *resurgent* expansions introduced by J. Ecalle around 1980. This implies that although divergent they can be *resummed*, defining analytic functions of $\hbar$ in complex sectors of the form[2] $\theta_0 < \arg \hbar < \theta_1$ (with $\theta_1 - \theta_0 < \pi$), $|\hbar| < \rho$. Ecalle's resummation procedure is a very natural variant of Borel's resummation procedure: the main difference is that instead of defining the resummed function by means of a Laplace integral along the positive real axis one may integrate along any half-line $[0, e^{i\theta}\infty[$ which meets no singularities of the "Borel transform"[3]. If two such "non singular" half-lines are separated by singularities of the Borel transform, the resulting resummations differ by small exponentials, which may be called *Stokes discontinuities*[4]. In all natural applications of the theory these "Stokes discontinuities" can be described in closed form by so-called "resurgence equations", which express them explicity in terms of the original expansions (in the same way as a differential equation expresses the derivative of a function in terms of the function itself).

Precise control of small exponential effects has been an important challenge for asymptoticians (whether physicists or applied mathematicians) in the recent years. Around 1990 the British physicist Michael Berry noticed that the ideas of Ecalle bore deep connection with some ideas of Dingle (whose book [7] is a

[2]In our case these functions also depend analytically on $x$ and on the coefficients of the polynomial $F$.

[3]For a brief account on Borel resummation, and its connection with the topics discussed here, cf. e.g. [11].

[4]Cf. the seminal article of Stokes on the Airy function ([12]), which can be understood as a first step into Ecalle's theory.

great classics among applied asymptoticians). More precisely, Berry noticed that in the examples studied by Dingle (which obviously belonged to the class introduced by Ecalle), the resurgence equations of Ecalle accounted for a phenomenon noticed by Dingle, namely the fact that if one truncates the divergent expansion in a suitable way ("truncating to the least term") the remainder term can be reexpressed explicitly in terms of the initial expansion. This allows us to iterate the "evaluation by truncation" process, yielding an impressively accurate scheme for numerical computation (for which M. Berry coined the term "hyperasymptotics" [1]). This way of understanding resurgences is now quite popular among a larger and larger community of applied asymptoticians, so that at the present time word *resurgence* is used by two communities, in two senses which are certainly deeply connected to each other (although this connection is not yet completely elucidated): a (small) community of geometers (pure mathematicians, if you prefer), who use Ecalle's theory to prove theorems; a (larger) community of "applied asymptoticians", who use the word "resurgence" in the sense of M. Berry, putting emphasis on numerical results. My works with Delabaere pertain to the first category, although numerical esperimentation is not absent from them.

## 3. The hierarchy of models

Comming back to the so-called "models" of Section 1, what I call the "hierarchy of models" is the following observation:

*Looking at a model of arbitrary order with a suitable magnifying glass, one "sees" a model of lower order.*

This vague and intuitive statement can be made precise in various ways. "Looking through a magnifying glass" means rescaling the differential equation in one of the ways mentioned at the beginning of Section 2. Depending on how the coefficients of $F$ are rescaled, the limit of $F$ as $\hbar \to 0$ may have zeros of various orders (each such zero is called a *turning point* of the rescaled differential equation). Outside turning points, the solutions of the rescaled equation "look like" sine or exponential functions models of order 0, in the terminology of Section 1; near a turning point of order 1, they "look like" solutions of the Airy equation; near a turning point of order 2, they "look like" solutions of the Weber equation; etc.

Apart from the "etc.". i.e. just considering turning point of order 1 or 2, the above idea is essentially well known, and has been extensively used by applied asymptoticians (who express it by saying that the Airy resp. Weber equation provides good "uniform approximations" for solutions of scaled differential equations near a turning point of order 1 resp. 2). The first exact (i.e. non approximate) formulation - using resurgence theory - has been given by Ahmedou ould Jidoumou in his thesis [9], and then improved by Eric Delabaere and myself [4].

In [10] I proposed a generalization for turning points of arbitrary order. Unfortunately the statement of that generalization was a bit technical. Nowadays I have a much better statement (stronger and simpler), which I found last November (two months after this colloquium) while delivering a postgraduate course in

Dalat on these questions: the statement (with a sketch of proof) can be found in the typewritten notes of my Dalat course (written by Trinh Duc Tai, in Vietnamese); a more detailed version will appears in the proceedings of the conference held in Kyoto in December 1998; *Towards the exact WKB analysis of differential equations...*, edited by T. Kawai and Y. Takei (to be published by Kyoto University Press).

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Université de Nice
Laboratoire J. A. Dieudonné/URM 6621
Parc Valrose 06180 Nice Cedex 2, France